Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Viorel Barbu, Giuseppe Da Prato, Arnaud Debussche

Essential m-dissipativity of Kolmogorov operators corresponding to periodic $2D$-Navier Stokes equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **15** (2004), n.1, p. 29–38.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2004_9_15_1_29_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2004.

Rend. Mat. Acc. Lincei
s. 9, v. 15:29-38 (2004)

Analisi matematica. — *Essential m-dissipativity of Kolmogorov operators corresponding to periodic 2D-Navier Stokes equations.* Nota (*) di Viorel Barbu, Giuseppe Da Prato e Arnaud Debussche, presentata dal Socio G. Da Prato.

Abstract. — We prove the essential m-dissipativity of the Kolmogorov operator associated with the stochastic Navier-Stokes flow with periodic boundary conditions in a space $L^2(H, \nu)$ where ν is an invariant measure.

Key words: Stochastic Navier-Stokes equations; Kolmogorov operators; Invariant measures.

Riassunto. — *Essenziale m-dissipatività dell'operatore di Kolmogorov associato all'equazione di Navier-Stokes stocastica.* Si dimostra l'essenziale m-dissipatività dell'operatore di Kolmogorov associato al flusso dell'equazione di Navier-Stokes stocastica con condizioni periodiche in uno spazio $L^2(H, \nu)$ dove ν è una misura invariante.

1. Introduction

We are here concerned with the following stochastic Navier-Stokes equation,

$$(1.1)\qquad \begin{cases} dX = (\nu_0 \Delta X + (X \cdot \nabla) X)\, dt + \nabla p dt + \sqrt{C} dW \quad \text{in } D \times \mathbb{R}^+, \\ \operatorname{div} X = 0 \quad \text{in } D \times \mathbb{R}^+, \\ X(t, \cdot) \text{ is periodic with period } 2\pi, \ \int_D X(t, \xi)\, d\xi = 0 \\ X(0, \xi) = x(\xi) \quad \text{in } D, \end{cases}$$

where D is the square $[0, 2\pi]^2$. The unknown X represents the velocity and p is the pressure, $C \in L(H)$ is a linear operator and W is a cylindrical Wiener process in H associated with a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$. The Hilbert space H is defined as

$$H = \left\{ x \in (L^2(D))^2 \colon \operatorname{div} x = 0 \ \text{ in } D, \int_D x(\xi)\, d\xi = 0 \right\}.$$

Moreover we set

$$V = (H^1_\#(D))^2 \cap H,$$

where the subscrit $\#$ means periodicity.

The operator $C \in L(H)$ is nonnegative, symmetric and of trace class and W is H-valued.

Moreover, we introduce the Stokes operator

$$A = \nu_0 P \Delta, \quad D(A) = V \cap (H^2_\#(D))^2, \quad V = \{ y \in (H^1_\#(D))^2 \colon \operatorname{div} y = 0 \},$$

(*) Pervenuta in forma definitiva all'Accademia il 30 ottobre 2003.

and set

$$(By, v) = b(y, y, v), \quad v \in V,$$

where

$$b(y, z, v) = \sum_{i,j=1}^{2} \int_D y_i D_i z_j v_j d\xi.$$

Then it is classical that problem (1.1) is equivalent to

$$\begin{cases} dX(t) = (AX(t) + BX(t))\, dt + \sqrt{C} dW(t) \\ X(0) = x. \end{cases} \tag{1.2}$$

We shall assume in all the paper that

HYPOTHESIS 1.1. $\mathrm{Tr}\,[C(-A)] < \infty$.

It is well known, see *e.g.* [6] that (1.1) has a unique solution $X = X(t, x)$. We shall denote by P_t the associated transition semigroup

$$P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad t \geqslant 0, \quad x \in H, \quad \varphi \in B_b(H), \tag{1.3}$$

where $B_b(H)$ is the space of all Borel bounded mappings $\varphi : H \to \mathbb{R}$ and $\mathbb{E}$ means expectation. Under Hypothesis 1.1, there exists at least one invariant measure ν for P_t. Even if this fact is well known, we shall present a proof in §2 for the reader's convenience and also because we shall obtain, as a byproduct, an estimate needed later. Moreover, under some mild non degeneracy conditions, this invariant measure is unique [2, 6-9, 11]. We do not need any non degeneracy condition here and fix one invariant measure ν from now on. Therefore, P_t can be uniquely extended to a contraction Markov semigroup in $L^2(H, \nu)$, still denoted by P_t. We shall denote by N its infinitesimal generator. The main result of this paper is that N is the closure in $L^2(H, \nu)$ of the Kolmogorov operator N_0

$$N_0 \varphi(x) = \frac{1}{2} \mathrm{Tr}\,[QD^2 \varphi(x)] - (Ax + B(x), D\varphi(x)), \quad x \in H, \quad \varphi \in \mathcal{E}_A(H), \tag{1.4}$$

where $\mathcal{E}_A(H)$ (the space of exponential functions) is the linear span of all functions (more precisely all real and imaginary parts of all functions)

$$\varphi_h(x) = e^{i(h, x)}, \quad x, h \in D(A).$$

In other words, we show that N is an extension of N_0 and that $\mathcal{E}_A(H)$ is a core for N.

This result generalizes a similar one, see [1] that was proved under the unpleasent assumption that the viscosity ν_0 is sufficiently large (notice, however, that the result in [1] is valid also for Dirichlet boundary conditions). In the present paper the viscosity ν_0 will not play any role, it will be put equal to 1 in what follows.

This result of *m*-dissipativity allows to identify the abstract generator N with the closure of a concrete differential operator. This has important consequences as the existence of strong solutions in the sense of Friedrichs of the elliptic Kolmogorov equation

$$\lambda\varphi - N\varphi = f. \tag{1.5}$$

This means that for any $\lambda > 0$ and $f \in L^2(H, \nu)$ there exists a sequence $\{\varphi_n\} \subset \mathcal{E}_A(H)$ such that

$$\lim_{n\to\infty} \varphi_n = \varphi, \quad \lim_{n\to\infty} N_0\varphi_n = N\varphi \quad \text{in } L^2(H, \nu).$$

Another consequence, see Theorem 4.2 below, is the following integration by parts formula

$$\int_H N\varphi\,\varphi d\nu = -\frac{1}{2}\int_H |C^{1/2} D\varphi|^2\, d\nu, \tag{1.6}$$

valid for all $\varphi \in D(N)$. Formula (1.6) is the starting point for studying the Sobolev space $W^{1,2}(H, \nu)$ and several properties of the measure ν. This will be the object of a future research.

Let us conclude this section with some technical remarks that will be used in what follows. First we recall that

$$b(y, z, z) = 0, \quad b(y, z, v) = -b(y, v, z), \quad y, v, z \in V \tag{1.7}$$

and, in view of the periodicity condition, we also have the basic identity

$$b(y, y, Ay) = 0, \quad y \in D(A). \tag{1.8}$$

We shall denote by $|\cdot|$ the norm in H, by $(\cdot, \cdot)$ the inner product in H and by $\|\cdot\|$ the norm in V. It is well known that

$$\|x\| \geqslant \pi^2 |x|. \tag{1.9}$$

Finally, we recall the Sobolev embedding theorem

$$H^\alpha(D) \subset L^{\frac{2}{1-\alpha}}(D), \quad \alpha \in (0, 1), \tag{1.10}$$

and the classical interpolatory estimate

$$\|x\|_{H^b(D)} \leqslant \|x\|_{H^a(D)}^{\frac{c-b}{c-a}} \quad \|x\|_{H^c(D)}^{\frac{b-a}{c-a}}, \quad 0 \leqslant a < b < c, \tag{1.11}$$

where $\|x\|_{H^b(D)} = |x|$.

2. Existence of invariant measure

It is convenient to introduce the following approximating problems,

$$\begin{cases} dX_\varepsilon(t) = AX_\varepsilon(t)\, dt + B_\varepsilon(X_\varepsilon(t))\, dt + \sqrt{C} dW(t) \\ X_\varepsilon(0) = x \in H \end{cases} \tag{2.1}$$

where

$$B_\varepsilon(x) = \begin{cases} B(x) & \text{if } \|x\| \leqslant \dfrac{1}{\varepsilon}, \\ \dfrac{B(x)}{\varepsilon^2 \|x\|^2} & \text{if } \|x\| > \dfrac{1}{\varepsilon}. \end{cases} \tag{2.2}$$

Since, for $\varepsilon > 0$, B_ε is regular and bounded, problem (2.2) has a unique solution which we denote by $X_\varepsilon(t, x)$. Moreover, it is well known that there exists an invariant measure ν_ε for problem (2.1).

We shall denote by N_ε the Kolmogorov operator,

$$N_\varepsilon(x) = L\varphi(x) - (B_\varepsilon(x), D\varphi(x)), \quad x \in H, \ \varphi \in \mathcal{E}_A(H), \tag{2.3}$$

where L is the Ornstein-Uhlenbeck operator in $C_b(H)$, the Banach space of all uniformly continuous and bounded real functions in H, and $D(L)$ is its domain,

$$L\varphi(x) = \frac{1}{2}\operatorname{Tr}[CD^2\varphi(x)] - (Ax, D\varphi(x)), \quad x \in H, \ \varphi \in D(L). \tag{2.4}$$

We recall that $D(L)$ can be defined troughout its resolvent as follows, see [3]; we consider the Laplace transform of P_t,

$$F(\lambda)\, f(x) := \int_0^{+\infty} e^{-\lambda t} P_t f(x)\, dt, \quad f \in C_b(H), \ \lambda > 0, \ x \in H.$$

Then, it is easy to see that $F(\lambda)$ is one-to-one and that fulfills the resolvent identity. Consequently, there exists a unique closed operator L in $C_b(H)$ such that $F(\lambda) = (\lambda - L)^{-1}$ for any $\lambda > 0$.

To prove the existence of an invariant measure ν for P_t we need the following lemma.

LEMMA 2.1. *Let $\delta < \delta_0 := \frac{1}{\|C\|}$. Then, there exists $C > 0$ independent of δ such that*

$$\int_H |Ax|^2 e^{\delta\|x\|^2} \nu_\varepsilon(dx) \leq C. \tag{2.5}$$

PROOF. Let us compute $N_\varepsilon \varphi$ for $\varphi(x) = e^{\delta\|x\|^2}$ with $\delta < \delta_0$. (Notice that φ does not belong to $D(L)$, however it can easily approximated by functions of $D(L)$).

We have

$$D\varphi(x) = -2\delta e^{\delta\|x\|^2} Ax, \quad D^2\varphi(x) = 4\delta^2 e^{\delta\|x\|^2} Ax \otimes Ax - 2\delta e^{\delta\|x\|^2} A.$$

It follows, in view of (1.8), that

$$N_\varepsilon \varphi = (\delta \operatorname{Tr}[(-A)C] + 2\delta^2 |C^{1/2}Ax|^2 - 2\delta |Ax|^2)\, e^{\delta\|x\|^2} \leq$$

$$\leq (\delta \operatorname{Tr}[(-A)C] + 2\delta^2 \|C\| |Ax|^2 - 2\delta |Ax|^2)\, e^{\delta\|x\|^2}.$$

Since $\int_H N_\varepsilon \varphi d\nu_\varepsilon = 0$ due to the invariance of ν_ε, we have

$$\int_H |Ax|^2 e^{\delta\|x\|^2} \nu_\varepsilon(dx) \leq \frac{\operatorname{Tr}[(-A)C]}{2(1 - \delta\|C\|)} \int_H e^{\delta\|x\|^2} \nu_\varepsilon(dx). \tag{2.6}$$

It follows that for any $R > 0$ we have

$$\int_H e^{\delta\|x\|^2}\nu_\varepsilon(dx) = \int_{\{\|x\|\leqslant R\}} e^{\delta\|x\|^2}\nu_\varepsilon(dx) + \int_{\{\|x\|>R\}} e^{\delta\|x\|^2}\nu_\varepsilon(dx) \leqslant$$

$$\leqslant e^{\delta R^2} + \frac{1}{R^2}\int_H \|x\|^2 e^{\delta\|x\|^2}\nu_\varepsilon(dx) \leqslant e^{\delta R^2} + \frac{1}{\pi^2 R^2}\int_H |Ax|^2 e^{\delta\|x\|^2}\nu_\varepsilon(dx).$$

Now, taking into account (2.6), it follows that

$$\int_H e^{\delta\|x\|^2}\nu_\varepsilon(dx) \leqslant e^{\delta R^2} + \frac{\mathrm{Tr}\,[(-A)\,C]}{2(\pi^2 R^2(1-\delta\|C\|))}\int_H e^{\delta\|x\|^2}\nu_\varepsilon(dx)$$

and choosing

$$R^2 = \frac{\mathrm{Tr}\,[(-A)\,C]}{2(\pi^2(1-\delta\|C\|))},$$

we see that

$$\int_H e^{\delta\|x\|^2}\nu_\varepsilon(dx) \leqslant 2\,e^{\frac{\delta \mathrm{Tr}\,[(-A)\,C]}{2(\pi^2(1-\delta\|C\|))}}.$$

Finally, the conclusion follows using again (2.6). □

Since the embedding $D(A)\subset H$ is compact, we find by (2.5) that $\{\nu_\varepsilon\}$ is tight and consequently it has a weak limit ν. The following result is straightforward.

PROPOSITION 2.2. *The measure ν is invariant for P_t. Moreover for any $\delta < \delta_0 :=$ $:= \dfrac{1}{\|C\|}$ we have*

$$\int_H |Ax|^2 e^{\delta\|x\|^2}\nu(dx) < +\infty\,. \tag{2.7}$$

3. A PRIORI ESTIMATES

LEMMA 3.1. *Assume that $\alpha \leqslant \dfrac{1}{\|C\|}$. Then we have*

$$\mathbb{E}\left(e^{\alpha\int_0^t |AX(s,\,x)|^2 ds}\right) \leqslant e^{\alpha\|x\|^2} e^{t\,\mathrm{Tr}\,[(-A)\,C]}, \quad x\in H,\ t\geqslant 0. \tag{3.1}$$

PROOF. Fix $\alpha \leqslant \dfrac{1}{\|C\|}$. Let us estimate, using the Itô formula,

$$\mathbb{E}\left(e^{\alpha\|X(t,\,x)\|^2 + \alpha\int_0^t |AX(s,\,x)|^2 ds}\right).$$

Set

$$V(t) = \|X(t)\|^2 + \int_0^t |AX(s)|^2 ds.$$

We have, taking into account (1.3),

$$d\|X(t)\|^2 = -2(AX(t), AX(t) + b(X(t)))\,dt + \mathrm{Tr}\,[(-A)\,C] - (AX(t), \sqrt{C}dW(t)) =$$
$$= -2\,|AX(t)|^2\,dt + \mathrm{Tr}\,[(-A)\,C] - (AX(t), \sqrt{C}dW(t)).$$

Consequently

$$dV(t) = -\,|AX(t)|^2\,dt + \mathrm{Tr}\,[(-A)\,C]\,dt - (AX(t), \sqrt{C}dW(t)).$$

Moreover

$$de^{\alpha V(t)} = \alpha e^{\alpha V(t)}\,dV(t) + \frac{1}{2}\alpha^2 e^{\alpha V(t)}\,|\sqrt{C}AX(t)|^2\,dt.$$

So,

$$de^{\alpha V(t)} = \alpha e^{\alpha V(t)}(-\,|AX(t)|^2\,dt + \mathrm{Tr}\,[(-A)\,C]\,dt - (AX(t), \sqrt{C}dW(t))) +$$
$$+\frac{1}{2}\alpha^2 e^{\alpha V(t)}\,|\sqrt{C}AX(t)|^2\,dt \leq$$
$$\leq \frac{1}{2}\alpha e^{\alpha V(t)}(\alpha\|C\| - 1)\,|AX(t)|^2\,dt + \mathrm{Tr}\,[(-A)\,C]\,dt - (AX(t), \sqrt{C}dW(t))) \leq$$
$$\leq \mathrm{Tr}\,[(-A)\,C]\,e^{\alpha V(t)}\,dt - (AX(t), \sqrt{C}dW(t)))\,e^{\alpha V(t)},$$

since $\alpha \leq \frac{1}{\|C\|}$. Consequently, we have that

$$\mathbb{E}(e^{\alpha V(t)}) \leq e^{\alpha|x|^2} + \mathrm{Tr}\,[(-A)\,C]\int_0^t \mathbb{E}(e^{\alpha V(s)})\,ds$$

and the conclusion follows from the Gronwall lemma. □

We now set, for any $h \in H$, $\eta^h(t, x) = D_x X(t, x)\cdot h$. It can be checked that $\eta^h(t, x)$ is a solution of the following problem

$$\left\{\begin{array}{l} \frac{d}{dt}\eta^h(t, x) = A\eta^h(t, x) + X(t, x)\cdot\nabla\eta^h(t, x) + \eta^h(t, x)\cdot\nabla X(t, x) \\ \eta^h_\varepsilon(0, s) = h. \end{array}\right. \tag{3.2}$$

LEMMA 3.2. *There exists* $\kappa > 0$, $q \in (0, 2)$ *such that*

$$|\eta^h(t, x)|^2 \leq e^{\kappa\int_0^t |AX(s, x)|^q ds}\,|h|^2, \quad x, h \in H. \tag{3.3}$$

PROOF. Multiplying both sides of (3.2) by $\eta^h(t, x)$ and taking into account (1.7) yields, for $p > 2$ and $p^{-1} + q^{-1} = 1$,

$$\frac{1}{2}\frac{d}{dt}|\eta^h(t, x)|^2 + \|\eta^h(t, x)\|^2 = b(\eta^h(t, x), X(t, x), \eta^h(t, x)) \leq \tag{3.4}$$
$$\leq |\nabla X(t, x)|_{L^p(D)}\,|(\eta^h(t, x))^2|_{L^q(D)} = |\nabla X(t, x)|_{L^p(D)}\,|\eta^h(t, x)|^2_{L^{2q}(D)}.$$

By the Sobolev embedding theorem (1.10) there exists $c > 0$ such that

$$|\nabla X(t, x)|_{L^p(D)} \leq c|AX(t, x)|$$

and

$$|\eta^h(t,x)|_{L^{2q}(D)} \leqslant c|\eta^h(t,x)|_{1-\frac{1}{q}}.$$

Moreover, in view of (1.11),

$$|\eta^h(t,x)|_{1-\frac{1}{q}} \leqslant c|\eta^h(t,x)|^{\frac{1}{q}} \|\eta^h(t,x)\|^{1-\frac{1}{q}}$$

and consequently

$$|\nabla X(t,x)|_{L^p(D)} |(\eta^h(t,x))^2|_{L^q(D)} \leqslant c|AX(t,x)|\,|\eta^h(t,x)|^{\frac{2}{q}} \|\eta^h(t,x)\|^{2(1-\frac{1}{q})}.$$

Since for any $a, b \geqslant 0$ the following inequality holds

$$ab \leqslant \frac{1}{q} a^q + \left(1 - \frac{1}{q}\right) b^{\frac{q}{q-1}},$$

there exists $c' > 0$ such that

$$|\nabla X(t,x)|_{L^p(D)} |(\eta^h(t,x))^2|_{L^q(D)} \leqslant c' \,|AX(t,x)|^q\,|\eta^h(t,x)|^2 + \frac{1}{2}\|\eta^h(t,x)\|^2.$$

Then by (3.4) we obtain that

$$\frac{d}{dt}|\eta^h(t,x)|^2 \leqslant 2c' \,|AX(t,x)|^q\,|\eta^h(t,x)|^2$$

and the conclusion follows. □

The following corollary is a straightforward consequence of Lemma 3.2.

Corollary 3.3. *For any $\sigma > 0$ there exists $\kappa_\sigma > 0$ such that*

$$|\eta^h(t,x)|^2 \leqslant e^{\kappa_\sigma t + \sigma \int_0^t |AX(s,x)|^2 ds} |h|^2, \quad x, h \in H. \tag{3.5}$$

By Corollary 3.3 and Lemma 3.1 we obtain immediately the following basic estimate.

Corollary 3.4. *For any $\alpha \leqslant \frac{1}{\|C\|}$ there exists $\omega_\alpha > 0$ such that*

$$\mathbb{E}\left(|\eta^h(t,x)|^2\right) \leqslant e^{2\omega_\alpha t} e^{\alpha\|x\|^2} |h|^2, \quad t \geqslant 0 \;\; x, h \in H. \tag{3.6}$$

4. m-DISSIPATIVITY

We now consider the transition semigroup P_t defined by (1.3) and a fixed invariant measure ν for P_t. It is well known that P_t can be uniquely extended to a strongly continuous semigroup of contractions in $L^2(H,\nu)$, still denoted by P_t. We shall denote by N its infinitesimal generator. We first prove that N is an extension of N_0.

Proposition 4.1. *For any $\varphi \in \mathcal{E}_A(H)$, we have $\varphi \in D(N)$ and $N\varphi = N_0\varphi$.*

Proof. Let $\varphi \in \mathcal{E}_A(H)$. Then by the Itô formula we have that

$$\lim_{h\to 0} \frac{1}{h}(P_h\varphi(x) - \varphi(x)) = N_0\varphi(x), \quad x \in H.$$

Therefore, it is enough to show that

$$\sup_{h\in(0,1]}\frac{1}{h}\|P_h\varphi-\varphi\|_{L^2(H,\nu)}<+\infty .$$

We have in fact, again by the Itô formula,

$$|P_h\varphi(x)-\varphi(x)|\le\|\varphi\|_1\int_0^h\mathbb{E}(|AX(s,x)|+|B(X(s,x))|)\,ds.$$

It follows that, by the invariance of ν, that

$$\|P_h\varphi-\varphi\|^2_{L^2(H,\nu)}\le h^2\|\varphi\|_1^2\int_H(|Ax|+|B(x)|)^2\nu(dx)<+\infty ,$$

thanks to (2.7). □

Finally, we can prove the following result.

THEOREM 4.2. *Assume that* $\mathrm{Tr}\,[(-A)C]<\infty$ *and that there exists an invariant measure* ν *for the transition semigroup* P_t *defined by* (1.3). *Let* N *be the infinitesimal generator of* P_t *in* $L^2(H,\nu)$. *Then* N *is the closure of the Kolmogorov operator* N_0 *defined by* (1.4).

Moreover, if $\varphi\in D(N)$ *then* $|C^{1/2}D\varphi|\in L^2(H,\nu)$ *and*

$$\int_H N\varphi\,\varphi d\nu=-\frac{1}{2}\int_H|C^{1/2}D\varphi|^2d\nu. \tag{4.1}$$

PROOF. We have to prove that $\overline{N_0}=N$. Let $f\in C_b^1(H)$ and $\lambda>0$. Since B_ε is bounded and regular, there exists a unique $\varphi_\varepsilon\in D(L)\cap C_b^1(H)$ such that

$$\lambda\varphi_\varepsilon(x)-L\varphi_\varepsilon(x)-\langle B_\varepsilon(x),D\varphi_\varepsilon(x)\rangle=f(x),\quad x\in H.$$

Moreover, φ_ε is given by

$$\varphi_\varepsilon(x)=\int_0^\infty e^{-\lambda t}\mathbb{E}[f(X_\varepsilon(t,x))]\,dt,\quad x\in H.$$

Fix $\alpha\in(0,\|C\|^{-1})$. Then, in view of Corollary 3.4, it follows that for any $\lambda>\omega_\alpha$ we have

$$(D\varphi_\varepsilon(x),h)=\mathbb{E}\int_0^\infty e^{-\lambda t}(Df(X_\varepsilon(t,x)),\eta_\varepsilon^h(t,x))\,dt,\quad x,h\in H$$

and

$$|(D\varphi_\varepsilon(x),h)|\le\frac{1}{\lambda-\omega_\alpha}e^{\frac{\alpha}{2}\|x\|^2}\|f\|_1\,|h|^2,\quad h,x\in H.$$

Thus, by the arbitrariness of h, we have that

$$|D\varphi_\varepsilon(x)|\le\frac{1}{\lambda-\omega_\alpha}e^{\frac{\alpha}{2}\|x\|^2}\|f\|_1,\quad x\in H. \tag{4.2}$$

We now fix $\lambda>\omega_\alpha$.

Claim 1. We have $\varphi_\varepsilon \in D(\overline{N_0})$ and

$$\lambda\varphi_\varepsilon - \overline{N_0}\varphi_\varepsilon = (B_\varepsilon(x) - B(x), D\varphi_\varepsilon)) + f.$$

In fact, by [4, Proposition 2.6], there exists a multisequence $\{\psi_{\vec{n}}\} \in \mathcal{E}_A(H)$ such that

$$\lim_{\vec{n}\to\infty} \psi_{\vec{n}}(x) = \varphi_\varepsilon(x), \quad x \in H,$$

$$\lim_{\vec{n}\to\infty} D\psi_{\vec{n}}(x) = D\varphi_\varepsilon(x), \quad x \in H,$$

$$\lim_{\vec{n}\to\infty} L\psi_{\vec{n}}(x) = L\varphi_\varepsilon(x), \quad x \in H.$$

By Lemma 2.1 and the dominated convergence theorem, it follows that

$$\lim_{\vec{n}\to\infty} N_0\psi_{\vec{n}}(x) = \overline{N_0}\varphi_\varepsilon(x) = L\varphi_\varepsilon + \langle B_\varepsilon(x), D\varphi_\varepsilon\rangle, \quad x \in H.$$

So, Claim 1 is proved.

Claim 2. We have

$$\lim_{\varepsilon\to 0} (B_\varepsilon(x) - B(x), D\varphi_\varepsilon)) = 0 \quad \text{in } L^2(H, \nu).$$

Once Claim 2 is proved, we deduce that the closure of the range of $\lambda - \overline{N_0}$ is dense in $L^2(H, \nu)$ and so, in view of a theorem by Lumer and Phillips, that $\overline{N_0} = N$ as claimed.

To prove Claim 2, we notice that, taking into account (4.2), that

$$\int_H |(B_\varepsilon(x) - B(x), D\varphi_\varepsilon))|^2 d\nu = \int_{\{\|x\| \geq 1/\varepsilon\}} |(B_\varepsilon(x) - B(x), D\varphi_\varepsilon))|^2 d\nu \leq$$

$$\leq \frac{1}{(\lambda - \omega_\alpha)^2}\|f\|_1^2 \int_{\{\|x\| \geq 1/\varepsilon\}} |B(x)|^2 \frac{\varepsilon\|x\|^2 - 1}{\varepsilon\|x\|^2} e^{\alpha|x|^2} d\nu \leq$$

$$\leq \frac{1}{(\lambda - \omega_\alpha)^2}\|f\|_1^2 \int_{\{\|x\| \geq 1/\varepsilon\}} |B(x)|^2 e^{\alpha|x|^2} d\nu.$$

Thus, it is enough to show that

$$\int_H |B(x)|^2 e^{\alpha|x|^2} \nu(dx) < +\infty. \tag{4.3}$$

We have in fact

$$|B(x)|^2 \leq \int_D |x|^2 |\nabla x|^2 d\xi \leq |Ax|^2 \|x\|^2.$$

It follows that if $\alpha' < \alpha$

$$\int_H |(B(x), D\varphi)|^2 \nu(dx) \leq \int_H |Ax|^2 \|x\|^2 e^{\alpha'|x|^2} d\nu \leq c\int_H |Ax|^2 e^{\alpha|x|^2} d\nu,$$

and the conclusion follows from (2.7). Finally, (4.1) follows integrating with respect to

ν the identity

$$N_0(\varphi^2) = 2\varphi N_0 \varphi + |Q^{1/2} D\varphi|^2.$$

The proof is complete. □

References

[1] V. Barbu - G. Da Prato, *The transition semigroup of stochastic Navier-Stokes equations in 2-D*. Infinite Dimensional Analysis, Quantum Probability and related topics, to appear.

[2] J. Bricmont - A. Kupiainen - R. Lefevere, *Exponential mixing of the 2D stochastic Navier-Stokes dynamics*. Commun. Math. Phys., 230, n. 1, 2002, 87-132.

[3] S. Cerrai, *A Hille-Yosida theorem for weakly continuous semigroups.* Semigroup Forum, 49, 1994, 349-367.

[4] G. Da Prato - L. Tubaro, *Some results about dissipativity of Kolmogorov operators*. Czechoslovak Mathematical Journal, 51, 126, 2001, 685-699.

[5] G. Da Prato - J. Zabczyk, *Second Order Partial Differential Equations in Hilbert spaces.* London Mathematical Society Lecture Notes n. 293, Cambridge University Press, 2002.

[6] F. Flandoli, *Dissipativity and invariant measures for stochastic Navier-Stokes equations.* NoDEA, 1, 1994, 403-423.

[7] F. Flandoli - B. Maslowski, *Ergodicity of the 2-D Navier-Stokes equation under random perturbations.* Commun. Math. Phys., 171, 1995, 119-141.

[8] S. Kuksin - A. Shirikyan, *A coupling approach to randomly forced nonlinear PDE's. I.* Commun. Math. Phys., 221, n. 2, 2001, 351-366.

[9] S. Kuksin - A. Piatniski - A. Shirikyan, *A coupling approach to randomly forced nonlinear PDE's. II.* Commun. Math. Phys., 230, n. 1, 2002, 81-85.

[10] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*. North-Holland, 1977.

[11] E. Weinan - J.C. Mattingly - Ya. Sinai, *Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation*. Commun. Math. Phys., 224, n. 1, 2001, 83-106.

Pervenuta l'8 settembre 2003,
in forma definitiva il 30 ottobre 2003.

V. Barbu:
University of Iasi
6600 Iasi (Romania)
barbu@uaic.ro

G. Da Prato:
Scuola Normale Superiore
Piazza dei Cavalieri, 7 - 56126 Pisa
daprato@sns.it

A. Debussche:
Ecole Normale Supérieure de Cachan
Antenne de Bretagne, Campus de Ker-Lann
35170 Bruz (Francia)
debussche@bretagne.ens-cachan.fr