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An example of a non-degenerate precession possessing two distinct pairs of axes

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Meccanica dei solidi. — *An example of a non-degenerate precession possessing two distinct pairs of axes.* Nota di Giancarlo Cantarelli e Corrado Risito, presentata (*) dal Socio S. Rionero.

Abstract. — In the present paper we provide an interesting example of a *non-degenerate* precession possessing two *distinct* pairs $(p, f), (p', f')$ of axes of precession and figure. Thus the problem arises of the existence of classes of precessions possessing a *unique* axis of precession and a *unique* axis of figure. In the fourth section we show that the class of non-degenerate *regular* precessions enjoys this property.

Key words: Rigid body; Fixed point; Precession.

Riassunto. — *Esempio di precessione non degenere con due coppie distinte di assi.* Nel presente lavoro si fornisce un interessante esempio di precessione *non degenere* che possiede due coppie *distinte* $(p, f), (p', f')$ di assi di precessione e di figura. Si pone perciò il problema dell'esistenza di classi di precessioni aventi un *solo* asse di precessione ed un *solo* asse di figura. Nel quarto paragrafo si dimostra che la classe delle precessioni *regolari* non degeneri gode della suddetta proprietà.

1. Introduction

In the Kinematics of rigid bodies a *precession* is the motion of a rigid body around a fixed point – the *centre* (or *pole*) of the precession – in which two distinct axes exist through the fixed point, forming a constant angle during the motion: an axis p, fixed in the frame of reference $\mathcal{R}$ (the *axis of precession*), and an axis f, fixed in the body (the *axis of figure*) [3, Chap. III, Section 11]. A precession is *non-degenerate* if it is not a rotational motion.

As a consequence of the above definition, the angular velocity $\vec{\omega}$ of a precession can be expressed, at any instant, as the vector sum of a vector $\vec{\omega}_1$ parallel to the precession axis p, and a vector $\vec{\omega}_2$ parallel to the figure axis f. Moreover, if the vector product of $\vec{\omega}_1$ and $\vec{\omega}_2$ is different from zero, at all times, then the precession is *non-degenerate*.

In the following section we give a simple example of a *non-degenerate* precession which possesses two *distinct* pairs of axes. In the third section we report a known theorem by Grioli, which provides a *necessary* and *sufficient* condition (recognizable on the vector function $\vec{\omega} = \vec{\omega}(t)$) for a rigid motion with a fixed point to be a precession. In the fourth section of the present paper we consider the class of non-degenerate *regular* precessions (*i.e.* the precessions where, during the motion, $\vec{\omega}_1(t)$ *is constant in* $\mathcal{R}$, whereas $\vec{\omega}_2(t)$ *is constant in the body*, with $\vec{\omega}_1(t) \times \vec{\omega}_2(t) \neq 0$ at any instant), and we prove that, for this class of precessions, *the precession axis and the figure axis are unique*.

(*) Nella seduta dell'11 aprile 2003.

2. Example

Consider a rigid plane lamina having the shape of a right-angled triangle ABC $(\widehat{A} = \alpha, \widehat{B} = \frac{\pi}{2}$: see fig. 1). Suppose the vertex A is constantly placed upon the origin O of a rectangular positively oriented system of axes $Oxyz$, with respective unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Let the cathetus AB of the lamina be constrained to move in the half-plane π_1: $y \sin\alpha + z \cos\alpha = 0$, $z \geqslant 0$, whereas the hypotenuse AC is constrained to move in the half-plane π_2: $z = 0$, $y \leqslant 0$.

In the figure we have denoted by $c_1 (\subset \pi_1)$ the half-circle on which B is constrained to move and by B_0, B_1 its end points, which belong to the x axis (see fig. 1). Moreover, we have denoted by $c_2 (\subset \pi_2)$ the arc of the circumference (with centre at O and radius $|AC|$) on which C is constrained to move, whose end points C_0, C_1 are symmetric with respect to the y axis, with $C_0\widehat{O}B_0 = B_1\widehat{O}C_1 = \alpha$ (see fig. 1).

A first pair of axes is given by: the z axis which is orthogonal to π_2 (the axis of precession p) and the line AC (the axis of figure f).

A second pair of axes is given by: the fixed line n_1 through the origin O, orthogonal to π_1 (the axis of precession p') and the line AB (the axis of figure f').

Note that the axes of each pair form a right-angle, and that α is the (minimum) angle between the two precession axes p, p' (because $\vec{n}_1 = \sin\alpha\, \vec{e}_2 + \cos\alpha\, \vec{e}_3$ is a unit vector orthogonal to π_1), as well as between the two figure axes f, f'.

We introduce now a rectangular positively oriented system of axes $O\xi\eta\zeta$, with respective unit vectors $\vec{i}, \vec{j}, \vec{k}$, rigidly connected with the lamina, choosing as *third* axis

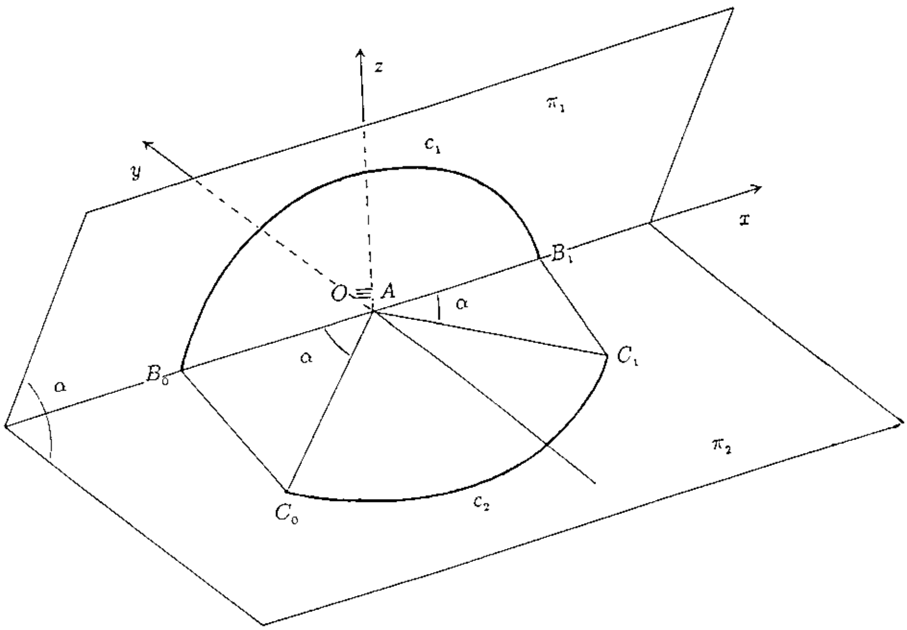

Fig. 1.

ζ the figure axis $f(\equiv AC)$, oriented from A to C, and as second axis η the line orthogonal to the lamina with the same orientation as $\overline{AC} \times \overline{AB}$ (it follows that the first axis ξ belongs to the plane of the lamina and its unit vector $\vec{i}$ forms the angle $\left(\frac{\pi}{2} - \alpha\right)$ with $\overline{AB}$). Let ϑ, ψ, φ be the three Euler angles of the *moving* axes ξ, η, ζ with respect to the *fixed* ones x, y, z. Since the angle of *nutation* ϑ coincides, at any instant of time, with the right-angle formed by p and f, the angular velocity $\vec{\omega}$ of the lamina, expressed by means of the Euler angles, takes up the following form

$$\vec{\omega} = \dot{\psi}\,\vec{e}_3 + \dot{\varphi}\,\vec{k} \tag{1}$$

where ψ is the angle of *precession*.

The last two Euler angles are not independent parameters, because the degree of freedom of the lamina is $n = 1$. The simplest way to get the constraint equation is to express the cartesian coordinates y and z of the vertex B as functions of ψ and φ (by means of a change of cartesian coordinates, it being known that: $\xi_B = |AB|\sin\alpha$, $\eta_B = 0$, $\zeta_B = |AB|\cos\alpha$, and expressing the direction cosines of $\vec{i}$ and $\vec{k}$ by means of the Euler angles), and substitute them into the equation: $y\sin\alpha + z\cos\alpha = 0$ of π_1, on which B is constrained to move. We get the following *constraint equation*

$$\cos\alpha\sin\varphi - \cos\alpha\cos\psi + \sin\alpha\sin\psi\cos\varphi = 0\,. \tag{2}$$

Note that while the vertex B describes the half-circle c_1, from B_1 to B_0, the third Euler angle φ ranges from zero to π. Moreover, the angle formed by the radius vector $\overline{OB}$ with the unit vector $\vec{e}_1$ of the fixed axis x, *coincides* with φ (see Remark 1). This observation enables us to set up a one to one correspondence between the points of c_1 and the values of φ in the interval $[0, \pi]$. For this reason it is more convenient to choose φ as lagrangian coordinate, rather than ψ.

Now, in order to show the existence of *non-degenerate* precessions of the lamina, we take a suitable *time equation* $\varphi = \varphi(t)$, defined on a time interval $I := [t_0, t_1]$, with $(0 \leqslant)\ t_0 < t_1 < \infty$, and of class $\mathcal{R}^1(I)$, satisfying the following conditions

$$\varphi(t_0) = \pi\,, \qquad \varphi(t_1) = \gamma\,, \qquad \dot{\varphi}(t) < 0 \quad \forall t \in I \tag{3}$$

where γ is an arbitrary (constant) *obtuse* angle: $\frac{\pi}{2} < \gamma < \pi$.

Finally, substituting $\varphi = \varphi(t)$ into (2), this equation determines *uniquely* the function of time $\psi = \psi(t)$ of class $\mathcal{R}^1(I)$, in the following form

$$\sin\psi(t) = \frac{\cos\alpha\cos\varphi(t)}{1 + \sin\alpha\sin\varphi(t)} \qquad \forall t \in I \tag{4}$$

$$\cos\psi(t) = \frac{\sin\alpha + \sin\varphi(t)}{1 + \sin\alpha\sin\varphi(t)} \qquad \forall t \in I\,. \tag{5}$$

In fact, substituting these two time functions, together with $\varphi = \varphi(t)$, into (2), the constraint equation is *identically* satisfied for all time instants t of I. Moreover, at the initial time instant t_0, the formulas (4) and (5) give

$$\sin\psi(t_0) = -\cos\alpha\,, \qquad \cos\psi(t_0) = \sin\alpha \tag{6}$$

which agrees with the initial value of the precession angle: $\psi(t_0) = \left(\frac{3\pi}{2} + \alpha\right)$, since in the initial configuration OB_0C_0 of the lamina (see fig. 1), corresponding to $\varphi(t_0) = \pi$, the cathetus OB_0 lies on the negative x semiaxis, and the moving axis ξ_0 is superposed on the nodal line, but is opposite to it (*i.e.* $\vec{N}_0 = -\vec{i}_0$).

Due to the time equation $\varphi = \varphi(t)$, the vertex B describes the arc B_0B_γ of the half-circle $c_1 (\subset \pi_1)$, where B_0 and B_γ are the points of c_1 corresponding to $\varphi = \pi$ and $\varphi = \gamma$, respectively, whereas the other vertex C describes the arc C_0C_γ of c_2, which belongs to the third quadrant of the coordinate plane xy, where, taking into account (6), the initial point C_0 is defined by $\vec{OC_0} = -|AC|\{\cos\alpha\,\vec{e}_1 + \sin\alpha\,\vec{e}_2\}$, and the end point C_γ is defined by

$$\vec{OC_\gamma} = |AC|\left\{\frac{\cos\alpha\cos\gamma\,\vec{e}_1 - (\sin\alpha + \sin\gamma)\,\vec{e}_2}{1 + \sin\alpha\sin\gamma}\right\}$$

taking into account (4), (5).

Finally, differentiating with respect to the time either the identity (4) or the identity (5), and then dividing both members by $\cos\psi(t)$ or $\sin\psi(t)$ respectively, we obtain

$$\dot\psi(t) = \frac{-\dot\varphi(t)\cos\alpha}{1 + \sin\alpha\sin\varphi(t)} > 0 \qquad \forall t \in I\,. \tag{7}$$

From (7) it follows that: $\left(\dot\psi(t)\,\vec{e}_3\right) \times \left(\dot\varphi(t)\,\vec{k}\right) \neq 0\ \forall t \in I$, which ensures that all the precessions of the rigid lamina ABC with a time equation $\varphi = \varphi(t)$ satisfying the conditions (3), are *non-degenerate*.

REMARK 1. The angle formed by the radius vector $\vec{OB}$ with the unit vector $\vec{e}_1$ of the x axis coincides with φ. In fact, since $\frac{\vec{OB}}{|OB|} = \sin\alpha\,\vec{i} + \cos\alpha\,\vec{k}$, we have

$$\frac{\vec{OB}}{|OB|}\cdot\vec{e}_1 = (\vec{e}_1\cdot\vec{i})\sin\alpha + (\vec{e}_1\cdot\vec{k})\cos\alpha = \sin\alpha\cos\psi\cos\varphi + \cos\alpha\sin\psi\,. \tag{8}$$

Now, eliminating $\sin\psi$ and $\cos\psi$ by means of (4) and (5), we get

$$\frac{\vec{OB}}{|OB|}\cdot\vec{e}_1 = \cos\varphi \tag{9}$$

which proves our statement.

Let us remark that the element a_{11} of the jacobian matrix, which is obtained by differentiating with respect to ψ the first member of the constraint equation (2), coincides with $\frac{\vec{OB}}{|OB|}\cdot\vec{e}_1$, as can be seen from (8). Therefore, taking into account (9), during the motion of the lamina $a_{11}(t)$ is given by

$$a_{11}(t) = \cos\varphi(t) < 0 \qquad \forall t \in I \tag{10}$$

which ensures that, at any instant t of the time interval I, the rank of the jacobian matrix is *maximum*.

On the other hand, for the value $\varphi = \frac{\pi}{2}$ (to which corresponds $\psi = 2\pi$) the rank of the jacobian matrix is zero. This is the case when B is at the middle point of the half-cir cle c_1, and C belongs to the negative y semiaxis. This is the reason why we have restricted the trajectory of the vertex B to the arc $B_0 B_\gamma$ of c_1 instead of extending it to the *whole* half-circl e c_1.

3. Grioli's formulas

Grioli has proved [1, 2] that the following identity

$$(11) \qquad \vec{\omega} \times \dot{\vec{\omega}} \cdot \vec{u} + (\vec{\omega} \cdot \vec{u}) \, \|\vec{\omega} \times \vec{u}\|^2 = \operatorname{cotg} \vartheta \|\vec{\omega} \times \vec{u}\|^3$$

is a *necessary* and *sufficient* condition which must be satisfied by the angular velocity $\vec{\omega}(t)$, during a *spherical rigid* motion, in order that the motion be a *precession*, where $\vec{u}$ is a unit vector of an (arbitrary) figure axis f. Moreover, the second Grioli's formula

$$(12) \qquad \vec{c} = \frac{\sin \vartheta}{\|\vec{\omega} \times \vec{u}\|} \left[\vec{\omega} - (\vec{\omega} \cdot \vec{u}) \ \vec{u} \right] + \cos \vartheta \, \vec{u}$$

allows us to determine a unit vector $\vec{c}$ of the precession axis p, *corresponding* to the figure axis f. In both the above formulas ϑ is the (*constant*) angle between $\vec{c}$ and $\vec{u}$.

We want to apply the formulas (11) and (12) to the example given in the previous Section 2. The axes of the first pair (p, f), which we have chosen as *third* axes z and ζ, form an angle $\vartheta = \frac{\pi}{2}$, and have unit vectors: $\vec{c} = \vec{e}_3$ and $\vec{u} = \vec{k}$ respectively. It is easy to verify that both the above formulas are satisfied, at every instant t of the time interval I.

Now, utilizing Grioli's formulas, we shall determine the unit vectors $\vec{c}\,'$ and $\vec{u}\,'$ of the second pair of axes (p', f') of the previous example, and the (*constant*) angle ϑ' between $\vec{c}\,'$ and $\vec{u}\,'$. In order to simplify the lengthy calculations, we choose the moving axis ξ in the plane ff', and moreover, we restrict ourselves to searching for a unit vector $\vec{u}\,'$ parallel to the coordinate plane $\xi\zeta$, by putting

$$(13) \qquad \vec{u}\,' = \sin \chi \, \vec{i} + \cos \chi \, \vec{k}$$

where $\chi (0 < \chi < \pi)$ is the *unknown* angle between $\vec{u}\,'$ and $\vec{k}$. We exclude: $\chi = 0$ and $\chi = \pi$, because we would get again the first pair of axes.

Expressing the components p, q, r of the angular velocity $\vec{\omega} \equiv \dot{\psi}(t) \ \vec{e}_3 + \dot{\varphi}(t) \ \vec{k}$ of the rigid lamina with respect to the moving axes ξ, η, ζ, as functions of the Euler angles, and then eliminating $\dot{\psi}(t)$ by means of formula (7), we obtain the following expression for the first member of (11)

$$(14) \qquad \vec{\omega} \times \dot{\vec{\omega}} \cdot \vec{u}\,' + (\vec{\omega} \cdot \vec{u}\,') \|\vec{\omega} \times \vec{u}\,'\|^2 = P(\sin \varphi) \left(\frac{\dot{\varphi}}{1 + \sin \alpha \sin \varphi} \right)^3$$

where $P(\sin\varphi)$ is the following polynomial of the third degree in $\sin\varphi$

(15) $$P(\sin\varphi) := c_0 + c_1 \sin\varphi + c_2 \sin^2\varphi + c_3 \sin^3\varphi \,.$$

The coefficients of this polynomial have the following expressions

(16) $$\begin{cases} c_0 = \dfrac{1}{2}\sin\chi\,[\sin(2\chi) - \sin(2\alpha)] \\ c_1 = 3\,\sin^2\chi \sin(\alpha-\chi) \\ c_2 = \dfrac{3}{2}\sin\chi \sin[2(\alpha-\chi)] \\ c_3 = \sin\chi \sin(2\alpha-\chi)\,\sin(\alpha-\chi) \end{cases}$$

which are all zero for $\chi = \alpha$. Therefore, for

(17) $$\vec{u}\,' = \sin\alpha\,\vec{i} + \cos\alpha\,\vec{k}$$

the identity (11) becomes: $0 = \operatorname{cotg}\vartheta'\,\|\vec{\omega}\times\vec{u}\,'\|^3\ \forall t \in I$, which implies $\vartheta' = \dfrac{\pi}{2}$, as $\vec{\omega}\times\vec{u}\,' \neq 0$, because the precession of the example given in Section 2 is a *non-degenerate* one.

By means of Grioli's first formula we have found the solution (17), which is a unit vector of the second figure axis f' (the cathetus AB), and we have also determined the angle $\vartheta' = \dfrac{\pi}{2}$ which $\vec{u}\,'$ forms with $\vec{c}\,'$. In order to identify a unit vector $\vec{c}\,'$ of p', we utilize Grioli's second formula (12), which now takes the simplified form $\left(\text{as } \vartheta' = \dfrac{\pi}{2}\right)$

(18) $$\vec{c}\,' = \frac{\vec{\omega} - (\vec{\omega}\cdot\vec{u}\,')\,\vec{u}\,'}{\|\vec{\omega}\times\vec{u}\,'\|}\,.$$

With simple calculations, taking into account the first seven formulas of the previous Section 2, we get

$$\vec{\omega}\cdot\vec{u}\,' = -\dot{\psi}\,, \qquad \|\vec{\omega}\times\vec{u}\,'\| = -\dot{\varphi}$$

from which it follows that

$$\vec{\omega} - (\vec{\omega}\cdot\vec{u}\,')\,\vec{u}\,' = \dot{\psi}\,\vec{e}_3 + \varphi\,\vec{k} + \dot{\psi}\,\vec{u}\,'.$$

Substituting $\vec{u}\,'$ by means of (17), thereafter expressing the direction cosines of $\vec{i}$ and $\vec{k}$ by means of the Euler angles, and finally eliminating $\sin\psi$, $\cos\psi$, $\dot{\psi}$ by means of (4), (5), (7), we obtain the following expression for the vector component of $\vec{\omega}$ orthogonal to f'

$$\vec{\omega} - (\vec{\omega}\cdot\vec{u}\,')\,\vec{u}\,' = -\dot{\varphi}(\sin\alpha\,\vec{e}_2 + \cos\alpha\,\vec{e}_3)$$

which inserted into the second member of (18) gives

(19) $$\vec{c}\,' = \sin\alpha\,\vec{e}_2 + \cos\alpha\,\vec{e}_3$$

which coincides with the unit vector $\vec{n}_1$ orthogonal to the half-plane π_1. Therefore the precession axis corresponding to the second figure axis f' is the line n_1 through the center O, orthogonal to π_1, *i.e.* the second precession axis p', which was found geometrically in the previous Section 2.

4. On the uniqueness of axes for the class of non-degenerate regular precessions

In this section we prove the following theorem.

THEOREM. *Every non-degenerate regular precession possesses a unique precession axis and a unique figure axis.*

Let $\mathcal{P}$ be any non-degenerate *regular* precession, where (p, f) is a pair of precession and figure axes satisfying the definition given in the Introduction. Let us introduce two positively oriented rectangular systems of axes, both with the origin at the centre O of $\mathcal{P}$: $Oxyz$, at rest in the frame $\mathcal{R}$, and $O\xi\eta\zeta$, rigidly connected to the body, where the *third* fixed axis z will be the *precession* axis p, oriented as $\vec{\omega}_1$, and the *third* moving axis ζ will be the *figure* axis f, oriented as $\vec{\omega}_2$. Then the angular velocity $\vec{\omega}(\equiv \vec{\omega}_1 + \vec{\omega}_2)$ of the body, expressed by means of the Euler angles, becomes

$$\vec{\omega} = \dot{\psi}_0 \vec{e}_3 + \dot{\varphi}_0 \vec{k} \qquad \forall t \in \Re \tag{20}$$

where $\dot{\psi}_0$ and $\dot{\varphi}_0$ are both *strictly positive* constants. Moreover, let ϑ $(0 < \vartheta < \pi)$ be the (*constant*) angle between the unit vectors $\vec{e}_3$ (of z) and $\vec{k}$ (of ζ).

Now, let us suppose, ab absurdo, that the given precession $\mathcal{P}$ possesses another pair of axes (p', f'), *different* from the previous one (p, f). Let $\vec{c}\,'$ and $\vec{u}\,'$ be two unit vectors of the axis of precession p' and the axis of figure f' respectively, and let us denote by ϑ' $(0 < \vartheta' < \pi)$ the (*constant*) angle between $\vec{c}\,'$ and $\vec{u}\,'$.

Consider first the particular case in which f' coincides with f, *i.e.* $\vec{u}\,' = \pm \vec{k}$. Then from the scalar identity (11) we get: $\vartheta' = \vartheta$ for $\vec{u}\,' = \vec{k}$ or $\vartheta' = \pi - \vartheta$ for $\vec{u}\,' = -\vec{k}$. Thereafter, from the vector identity (12) we obtain $\vec{c}\,' = \vec{e}_3$, both in the case: $\vec{u}\,' = \vec{k}$, $\vartheta' = \vartheta$ and in the case: $\vec{u}\,' = -\vec{k}$, $\vartheta' = \pi - \vartheta$. Thus we have proved that: *if f' coincides with f, then also p' coincides with p*, and so we get again the first pair of axes.

Now we consider the *general* case: $f' \neq f$. Denoting by λ the (*constant*) angle between $\vec{u}\,'$ and $\vec{k}$, we have: $0 < \lambda < \pi$, as $\vec{u}\,' \times \vec{k} \neq 0$. In order to simplify the subsequent long calculations, we choose without loss of generality the first moving axis ξ belonging to the moving plane ff' and so oriented that: $\vec{u}\,' \cdot \vec{i} > 0$. This implies

$$\vec{u}\,' = \sin\lambda\, \vec{i} + \cos\lambda\, \vec{k}\,. \tag{21}$$

Expressing the components p, q, r of $\vec{\omega}$ as functions of the Euler angles, we get

$$p = \dot{\psi}_0 \sin\vartheta \sin\varphi\,, \qquad q = \dot{\psi}_0 \sin\vartheta \cos\varphi\,, \qquad r \equiv r_0 := \dot{\psi}_0 \cos\vartheta + \dot{\varphi}_0 \tag{22}$$

where φ is a *linear* function of the time (whereas: ϑ, $\dot{\psi}_0$ and $\dot{\varphi}_0$ are real constants). Therefore, taking into account (21) and (22), and putting

$$a := \dot{\psi}_0 \sin\vartheta \sin\lambda \quad (= const. > 0) \tag{23}$$

we have

$$\vec{\omega} \cdot \vec{u}\,' = a \sin\varphi + r_0 \cos\lambda \,. \tag{24}$$

Moreover:

$$\|\vec{\omega} \times \vec{u}\,'\| = \sqrt{-a^2 \sin^2\varphi - 2ar_0 \cos\lambda \sin\varphi + (r_0^2 \sin^2\lambda + \dot{\psi}_0^2 \sin^2\vartheta)^2} \tag{25}$$

and

$$\vec{\omega} \times \dot{\vec{\omega}} \cdot \vec{u}\,' = ar_0 \dot{\varphi}_0 \sin\varphi - \dot{\psi}_0^2 \dot{\varphi}_0 \sin^2\vartheta \cos\lambda \,. \tag{26}$$

Putting the three expressions (24), (25), (26) into Grioli's scalar identity (11) (where $\vec{u}$ is replaced by $\vec{u}\,'$ and $\text{cotg}\,\vartheta$ by $\text{cotg}\,\vartheta'$), we obtain in the first member a polynomial $P_3(\sin\varphi)$ of the third degree in $\sin\varphi$, whereas in the second member appears the square root of a polynomial $P_6(\sin\varphi)$ of the sixth degree in $\sin\varphi$. Rationalizing the above identity (*i.e.* squaring both members), and thereafter assembling together the terms of the same degree in $\sin\varphi$, we obtain an *algebraic equation* of the sixth degree for the *unknown* $x := \sin\varphi$, where all the coefficients are *constant* during the motion. In particular, the coefficient of x^6 is

$$a^6(1 + \text{cotg}^2\,\vartheta')$$

which is a *finite, non-zero* real number, owing to (23) and to the non vanishing of $\sin\vartheta'$. Thus, the algebraic equation is not an identity, and therefore possesses six *constant* roots (real or complex). But this is in contradiction with the fact that $\dot{\varphi}_0$ is a *strictly positive* constant, and so the theorem is proved.

5. Final remarks

The aim of the present section is to show that the *non-degenerate* precessional motions of the rigid lamina ABC, with two pairs of precession and figure axes (see Section 2), can be defined also on a *unbounded* interval of time.

In fact, let us consider as an example, the following *time equations*

(i) $\varphi(t) = \gamma + (\pi - \gamma)\, e^{-k(t-t_0)} \quad \forall\, t \in [t_0, \infty) \quad (k > 0)$

(ii) $\varphi(t) = \pi - (\pi - \gamma) \sin^2 k(t - t_0) \quad \forall\, t \in \Re \quad (k = 1)$

where k is a constant, whose dimensions are $[k] = [T^{-1}]$.

During the precession of the lamina with the time equation (i), the vertex B describes the arc $B_0 B_\gamma$ of the half-circle c_1 (see the figure of Section 2), starting at B_0 and tending asymptotically to B_γ, without inversions of the motion, as $\dot{\varphi}(t) < 0 \;\forall t \geqslant t_0$. Therefore, due to (7), we have: $\vec{\omega}_1(t) \times \vec{\omega}_2(t) \neq 0 \;\forall t \geqslant t_0$, which ensures that the precession is a *non-degenerate* one.

On the other hand, during the precession of the lamina with the time equation (*ii*), the vertex B oscillates indefinitely between the end points B_0 and B_γ of the arc $B_0 B_\gamma$, because the motion is periodic. It is easy to recognize that we have: $\vec{\omega}_1(t) \times \vec{\omega}_2(t) = 0$ *only* at the instants of time at which the vertex B is on the end points of the arc $B_0 B_\gamma$. But these instants of time are *isolated points* of the time interval $\Re$, and therefore the *periodic motion* of the lamina is a *non-degenerate* precession (possessing two *distinct* pairs of axes of precession and figure).

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