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On the space of real algebraic morphisms

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Geometria. — *On the space of real algebraic morphisms.* Nota di RICCARDO GHILONI, presentata (*) dal Socio C. De Concini.

ABSTRACT. — In this *Note*, we announce several results concerning basic properties of the spaces of morphisms between real algebraic varieties. Our results show a surprising intrinsic rigidity of Real Algebraic Geometry and illustrate the great distance which, in some sense, exists between this geometry and Real Nash one. Let us give an example of this rigidity. An affine real algebraic variety X is rigid if, for each affine irreducible real algebraic variety Z, the set of all nonconstant regular morphisms from Z to X is finite. We are able to prove that, given a compact smooth manifold M of positive dimension, there exists an uncountable family $\{M_i\}_{i \in I}$ of rigid affine nonsingular real algebraic varieties diffeomorphic to M such that, for each $i \neq j$ in I, M_i is not biregularly isomorphic to M_j.

KEY WORDS: Real algebraic morphisms; Real algebraic rigidity; Arithmetic of dominating real algebraic morphisms.

RIASSUNTO. — *Sullo spazio dei morfismi algebrici reali.* In questa *Nota*, annunciamo alcuni risultati riguardanti proprietà basilari degli spazi di morfismi tra varietà algebriche reali. I nostri risultati mostrano una sorprendente rigidità intrinseca della Geometria Algebrica Reale ed illustrano la grande distanza che, in un certo senso, esiste tra questa geometria e quella Nash reale. Diamo un esempio di questa rigidità. Una varietà algebrica reale affine X è rigida se, per ogni varietà algebrica reale affine irriducibile Z, l'insieme dei morfismi regolari noncostanti da Z in X è finito. Siamo in grado di dimostrare che, data una varietà differenziabile compatta M di dimensione positiva, esiste una famiglia non-numerabile $\{M_i\}_{i \in I}$ di varietà algebriche reali affini nonsingolari rigide diffeomorfe a M tali che, per ogni $i \neq j$ in I, M_i non è biregolarmente isomorfa a M_j.

1. INTRODUCTION

In this *Note*, we announce several results concerning basic properties of the spaces of morphisms between real algebraic varieties. Making use of new algebraic invariants, we study the topology and the finiteness properties of these spaces, and the arithmetic of dominating morphisms. Our results show a surprising intrinsic rigidity of Real Algebraic Geometry and illustrate the great distance which, in some sense, exists between this geometry and Real Nash one. In order to simplify the exposition, we will present only a part of our results giving our definitions and theorems in a weak form: for example, we will consider only affine real algebraic varieties. Furthermore, for each new result, we will give a sketch of the proof.

Let us start recalling some classical notions.

A *real algebraic subset* of $\mathbb{R}^n$ is defined as the set of zeros of a collection of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. Let S be a subset of $\mathbb{R}^n$, let T be a subset of $\mathbb{R}^m$ and let $f : S \to T$ be a map. We say that f is *regular* if, for each $x \in S$, there are a Zariski neighborhood Ω of x in $\mathbb{R}^n$, two polynomials $p : \mathbb{R}^n \to \mathbb{R}^m$ and $q : \mathbb{R}^n \to \mathbb{R}$ such that $q^{-1}(0) \cap S \cap \Omega = \emptyset$ and $f = p/q$ on $S \cap \Omega$. An *affine real algebraic variety* is a topological space X equipped with a sheaf $\mathcal{R}_X$ of real-valued functions, isomorphic to a real algebraic sub-

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set of some $\mathbb{R}^n$ with its Zariski topology, equipped with its sheaf of regular functions. The topology of X is called *Zariski topology*. Given a subset Y of X, we indicate by $\mathrm{Zcl}_X(Y)$ the Zariski closure of Y in X. A point x of X is *nonsingular of dimension d* if the ring of germs $\mathcal{R}_{X,x}$ is a regular local ring of dimension d. The *dimension* $\dim(X)$ of X is the largest dimension of nonsingular points of X. We denote by $\mathrm{Nonsing}(X)$ the set of all nonsingular points of X of dimension $\dim(X)$ and we say that X is *nonsingular* if $X = \mathrm{Nonsing}(X)$. A morphism between two affine real algebraic varieties is called a *regular map*. Let Y and Z be two affine real algebraic varieties. We will indicate by $\mathcal{R}(Y, Z)$ the set of all regular maps from Y to Z. A *projective real algebraic set* and a *projective real algebraic variety* can be defined in a similar way; in any case, such real algebraic varieties are always affine.

A 1-dimensional affine real algebraic variety is called a *real algebraic curve* and a 1-dimensional Zariski closed subset of an affine real algebraic variety X is called a *real algebraic curve of* X. Let C be an irreducible real algebraic curve. By normalization, there exists a projective nonsingular irreducible complex algebraic curve $C_{\mathbb{C}}$ defined over $\mathbb{R}$ such that C is birationally isomorphic to the set of all real points $C_{\mathbb{C}}(\mathbb{R})$ of $C_{\mathbb{C}}$ where $C_{\mathbb{C}}(\mathbb{R})$ is viewed as a projective real algebraic curve. We say that $C_{\mathbb{C}}$ is a *smooth complexification of* C. Since $C_{\mathbb{C}}$ is unique up to birational isomorphism over $\mathbb{R}$, we may define the *genus* $g(C)$ *of* C as the genus of $C_{\mathbb{C}}$ (see [7, Proposition 7.1]). Furthermore, the following is true. Let $r : C \to D$ be a nonconstant regular map between irreducible real algebraic curves and let $C_{\mathbb{C}}$ and $D_{\mathbb{C}}$ be smooth complexifications of C and D respectively. Let $H := \mathrm{Nonsing}(C) \cap r^{-1}(\mathrm{Nonsing}(D))$ and let $r' : H \to \mathrm{Nonsing}(D)$ be the restriction of r to H. Identify H with a Zariski open subset of $C_{\mathbb{C}}(\mathbb{R})$ and $\mathrm{Nonsing}(D)$ with a Zariski open subset of $D_{\mathbb{C}}(\mathbb{R})$. Then there is an unique nonconstant complex regular map $r_{\mathbb{C}}: C_{\mathbb{C}} \to D_{\mathbb{C}}$ which extends r'. We call $r_{\mathbb{C}}$ a *complexification of* r.

Observe that every affine real algebraic variety can be equipped with a topology finer than Zariski's one, *i.e.*, the *euclidean topology* induced by $\mathbb{R}^n$. Unless otherwise indicated, all affine real algebraic varieties used below will be considered equipped with the euclidean topology.

1. Sectional curve genus and embedded toric genus

Let X be an affine real algebraic variety of positive dimension, let $x \in X$ and let $k \in \mathbb{N}$ where $\mathbb{N}$ is the set of all nonnegative integers. We denote by $\mathcal{C}^k_x(X)$ the set of all irreducible real algebraic curves C of X such that $C \ni x$ and $g(C) = k$.

Lemma 1. *Let X be an affine real algebraic variety of positive dimension and let* $p \in \mathrm{Nonsing}(X)$. *Suppose X is irreducible. Then there is an integer k such that* $\bigcup_{C \in \mathcal{C}^k_p(X)} C$ *is Zariski-dense in* X.

Proof. Suppose X is a r-dimensional algebraic subset of $\mathbb{R}^n$. Let T be the Zariski tangent space of X at p, let N be the orthogonal vector space of T in $\mathbb{R}^n$, let $\mathbb{P}(T)$ be the projective space associated with T and let $\sigma : T\setminus\{0\} \to \mathbb{P}(T)$ be the natural projection. Fix $\nu \in \mathbb{P}(T)$. Let $\langle N, \nu \rangle$ be the vector subspace of $\mathbb{R}^n$ generated by N and a vector $v \in T \setminus \{0\}$ such that $\sigma(v) = \nu$ and, for each $x \in X$, let $N_{\nu, x}$ be the affine $(n - r + 1)$-plane of $\mathbb{R}^n$ defined by $N_{\nu, x} := x + \langle N, \nu \rangle$. By transversality, the intersection $N_{\nu, p} \cap X$ is nonsingular of dimension 1 at p. Denote by C_ν the irreducible component of such an intersection containing p. From the Implicit Function Theorem, it follows the existence of a neighborhood U of p in X such that $U \subset \bigcup_{\nu \in \mathbb{P}(T)} C_\nu \subset \bigcup_{k \in \mathbb{N}} \bigcup_{C \in \mathcal{C}_p^k(X)} C$. In particular, setting $Z_k := \mathrm{Zcl}_X\left(\bigcup_{C \in \mathcal{C}_p^k(X)} C \right)$ for each $k \in \mathbb{N}$, it follows that $U \subset \bigcup_{k \in \mathbb{N}} Z_k$. By the Baire theorem, we have that there is an integer k such that $\dim Z_k = \dim X$, hence Z_k coincides with X. □

The previous lemma ensures that the following definition is consistent.

DEFINITION (sectional curve genus). Let X be an affine real algebraic variety of positive dimension. First, suppose X is irreducible. The *sectional curve genus* $p_s(X)$ of X is defined as follows:

$$p_s(X) := \min_{x \in \mathrm{Nonsing}(X)} \min \left\{ k \in \mathbb{N} \,\middle|\, \mathrm{Zcl}_X\left(\bigcup_{C \in \mathcal{C}_x^k(X)} C \right) = X \right\}.$$

If X is reducible and $X_1, \ldots, X_r$ are its irreducible components of positive dimension, then we put $p_s(X) := \max_{i \in \{1,2,\ldots,r\}} p_s(X_i)$. □

Let X be an affine real algebraic variety, let $r := \dim(X)$ and let e be the maximum dimension of the Zariski tangent spaces of X. We define the *Whitney-Lluis number* $WL(X)$ *of* X by $WL(X) := \max\{2r + 1, r + e - 1\}$. Let Y be an affine real algebraic variety and let $\varphi : X \to Y$ be a topological embedding, *i.e.*, a homeomorphism onto its image. If $\varphi(X)$ is Zariski locally closed in Y and both φ and $\varphi^{-1}: \varphi(X) \to X$ are regular, then we say that φ is a *biregular embedding of* X *into* Y. Let $m := WL(X)$. Suppose X is an algebraic subset of $\mathbb{R}^n$ and $n \geq m$. By the Generic Projection Theorem (see [6] and [8, Section 2]), we know that a generic projection $p : \mathbb{R}^n \to \mathbb{R}^m$ restricted to X is a biregular embedding of X into $\mathbb{R}^m$. This fact permits to give the following definition (recall that $\mathbb{R}$ is a nonsingular irreducible real algebraic curve).

DEFINITION (embedded toric genus). Let X be an affine real algebraic variety of positive dimension. The *embedded toric genus* $ep_t(X)$ *of* X is the largest integer h with the following properties: there are a positive integer $s \leq WL(X)$, nonsingular irreducible real algebraic curves $C_1, C_2, \ldots, C_s$ with $\min_{i \in \{1,2,\ldots,s\}} g(C_i) = h$ and a biregular embedding of X into the s-torus $\prod_{i=1}^{s} C_i$. □

The following lemma gives a basic relation between sectional curve genus and embedded toric genus of an affine real algebraic variety.

LEMMA 2. *Let* X *be an affine real algebraic variety of positive dimension. It holds*: $ep_t(X) \leq p_s(X)$.

PROOF. Let $k := p_s(X)$ and $h := ep_t(X)$. For simplicity, we may suppose that X is an irreducible real algebraic subset of a s-torus $T := \prod_{i=1}^{s} C_i$ with $\min_i g(C_i) = h$. Fix $p \in \mathrm{Nonsing}(X)$ such that $\mathcal{C}_p^k(X) \neq \emptyset$ and let $C \in \mathcal{C}_p^k(X)$. Choose $i \in \{1, 2, \ldots, s\}$ such that the natural projection π of T onto C_i restricted to C is nonconstant. Define $\varrho : C \to C_i$ by $\varrho := \pi|_C$ and consider a complexification $\varrho_{\mathbb{C}}: C_{\mathbb{C}} \to C_{i,\mathbb{C}}$ of ϱ. Since $\varrho_{\mathbb{C}}$ is nonconstant, by Riemann-Hurwitz formula, we have that $k = g(C_{\mathbb{C}}) \geq g(C_{i,\mathbb{C}}) = g(C_i) \geq h$. □

Let X and $\widetilde{X}$ be two affine real algebraic varieties. We say that $\widetilde{X}$ *is obtainable by a weak change of the algebraic structure of* X if there exists a Nash isomorphism $\varphi : \widetilde{X} \to X$ such that φ is a regular map and $\varphi(\mathrm{Sing}(\widetilde{X})) \subset \mathrm{Sing}(X)$. For short, we will indicate such a change of the algebraic structure of X by the symbol $\widetilde{X} \tilde{\to} X$.

Up to weak change of the algebraic structure, the embedded toric genus (and hence the sectional curve genus) of an affine real algebraic variety can assume arbitrarily large values.

LEMMA 3. *Let X be an affine real algebraic variety of positive dimension and let $h \in \mathbb{N}$. Then there exists a weak change $\widetilde{X} \tilde{\to} X$ of the algebraic structure of X such that $ep_t(\widetilde{X}) \geq h$.*

PROOF. We may suppose that X is a bounded algebraic subset of $\mathbb{R}^n$ where n is less than or equal to $WL(X)$ and $X \cap \bigcup_{i=1}^{n} \{x_i = 1\} = \emptyset$. Let d be an odd integer such that $(d-1)(d-2)/2 \geq h$, let C be the nonsingular irreducible real algebraic curve defined by $C := \{(x, y) \in \mathbb{R}^2 \mid x^d + y^d = 1\}$ and let $\varphi : C \to \mathbb{R}$ be the restriction to C of the projection $\pi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\pi(x, y) := x$. The product map $\varphi^n: C^n \to \mathbb{R}^n$ restricted from $\widetilde{X} := (\varphi^n)^{-1}(X)$ to X is the desired weak change of the algebraic structure of X. □

2. Topology of real algebraic morphism space

Let N and M be metric spaces (resp. smooth manifolds). We denote by $C^0(N, M)$ (resp. $C^\infty(N, M)$) the set of all continuous (resp. smooth) maps from N to M equipped with the topology of uniform convergence (resp. the Whitney C^∞-topology, see [5]). We need the notion of trivial subset of $C^0(N, M)$.

DEFINITION. Let N and M be metric spaces. We indicate by $\mathrm{Isol}(N)$ the set of all isolated points of N. The *trivial subset* $\mathrm{Triv}(N, M)$ of $C^0(N, M)$ is the set of the maps $f \in C^0(N, M)$ such that $f(N \setminus \mathrm{Isol}(N)) \subset \mathrm{Isol}(M)$. □

The trivial set defined above is easily described.

LEMMA 4. *Let N and M be two metric spaces such that* $\mathrm{Isol}(N)$, $\mathrm{Isol}(M)$ *and the set $\mathcal{N}$ of all connected components of* $N \setminus \mathrm{Isol}(N)$ *are finite* (*for example when N and M are affine real algebraic varieties*). *Suppose* $\sharp\, \mathrm{Isol}(N) = a$, $\sharp\, \mathrm{Isol}(M) = b$ *and* $\sharp\, \mathcal{N} = c$.

Then Triv(N, M) *is open and closed in* $C^0(N, M)$ *and is homeomorphic to the product* $M^a \times \{1, 2, \ldots, b\}^{\{1, 2, \ldots, c\}}$ *where both* M^0 *and* $\emptyset^\emptyset$ *are considered equal to a point.*

DEFINITION. Let N and M be metric spaces and let L be a subset of $C^0(N, M)$. We say that L is *nowhere dense up to trivial points in* $C^0(N, M)$ if $L \setminus \mathrm{Triv}(N, M)$ is nowhere dense in $C^0(N, M)$, *i.e.*, the interior of the closure of $L \setminus \mathrm{Triv}(N, M)$ in $C^0(N, M)$ is void. □

Before stating our main topological results, we give some other definitions.

DEFINITION. Let N be a topological space. We say that N is *euclidean* if there exists a topological embedding of N into some $\mathbb{R}^n$. Suppose N is euclidean. The *algebraic dimension* $\dim_{\mathrm{alg}} N$ of N is defined as the following integer:

$$\min\{d \in \mathbb{N} \mid \exists \text{ a topol. embedding } h : N \hookrightarrow \mathbb{R}^n \text{ with } \dim \mathrm{Zcl}_{\mathbb{R}^n}(h(N)) = d\}. \qquad \square$$

REMARK. If V is an algebraic subset of $\mathbb{R}^n$, then it is evidently euclidean and $\dim(V)$ is equal to the algebraic dimension of V viewed as an euclidean space.

The following two theorems give an accurate description of the relative topology of the spaces of regular maps between affine real algebraic varieties N and M (resp. affine nonsingular real algebraic varieties) induced by $C^0(N, M)$ (resp. $C^\infty(N, M)$).

THEOREM 5 (general case). *Let* X *and* Y *be affine real algebraic varieties* (*possibly singular*) *of positive dimension. Then the following is true.*

a) *If* $p_s(X) < ep_t(Y)$, *then every regular map from* X *to* Y *is Zariski-locally constant.*

b) *If* $ep_t(Y) \geqslant 1$, *then* $\mathcal{R}(X, Y)$ *is nowhere dense up to trivial points in* $C^0(X, Y)$.

c) *Let* $p_s(X) \geqslant ep_t(Y) \geqslant 2$ *and suppose* Y *compact. Let* $m := \dim(Y)$, *let* k *be the number of Zariski-connected components of* X *and let* r *be the number of irreducible components of* X. *Equip* $\mathcal{R}(X, Y)$ *with the relative topology induced by* $C^0(X, Y)$. *Then* $\mathcal{R}(X, Y)$ *is euclidean and* $km \leqslant \dim_{\mathrm{alg}} \mathcal{R}(X, Y) \leqslant rm$. *Furthermore, if* $k = r$ (*for example when* X *is irreducible*), *then* $\mathcal{R}(X, Y)$ *is closed in* $C^0(X, Y)$.

THEOREM 5′ (nonsingular case). *Let* X *and* Y *be compact affine nonsingular real algebraic varieties of positive dimension. Then previous properties* a) *and* c) *with* $C^0(X, Y)$ *replaced by* $C^\infty(X, Y)$ *hold again, while property* b) *must be modified as follows.*

b′) *If* $ep_t(Y) = 1$, *then* $\mathcal{R}(X, Y)$ *is not dense in* $C^\infty(X, Y)$. *If* $ep_t(Y) \geqslant 2$, *then* $\mathcal{R}(X, Y)$ *is nowhere dense in* $C^\infty(X, Y)$.

Let us give an idea of the proof of the previous theorems.

SKETCH OF THE PROOF OF THEOREM 5 (general case). For simplicity, suppose that X and Y are irreducible. Let $k := p_s(X)$ and $h := ep_t(Y)$. Fix $p \in \mathrm{Nonsing}(X)$ such that $\mathrm{Zcl}_X\Big(\bigcup_{C \in \mathcal{C}_p^k(X)} C\Big) = X$. Suppose Y is an algebraic subset of a s-torus $T := \prod_{i=1}^{s} C_i$ where each C_i is a nonsingular irreducible real algebraic curve and $\min_i g(C_i) = h$.

a) Suppose that there exists a nonconstant regular map f from X to Y. Let $f[p] := = f^{-1}(f(p))$. Since $f[p]$ is a proper algebraic subset of X and $\bigcup_{C \in \mathcal{C}_p^k(X)} C$ is Zariski-dense in X, there is $C \in \mathcal{C}_p^k(X)$ such that $C \not\subset f[p]$, so the regular map $f' : C \to T$, defined by $f'(x) := f(x)$ for each $x \in C$, is nonconstant. In particular, there exists $i \in \{1, 2, \dots, s\}$ such that, if $\pi_i : T \to C_i$ is the natural projection of T onto C_i, then $\pi_i \circ f' : C \to C_i$ is nonconstant. By Riemann-Hurwitz formula, we have $k = g(C) \geqslant \geqslant g(C_i) \geqslant h$ which contradicts the assumption of part a) of the theorem.

b) Let d_T be a metric for T which induces the topology of T. Fix $f \in \mathcal{R}(X, Y) \setminus \mathrm{Triv}(X, Y)$. Let $X' := \mathrm{Isol}(X) \cup f^{-1}(\mathrm{Isol}(Y))$ and $p \in X \setminus X'$. Let $q := = f(p)$, let G be an irreducible real algebraic curve of X such that p is an accumulation point of $\mathrm{Nonsing}(G)$ (use the Nash Curve Selection Lemma) and let $\varrho : U \to Y$ be a continuous retraction from an open neighborhood U of Y in T onto Y (recall that the Alexandrov compactification of every real algebraic set is a polyedron and each compact polyedron is an absolute neighborhood retract). Fix a point $p' \in \mathrm{Nonsing}(G)$ arbitrarily close to p and a relatively compact neighborhood Ω of p' in X such that $\Omega \cap G \subset \mathrm{Nonsing}(G)$. Let $d := \max\{0, 2g(G) - 1\}$ and let ε be a positive real number. Using an adequate partition of unity on X and the retraction ϱ, we can define a continuous map $g : X \to Y$ such that: $g = f$ on $X \setminus \Omega$, $\sup_{x \in \Omega} d_T(g(x), f(x)) < \varepsilon$ and, for some $i \in \{1, 2, \dots, s\}$, the map $g' := \pi_i \circ g_{|\mathrm{Nonsing}(G)} : \mathrm{Nonsing}(G) \to C_i$ is smooth, nonconstant and has at least d distinct critical points of order 2. We will prove that g is not approximable in $C^0(X, Y)$ by regular maps completing the proof of part b). Suppose on the contrary that there exists a regular map $R : X \to Y$ arbitrarily close to g in $C^0(X, Y)$. Choosing R sufficiently close to g and using Rolle's theorem, we have that the map $R' : \mathrm{Nonsing}(G) \to C_i$, defined by $R' := \pi_i \circ R$, is regular, nonconstant and has at least d distinct critical points. This is impossible because, fixed a complexification $R'_{\mathbb{C}} : G_{\mathbb{C}} \to C_{i,\mathbb{C}}$ of R', $R'_{\mathbb{C}}$ does not satisfy the Riemann-Hurwitz formula.

c) STEP I. Fix $i \in \{1, 2, \dots, s\}$. By hypothesis, $g(C_i) \geqslant 2$. Let us recall an improved version of de Franchis' finiteness theorem (see [9] and the next section of this paper): *The number of all nonconstant holomorphic maps from a Riemann surface A of genus a to a Riemann surface B of genus $b \geqslant 2$ is less than or equal to an integer $N(a, b)$ depending only on a and b*. Let $N_i := N(k, g(C_i))$. Let us prove that the number of all nonconstant regular maps from X to C_i is less than or equal to N_i. Suppose on the contrary that there exist $N_i + 1$ distinct nonconstant regular maps $f_1, f_2, \dots, f_{N_i+1}$ from X to C_i. Let $\Delta := \bigcup_{j=1}^{N_i+1} f_j^{-1}(f_j(p)) \cup \bigcup_{j \neq j'} \{f_j = f_{j'}\}$. Δ is a proper algebraic subset of X so there exist $C \in \mathcal{C}_p^k(X)$ such that $C \not\subset \Delta$. It follows that $f'_1 := f_1|_C, \dots, f'_{N_i+1} := f_{N_i+1}|_C$ are distinct nonconstant regular maps from C to C_i, so the corresponding complexifications $f'_{1,\mathbb{C}}, \ldots, f'_{N_i+1,\mathbb{C}}$ are distinct nonconstant complex regular maps from $C_{\mathbb{C}}$ to $C_{i,\mathbb{C}}$. This contradicts the mentioned improved version of de Franchis' theorem.

STEP II. For each $i \in \{1, 2, \ldots, s\}$, let ϱ_i: $Y \to C_i$ be the restriction to Y of the natural projection of T onto C_i and let $\{g_{i,1}, \ldots, g_{i,n_i}\}$ be the set of all nonconstant regular maps from X to C_i. For each $\chi \subset \{1, 2, \ldots, s\}$, let $\chi^* := \{1, 2, \ldots, s\} \setminus \chi$, let $T_{\chi^*} := \prod_{i \in \chi^*} C_i$, let ϱ_{χ^*}: $Y \to T_{\chi^*}$ be the restriction to Y of the natural projection of T onto T_{χ^*} and, if $\chi \neq \emptyset$, let $F(\chi)$ be the set of all functions $\psi : \chi \to \mathbb{N}$ such that, for each $i \in \chi$, $\psi(i) \in \{1, 2, \ldots, n_i\}$. For each $\chi \subset \{1, 2, \ldots, s\}$ with $\chi \neq \emptyset$ and for each $\psi \in F(\chi)$, define

$$\mathcal{R}_{\chi,\psi} := \{f \in \mathcal{R}(X, Y) \,|\, \varrho_i \circ f = g_{i,\psi(i)} \; \forall i \in \chi, \; \varrho_i \circ f \text{ is constant } \forall i \in \chi^*\}.$$

If $\chi = \emptyset$, then we denote by $\mathcal{R}_\emptyset$ the set of all constant maps from X to Y. We have that: $\mathcal{R}(X, Y) = \mathcal{R}_\emptyset \sqcup \bigsqcup_{\emptyset \neq \chi \subset \{1,2,\ldots,s\}, \psi \in F(\chi)} \mathcal{R}_{\chi,\psi}$, $\mathcal{R}_\emptyset$ and every $\mathcal{R}_{\chi,\psi}$ are open and closed in $C^0(X, Y)$ and $\mathcal{R}_\emptyset$ is homeomorphic to Y. In this way, it suffices to prove that each $\mathcal{R}_{\chi,\psi}$ is euclidean and its algebraic dimension is less than or equal to dim Y. Fix $\chi \neq \emptyset$ and $\psi \in F(\chi)$. For each $f \in \mathcal{R}_{\chi,\psi}$, define $\alpha(f)$ as the point of T_{χ^*} such that $\varrho_{\chi^*}(f(X)) = \{\alpha(f)\}$. The map $\varphi : \mathcal{R}_{\chi,\psi} \to T_{\chi^*}$ defined by $\varphi(f) := \alpha(f)$ is a topological embedding, so $\mathcal{R}_{\chi,\psi}$ is euclidean and $\dim_{\text{alg}} \mathcal{R}_{\chi,\psi} \leqslant \dim \varrho_{\chi^*}(Y) \leqslant \dim Y$. □

Piecing together Lemma 3 and Theorem 5, we have the following result.

COROLLARY 6. *Let X and Y be affine real algebraic varieties (possibly singular) of positive dimension. Then we have:*

a) *there exists a weak change $\widetilde{Y} \xrightarrow{\sim} Y$ of the algebraic structure of Y such that every regular map from X to $\widetilde{Y}$ is Zariski-locally constant,*
b) *if $\mathcal{R}(X, Y)$ is dense in $C^0(X, Y)$ (or in $C^\infty(X, Y)$ if both X and Y are nonsingular), then $ep_t(Y) = 0$.*

2. ARITHMETIC OF DOMINATING REAL ALGEBRAIC MORPHISMS

In this section, we will present some real versions of the following famous theorem of de Franchis [4].

THEOREM 7 (de Franchis). *Let C and D be two compact Riemann surfaces. Suppose the genus $g(D)$ of D is greater than or equal to 2. Then the set of all nonconstant (and hence surjective) holomorphic maps from C to D is finite.*

In the situation of the previous theorem, it is possible to estimate the number of nonconstant holomorphic maps by means of $g(C)$ and $g(D)$. In fact, in [9], it is proved the following result (see the references of [9] also).

THEOREM 8 (Tanabe). *Let C and D be compact Riemann surfaces with* $g(D) \geq 2$. *Define the Hurwitz-Tanabe function* $HT : \mathbb{N} \times (\mathbb{N} \setminus \{0, 1\}) \to \mathbb{N}$ *by*

$$HT(a, b) := \begin{cases} 84(a-1) & \text{if } a = b \geq 2 \\ 2(2b-1)(a-1)\left(4\,\dfrac{a-1}{b-1} + 1\right)^{2a} & \text{if } a > b \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then the number of nonconstant holomorphic maps from C to D is less than or equal to $HT(g(C), g(D))$.

We will consider two kinds of dominating maps: the «weakly dominating regular maps» and the «weakly open regular maps». Let us give precise definitions.

DEFINITION. Let X and Y be affine algebraic varieties and let $f \in \mathcal{R}(X, Y)$. We say that f is *weakly dominating* if, for each irreducible component X' of X, there exists an irreducible component Y' of Y such that $\mathrm{Zcl}_Y(f(X')) = Y'$. We denote by $\mathrm{wDom}\,\mathcal{R}(X, Y)$ the set of all weakly dominating regular maps from X to Y. □

DEFINITION. Let X and Y be affine algebraic varieties. A regular map f from X to Y is *weakly open* if, for each irreducible component X' of X, there exists an irreducible component Y' of Y such that $f(X') \subset Y'$ and the interior of $f(\mathrm{Nonsing}(X'))$ in Y' is nonvoid. We indicated by $\mathrm{wo}\,\mathcal{R}(X, Y)$ the set of all weakly open regular maps from X to Y. □

REMARK. If a regular map is open in the usual sense, then it is weakly open.

Let us state our real generalizations of de Franchis' theorem.

THEOREM 9 (weakly dominating regular maps). *Let X and Y be affine algebraic varieties of positive dimension with* $ep_t(Y) \geq 2$. *Suppose that: a is the number of positive-dimensional irreducible components of X, b is the number of 0-dimensional irreducible components of X, c is the number of positive-dimensional Zariski-connected components of X, A is the number of positive-dimensional irreducible components of Y, B is the number of 0-dimensional irreducible components of Y and C is the number of positive-dimensional Zariski-connected components of Y. Then* $\mathrm{wDom}\,\mathcal{R}(X, Y)$ *is finite and it holds:*

$$\sharp\, \mathrm{wDom}\,\mathcal{R}(X, Y) \leq A^a B^b C^c (B + HT(p_s(X), ep_t(Y))^{WL(Y)})^a$$

where 0^0 *is considered equal to* 1.

We sketch the proof of this theorem supposing that X and Y are irreducible.

SKETCH OF THE PROOF OF THEOREM 9. Let $k := p_s(X)$ and $h := ep_t(Y)$. Let s be the smallest positive integer such that $s \leq WL(Y)$ and there is a biregular embedding of Y

into a s-torus $\prod_{i=1}^{s} C_i$ where $C_1, \ldots, C_s$ are nonsingular irreducible real algebraic curves with $\min_i g(C_i) = h$. Fix $i \in \{1, 2, \ldots, s\}$. Piecing together Step I of the proof of part *c*) of Theorem 5 with Theorem 8, it follows that the number of all nonconstant regular maps from X to C_i is less than or equal to $HT(k, g(C_i))$. By an explicit calculation, it is possible to verify that $HT(k, g(C_i)) \leqslant HT(k, h)$. Using the notations of Step II of the proof of part *c*) of Theorem 5 and bearing in mind both the definition of s and the definition of weakly dominating regular map, we have that $\mathrm{wDom}\, \mathcal{R}(X, Y) \subset \bigsqcup_{\psi \in F(\{1,2,\ldots,s\})} \mathcal{R}_{\{1,2,\ldots,s\},\psi}$ so $\sharp\, \mathrm{wDom}\, \mathcal{R}(X, Y) \leqslant \prod_{i=1}^{s} HT(k, g(C_i)) \leqslant$ $\leqslant HT(k, h)^s \leqslant HT(k, h)^{WL(Y)}$ as required. □

THEOREM 10 (weakly open regular maps). *Let X and Y be affine algebraic varieties of positive dimension with $ep_t(Y) \geqslant 2$. Then* $\mathrm{wo}\, \mathcal{R}(X, Y)$ *is finite.*

The proof of the latter theorem is quite complicated so we omit it.

COROLLARY 11 (biregular automorphisms). *Let X be an affine nonsingular irreducible algebraic variety of positive dimension with $ep_t(X) \geqslant 2$. Then the set* $\mathrm{Aut}(X)$ *of all biregular automorphisms of X is finite and it holds*:

$$\sharp\, \mathrm{Aut}(X) \leqslant HT(p_s(X), ep_t(X))^{2\, \dim(X) + 1}.$$

Piecing together Lemma 3 and Theorem 9, we obtain the following result.

COROLLARY 12. *Let X be an affine nonsingular irreducible algebraic variety of positive dimension. Then there exists a weak change $\widetilde{X} \rightrightarrows X$ of the algebraic structure of X such that $Aut(\widetilde{X})$ is finite.*

3. RIGIDITY OF REAL ALGEBRAIC MORPHISMS

First, let us give the notion of rigid varieties.

DEFINITION. An affine real algebraic variety X is *rigid* if, for each affine irreducible real algebraic variety Z, the set of all nonconstant regular maps from Z to X is finite. □

THEOREM 13 (rigidity). *Each affine real algebraic variety X admits a weak change $\widetilde{X} \rightrightarrows X$ of its algebraic structure in such a way that $\widetilde{X}$ is rigid.*

In order to give an idea of the proof of the previous rigidity theorem, we need some preliminary notions and results.

Definition. Let X be a r-dimensional Zariski-locally closed subset of $\mathbb{R}^n$, let $j \in \{0, 1, \dots, n-r\}$ and let L be a j-dimensional vector plane of $\mathbb{R}^n$. We say that L *is good for* X *in* $\mathbb{R}^n$ if $\sup_{x \in \mathbb{R}^n} \sharp\{(x+L) \cap X\}$ is finite. □

Definition. Let X be a r-dimensional Zariski-locally closed subset of $\mathbb{R}^n$ and let B be a base of $\mathbb{R}^n$. Let $j \in \{1, \dots, n\}$. A *coordinate* j*-plane of* B is a vector subspace of $\mathbb{R}^n$ generated by j vectors of B. The *coordinate* 0*-plane of* B is defined to be $\{0\}$. We say that B *is good for* X *in* $\mathbb{R}^n$ if, for each $j \in \{0, 1, \dots, n-r\}$, all coordinate j-planes of B are good for X in $\mathbb{R}^n$. □

Lemma 14. *Let* X *be a* r*-dimensional Zariski-locally closed subset of* $\mathbb{R}^n$*. Then there is a base* B *of* $\mathbb{R}^n$ *good for* X *in* $\mathbb{R}^n$*.*

Proof. A proof of this lemma can be obtained by means of arguments similar to the ones used by Whitney in [11, Section 10, Chapter 7]. □

Definition. Let $T := \prod_{i=1}^{s} C_i$ be a s-torus where $C_1, \dots, C_s$ are real algebraic curves and let X be a r-dimensional Zariski-locally closed subset of T. Let $\chi \subset \{1, 2, \dots, s\}$. Define $T_\chi := \prod_{i \in \chi} C_i$ where $T_\emptyset$ is considered to be a point and let ϱ_χ be the natural projection of T onto T_χ. We say that X *is in good position into* T if, for each $\chi \subset \{1, 2, \dots, s\}$ with $\sharp\chi \geq r$, the following is true: $\sup_{x \in T_\chi} \sharp\{X \cap \varrho_\chi^{-1}(x)\}$ is finite. □

Lemma 15. *Let* X *be a* r*-dimensional Zariski-locally closed subset of* $\mathbb{R}^n$*. Then there are a nonsingular irreducible real curve* C *with* $g(C) \geq 2$ *and a Zariski-locally closed subset* $\widetilde{X}$ *of the* n*-torus* C^n *in good position into* C^n *such that* $\widetilde{X}$ *is obtainable by a weak change of the algebraic structure of* X*.*

Proof. By Lemma 14, we may suppose that the canonical base of $\mathbb{R}^n$ is good for X in $\mathbb{R}^n$. Now, repeating the proof of Lemma 3, we have this lemma. □

Sketch of the Proof of Theorem 13. Suppose X is a r-dimensional algebraic subset of $\mathbb{R}^n$ and $n \geq 2r-1$. Let us apply Lemma 15 to X obtaining a nonsingular irreducible real curve C with $g(C) \geq 2$ and a Zariski-locally closed subset $\widetilde{X}$ of the n-torus $T := C^n$ in good position into T such that $\widetilde{X}$ is obtainable by a weak change of the algebraic structure of X. Fix an affine irreducible real algebraic variety Z of positive dimension. Let $\mathcal{R}^*(Z, X)$ be the set of all nonconstant regular maps from Z to X. Recalling Step II of the proof of part *c*) of Theorem 5, we know that $\mathcal{R}^*(Z, X) \subset \bigsqcup_{\emptyset \neq \chi \subset \{1,2,\dots,s\}, \psi \in F(\chi)} \mathcal{R}_{\chi,\psi}$. Using the good position of $\widetilde{X}$ into T and the inequality $n \geq 2r-1$, we see that each $\mathcal{R}_{\chi,\psi} \cap \mathcal{R}^*(Z, X)$ is finite. □

Let X be an affine real algebraic variety of positive dimension. Applying Theorem 13 to $X \times \mathbb{R}$, we can prove the following theorem.

THEOREM 16 (alg↔nash). *Let X be an affine real algebraic variety of positive dimension. Then there exists an uncountable family $\{X_i\}_{i \in I}$ of rigid affine real algebraic varieties such that each X_i is Nash isomorphic to X, but, for each $i \neq j$ in I, X_i is not biregularly isomorphic to X_j. More precisely, each X_i is obtainable by a weak change of the algebraic structure of X and, for each $i \neq j$ in I, X_i is not birationally isomorphic to X_j.*

The latter result, together with theorems of Tognoli [10] and Akbulut and King [1], generalizes a well-known result by Bochnak and Kucharz [3] and Ballico [2].

COROLLARY 17. *Let M be a smooth manifold of positive dimension. Suppose M is diffeomorphic to the interior of a compact smooth manifold with (possibly empty) boundary. Then there exists an uncountable family $\{M_i\}_{i \in I}$ of rigid affine nonsingular real algebraic varieties such that each M_i is diffeomorphic to M, but, for each $i \neq j$ in I, M_i is not birationally isomorphic to M_j.*

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