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Equazioni a derivate parziali. — *Effective saturation for composite porous media.*
Nota (*) di Micol Amar e Roberto Gianni, presentata dal Socio M. Primicerio.

Abstract. — This paper is devoted to the study of the homogenization of a porous medium, composed of different materials arranged in a periodic structure. This provides the profile of the saturation function for the limit material.

Key words: Homogenization; Parabolic equations; Filtration; Porous media.

Riassunto. — *Saturazione effettiva per mezzi porosi compositi.* Questo lavoro è dedicato allo studio dell'omogeneizzazione di un mezzo poroso, costituito da materiali diversi disposti in struttura periodica. Questo fornisce il profilo della funzione di saturazione del materiale limite.

1. Introduction

It is well known that, in absence of gravity, the flow of an incompressible liquid in a partially saturated porous media is ruled by the equation

$$\sigma'(x, u)\,\frac{\partial u}{\partial t} = \Delta u \quad \text{in } \Omega_T, \tag{1.1}$$

where σ is a bounded function, non decreasing and Lipschitz-continuous with respect to the second entry. Moreover, for physical reasons, we assume the strict monotonicity of $\sigma(x, \cdot)$ for $s < 0$.

Equation (1.1) is a simple consequence of the mass balance equation and of the Darcy's law. Here, the unknown u is related to the liquid pressure in the porous medium, while σ is the saturation profile, giving the volume fraction occupied by the liquid as a function of u. Moreover, the spatial dependence of σ corresponds to the fact that the properties of the material are changing with the position.

In particular, we can think of a composite porous medium, made of finely mixed materials, each of them with a different saturation profile. In this case, $\sigma(x, s) = \sum \sigma_i(s)\,\chi_{E_i}(x)$, where each E_i represents the i^{th}-material. Assuming a microscopic periodic structure of width ε, it appears a small parameter in the problem. We investigate the homogenization limit for $\varepsilon \to 0$, proving that the pressure inside the resulting material satisfies an equation similar to (1.1), where σ is replaced by the effective saturation $\sigma_0(s) = \sum \sigma_i(s)\,|E_i|$. For a general survey on the homogenization in porous media, we refer to [6], and for more details on this topic, see, *e.g.* [4, 8].

It has been discussed for a long time if it makes sense to assume that the limit for $s \to 0^-$ of the derivative of the saturation profile is strictly positive (see, *e.g.*, [3, 7]). From the mathematical point of view, this is a relevant question, since it guarantees the strict parabolicity of the equation (1.1) in the unsaturated region (see, *e.g.*, [5, 7]). Our result seems to indicate that, at least for finely mixed materials, this should be the case, since it suffices that this property

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holds for at least one of the components of the medium, in order to have that it is satisfied by the homogenized composite.

The paper is organized as follows: Section 2 is devoted to some preliminary estimates for the solution of (1.1), which are crucial for the homogenization result. In Section 3 we find the homogenized problem in the case where the saturation has the general form $\sigma(\varepsilon^{-1}x, s)$, with $\sigma(\cdot, s)$ periodic.

2. Preliminaries

Throughout this paper, C will denote a positive constant which may vary each time. We denote by $\mathrm{Lip}_L(\mathbb{R})$ the set of those functions which are Lipschitz-continuous on $\mathbb{R}$, with Lipschitz constant given by L. Set $Y=(0,1)^n$; we say that a function defined on $\mathbb{R}^n$ is Y-periodic if it is periodic of period 1 with respect to each variable.

Let $\Omega\subset\mathbb{R}^n$, $n\geq 1$, be an open bounded set with Lipschitz boundary and T be a fixed positive number. We set $\Omega_T=\Omega\times(0,T)$. We denote by $L^2(0,T;H^1(\Omega))$ the Sobolev space of all L^2-functions g, such that $g(\cdot,t)\in H^1(\Omega)$ for a.e. $t\in(0,T)$, equipped with the natural norm

$$\|g\|_{L^2(0,T;H^1(\Omega))}=\left(\int_0^T\left[\int_\Omega|\nabla_x g(x,t)|^2\,dx\right]dt\right)^{1/2}+\left(\int_0^T\left[\int_\Omega|g(x,t)|^2\,dx\right]dt\right)^{1/2}.$$

Let $\tilde{\sigma}:\mathbb{R}\to\mathbb{R}$ be a non-decreasing function belonging to $\mathrm{Lip}_L(\mathbb{R})$, such that $\tilde{\sigma}(s)\leq 1$, for every $s\in\mathbb{R}$, $\tilde{\sigma}(s)\equiv 1$, for every $s\geq 0$, and $\tilde{\sigma}$ is strictly increasing for $s<0$. Let $\sigma:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ be a measurable function satisfying the following conditions:

$$\begin{array}{llll} & H1) & \sigma(x,\cdot)\in\mathrm{Lip}_L(\mathbb{R}) & \text{for a.e. } x\in\mathbb{R}^n;\\ & H2) & \sigma(x,s)\equiv 1 & \forall s\geq 0,\ \text{for a.e. } x\in\mathbb{R}^n;\\ (2.1) & H3) & \sigma(x,s)\leq\tilde{\sigma}(s) & \forall s\in\mathbb{R},\ \text{for a.e. } x\in\mathbb{R}^n;\\ & H4) & \tilde{\sigma}'(s)\leq\dfrac{\partial}{\partial s}\sigma(x,s) & \forall s\in\mathbb{R},\ \text{for a.e. } x\in\mathbb{R}^n. \end{array}$$

Note that previous assumptions imply also

$$\begin{array}{llll} & H5) & \sigma(x,s)\leq 1 & \forall s\in\mathbb{R},\ \text{for a.e. } x\in\mathbb{R}^n;\\ (2.2) & H6) & 0\leq\tilde{\sigma}'(s)\leq\dfrac{\partial}{\partial s}\sigma(x,s)\leq L & \forall s\in\mathbb{R},\ \text{for a.e. } x\in\mathbb{R}^n;\\ & H7) & \tilde{\sigma}'(s)>0,\quad\dfrac{\partial}{\partial s}\sigma(x,s)>0 & \forall s<0,\ \text{for a.e. } x\in\mathbb{R}^n. \end{array}$$

For the sake of simplicity, in the sequel we will write ∇ instead of ∇_x and $\sigma'(x,s)$ instead of $\dfrac{\partial}{\partial s}\sigma(x,s)$.

Let $\phi\in H^1(\Omega)\cap L^\infty(\Omega)$ and $\Phi\in H^2(\Omega_T)\cap L^\infty(\Omega_T)$ be such that $\Delta\Phi=0$ in

Ω_T; consider the problem

$$(2.3)\qquad \begin{cases} \sigma'(x,u)\dfrac{\partial u}{\partial t} = \Delta u & \text{in } \Omega_T; \\ u(x,0) = \phi(x) & \text{on } \Omega\,; \\ u(x,t) = \Phi(x,t) & \text{on } \partial\Omega \times (0,T)\,. \end{cases}$$

Note that the first equation in (2.3) can be written also in the form

$$\frac{\partial \sigma}{\partial t}(x,u) = \Delta u \quad \text{in } \Omega_T,$$

which is more convenient in order to state the weak formulation of (2.3). Indeed, we will say that a function $u \in L^2(0,T;H^1(\Omega))$ is a solution of (2.3), if

$$(2.4)\qquad \int_{\Omega_T} \sigma(x,u)\frac{\partial \psi}{\partial t}(x,t)\,dx\,dt + \int_{\Omega} \sigma(x,\phi(x))\,\psi(x,0)\,dx = \int_{\Omega_T} \nabla u \nabla \psi\,dx\,dt$$

for every $\psi \in C^1(\Omega_T)$, such that $\psi(x,T)=0$ on Ω and $\psi(x,t)=0$ on $\partial\Omega$, and $u=\Phi$ on $\partial\Omega \times (0,T)$ in the sense of traces.

By (2.4) it follows that the second equation in (2.3) stands for

$$(2.5)\qquad \sigma(x,u(x,0)) = \sigma(x,\phi(x)) \quad \text{on } \Omega\,,$$

which implies that u and ϕ have the same sign. Then, (2.5) is always satisfied on the set where u and ϕ are nonnegative, since in such a case $\sigma \equiv 1$, while it reduces to $u(x,0)=\phi(x)$ on the set where u and ϕ are strictly negative, since there $\sigma(x,\cdot)$ is strictly increasing and then invertible.

We recall that problem (2.3) has always a unique solution $u \in L^2(0,T;H^1(\Omega))$, in the sense of (2.4), whose trace equals Φ on $\partial\Omega \times (0,T)$ (see, *e.g.*, [1, 2]).

If we set $v = u - \Phi$, (2.3) can be equivalently written in the form

$$(2.6)\qquad \begin{cases} \sigma'(x,u)\dfrac{\partial v}{\partial t} = \Delta v - \sigma'(x,u)\dfrac{\partial \Phi}{\partial t} & \text{in } \Omega_T; \\ v(x,0) = \phi(x) - \Phi(x,0) & \text{on } \Omega\,; \\ v(x,t) = 0 & \text{on } \partial\Omega \times (0,T)\,. \end{cases}$$

Multiplying by v_t the first equation in (2.6) and integrating by parts over $\Omega \times (0,t)$, we obtain

$$(2.7)\qquad 0 = \int_{\Omega\times(0,t)} \sigma'(x,u)\,v_t^2\,dx\,d\tau + \int_{\Omega\times(0,t)} \nabla v \nabla v_t\,dx\,d\tau + \int_{\Omega\times(0,t)} \sigma'(x,u)\,\Phi_t v_t\,dx\,d\tau =$$

$$= \int_{\Omega\times(0,t)} \sigma'(x,u)\,v_t^2\,dx\,d\tau + \frac{1}{2}\int_{\Omega\times(0,t)} \frac{\partial}{\partial t}|\nabla v|^2\,dx\,d\tau + \int_{\Omega\times(0,t)} \sigma'(x,u)\,\Phi_t v_t\,dx\,d\tau =$$

$$= \int_{\Omega\times(0,t)} \sigma'(x,u)\,v_t^2\,dx\,d\tau + \frac{1}{2}\int_{\Omega} |\nabla v|^2(x,t)\,dx -$$

$$- \frac{1}{2}\int_{\Omega} |\nabla\phi(x) - \nabla\Phi(x,0)|^2\,dx + \int_{\Omega\times(0,t)} \sigma'(x,u)\,\Phi_t v_t\,dx\,d\tau\,.$$

Recalling that $\sigma' \geqslant 0$ and using Young inequality, this implies

$$\|u\|_{L^2(0,T;H^1(\Omega))} \leqslant \|v\|_{L^2(0,T;H_0^1(\Omega))} + \|\Phi\|_{H^1(\Omega_T)} \leqslant C\big(\|\phi\|_{H^1(\Omega)} + \|\Phi\|_{H^1(\Omega_T)}\big) \tag{2.8}$$

where C depends only on the dimension n, Ω and T. Moreover, taking into account $H6$) and $H7$) of (2.2), by (2.7) it follows

$$\begin{aligned}\int_{\Omega_T} \Big(\frac{\partial}{\partial t}\tilde\sigma(u)\Big)^2 dx\,dt = \int_{\Omega_T} \big(\tilde\sigma'(u)\big)^2 u_t^2\,dx\,dt \leqslant \\ \leqslant C\left[\int_{\Omega_T}\big(\tilde\sigma'(u)\big)^2 v_t^2\,dx\,dt + \int_{\Omega_T}\big(\tilde\sigma'(u)\big)^2 \Phi_t^2\,dx\,dt\right] \leqslant C\big(\|\phi\|_{H^1(\Omega)} + \|\Phi\|_{H^1(\Omega_T)}\big)\end{aligned} \tag{2.9}$$

where C depends only on L, n, Ω and T. Finally,

$$\int_{\Omega_T} |\nabla\tilde\sigma(u)|^2\,dx\,dt = \int_{\Omega_T}\big(\tilde\sigma'(u)\big)^2 |\nabla u|^2\,dx\,dt \leqslant C\big(\|\phi\|_{H^1(\Omega)} + \|\Phi\|_{H^1(\Omega_T)}\big) \tag{2.10}$$

where, again, C depends only on L, n, Ω and T. Note that estimates (2.8), (2.9) and (2.10) actually hold for $\phi < 0$ on Ω, but they can be achieved by approximation and semicontinuity also for general ϕ.

3. Homogenization

The aim of this paper is to study an hogenization problem related to (2.3). To this purpose, let $Y = (0,1)^n$ be the unit cell in $\mathbb{R}^n$. A function f, defined on $\mathbb{R}^n$, is said to be Y-periodic if it is periodic of period 1 with respect to each variable x_i, with $1 \leqslant i \leqslant n$. Assume that $\sigma: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying the previous assumptions $H1$)-$H4$) in (2.1), which is also Y-periodic with respect to the variable $x \in \mathbb{R}^n$. Let $\varepsilon > 0$, define $\sigma_\varepsilon(x,s) = \sigma(\varepsilon^{-1}x, s)$ and consider the family of problems

$$\begin{cases} \sigma'_\varepsilon(x,u_\varepsilon)\dfrac{\partial u_\varepsilon}{\partial t} = \Delta u_\varepsilon & \text{in } \Omega_T; \\ u_\varepsilon(x,0) = \phi(x) & \text{on } \Omega; \\ u_\varepsilon(x,t) = \Phi(x,t) & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{3.1}$$

By (2.8), which is clearly independent of ε, we obtain that there exists a function $u \in L^2(0,T;H_0^1(\Omega))$ such that, up to a subsequence,

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \quad \text{when } \varepsilon \to 0^+, \tag{3.2}$$

and $u = \Phi$ on $\partial\Omega \times (0,T)$ in the sense of traces.

Moreover, by (2.9) and (2.10) it follows that $\|\tilde\sigma(u_\varepsilon)\|_{H^1(\Omega_T)} \leqslant C$, where C depends on L, ϕ and Φ, but not on ε. Hence, there exists a function $v \in H^1(\Omega_T)$ such that, up to a subsequence,

$$\tilde\sigma(u_\varepsilon) \to v \quad \text{strongly in } L^2(\Omega_T), \quad \text{when } \varepsilon \to 0^+. \tag{3.3}$$

Note that $v \leqslant 1$ almost everywhere in Ω_T.

PROPOSITION. 3.1. *Let* $v \in H^1(\Omega_T)$ *and* $u \in L^2(0, T; H_0^1(\Omega))$ *be the functions defined in* (3.2) *and* (3.3), *respectively. Then*

$$v = \tilde{\sigma}(u).$$

PROOF. Since $\tilde{\sigma}(u_\varepsilon) \to v$ strongly in $L^2(\Omega_T)$, when $\varepsilon \to 0^+$, for every $\delta > 0$ we may found a set $N_\delta \subset \Omega$, with $|N_\delta| < \delta$, in such a way that $\tilde{\sigma}(u_\varepsilon) \to v$ uniformly in $\Omega_T^\delta := \Omega_T \setminus N_\delta$. Let us fix $\eta > 0$ and define

$$\Omega_{T,\eta}^{\delta,1} = \{(x, t) \in \Omega_T^\delta : v \leqslant 1 - \eta\},$$

$$\Omega_{T,\eta}^{\delta,2} = \{(x, t) \in \Omega_T^\delta : v \geqslant 1 - 2\eta\}.$$

Clearly, $\Omega_{T,\eta}^{\delta,1} \cup \Omega_{T,\eta}^{\delta,2} = \Omega_T^\delta$ and their intersection is in general non empty.

By the uniform convergence, it follows that there exists ε_0 sufficiently small and depending on δ and η, such that, for every $\varepsilon \leqslant \varepsilon_0$, $\tilde{\sigma}(u_\varepsilon) \leqslant 1 - \eta/2$ in $\Omega_{T,\eta}^{\delta,1}$. This implies that

(*i*) $u_\varepsilon \leqslant -C_1(\eta)$ almost everywhere in $\Omega_{T,\eta}^{\delta,1}$, for a proper value $C_1(\eta) > 0$;

(*ii*) $u_\varepsilon = \tilde{\sigma}^{-1}(\tilde{\sigma}(u_\varepsilon)) \to \tilde{\sigma}^{-1}(v)$ uniformly in $\Omega_{T,\eta}^{\delta,1}$;

(*iii*) since $u_\varepsilon \rightharpoonup u$ weakly in $L^2(0, T; H_0^1(\Omega))$, we have that, almost everywhere in $\Omega_{T,\eta}^{\delta,1}$, $\tilde{\sigma}^{-1}(v) = u$; *i.e.*,

$$v = \tilde{\sigma}(u) \quad \text{a.e. in } \Omega_{T,\eta}^{\delta,1}. \tag{3.4}$$

Moreover, by (*ii*) and (*iii*), it follows

$$u_\varepsilon \to u \qquad \text{uniformly in } \Omega_{T,\eta}^{\delta,1}. \tag{3.5}$$

In $\Omega_{T,\eta}^{\delta,2}$, still using the uniform convergence of $\tilde{\sigma}(u_\varepsilon) \to v$, we have that there exists ε_0 sufficiently small and depending on δ and η, such that, for every $\varepsilon \leqslant \varepsilon_0$, $\tilde{\sigma}(u_\varepsilon) \geqslant 1 - 3\eta$ in $\Omega_{T,\eta}^{\delta,2}$. This implies that

(*iv*) $u_\varepsilon \geqslant -C_2(\eta)$ almost everywhere in $\Omega_{T,\eta}^{\delta,2}$, for a proper value $C_2(\eta) > 0$;

(*v*) by the weak convergence, $u \geqslant -C_2(\eta)$ almost everywhere in $\Omega_{T,\eta}^{\delta,2}$;

(*vi*) by the continuity of $\tilde{\sigma}$, it follows that, for a proper $C_3(\eta) > 0$, $\tilde{\sigma}(u) \geqslant 1 - C_3(\eta)$, which gives

$$\|\tilde{\sigma}(u_\varepsilon) - \tilde{\sigma}(u)\|_{L^\infty(\Omega_{T,\eta}^{\delta,2})} \leqslant 3\eta + C_3(\eta).$$

This implies that, for every $\delta > 0$, every $\eta > 0$ and every $\varepsilon \leqslant \varepsilon_0$,

$$\begin{aligned}
\int_{\Omega_T} |\tilde{\sigma}(u) - v|\, dx\, dt &= \int_{\Omega_T^\delta} |\tilde{\sigma}(u) - v|\, dx\, dt + \int_{N_\delta} |\tilde{\sigma}(u) - v|\, dx\, dt \leqslant \\
&\leqslant \int_{\Omega_{T,\eta}^{\delta,1}} |\tilde{\sigma}(u) - v|\, dx\, dt + \int_{\Omega_{T,\eta}^{\delta,2}} |\tilde{\sigma}(u) - v|\, dx\, dt + \int_{N_\delta} |\tilde{\sigma}(u) - v|\, dx\, dt \leqslant \\
&\leqslant 0 + \int_{\Omega_{T,\eta}^{\delta,2}} |\tilde{\sigma}(u) - \tilde{\sigma}(u_\varepsilon)|\, dx\, dt + \int_{\Omega_{T,\eta}^{\delta,2}} |\tilde{\sigma}(u_\varepsilon) - v|\, dx\, dt + \int_{N_\delta} |\tilde{\sigma}(u) - v|\, dx\, dt \leqslant \\
&\leqslant [3\eta + C_3(\eta)]\, |\Omega_T| + \int_{\Omega_T} |\tilde{\sigma}(u_\varepsilon) - v|\, dx\, dt + \int_{N_\delta} |\tilde{\sigma}(u) - v|\, dx\, dt,
\end{aligned}$$

where we used (3.4). Using 3.3 and letting first $\varepsilon \to 0^+$, then $\eta \to 0^+$ and finally $\delta \to 0^+$, we obtain that $\tilde{\sigma}(u) = v$ almost everywhere in Ω_T, and this concludes the proof. $\square$

For every $s \in \mathbb{R}$, assume that $\sigma(\cdot, s)$ is Y-periodic and define

$$\sigma_0(s) = \int_Y \sigma(y, s)\, dy\,;$$

i.e., the mean value of $\sigma(\cdot, s)$, with respect to the first variable, when s is fixed.

REMARK 3.2. Clearly, $\sigma_0 \in \mathrm{Lip}_L(\mathbb{R})$. It is well known that, due to the periodicity of the function $\sigma(\cdot, s)$, for every $s \in \mathbb{R}$, $\sigma(\varepsilon^{-1}x, s) \rightharpoonup \sigma_0(s)$ $*$-weakly in $L^\infty(\mathbb{R})$, when $\varepsilon \to 0^+$. Moreover, taking into account the Lipschitz continuity of $\sigma(\varepsilon^{-1}x, \cdot)$ and $\sigma_0(\cdot)$ and recalling that every L^2-function can be strongly approximated by step functions, it is possible to prove that, for every $u \in L^2(\Omega_T)$,

$$\sigma(\varepsilon^{-1}x, u) \rightharpoonup \sigma_0(u) \qquad *\text{-weakly in } L^\infty(\Omega_T)$$

when $\varepsilon \to 0^+$.

In order to prove the homogenization result (see Theorem (3.4) below), we need the following lemma.

LEMMA 3.3. *Assume that $\sigma : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying all the hypotheses in* (2.1) *and such that $\sigma(\cdot, s)$ is Y-periodic. For every $\varepsilon > 0$, set $\sigma_\varepsilon(x, s) = \sigma(\varepsilon^{-1}x, u_\varepsilon)$ and let $u_\varepsilon \in L^2(0, T; H_0^1(\Omega))$ be the solution of* (3.1). *Then, for every $\varepsilon \to 0^+$, we have*

$$\sigma_\varepsilon(x, u_\varepsilon) \rightharpoonup \sigma_0(u) \qquad *\text{-weakly in } L^\infty(\Omega_T)\,.$$

PROOF. We have that $\|u_\varepsilon\|_{L^\infty(\Omega_T)} \leq M$, with M in dependent of ε, which is a consequence of the maximum principle. This implies that there exists $\widetilde{M} > 0$, such that $\|\sigma_\varepsilon(x, u_\varepsilon)\|_{L^\infty(\Omega_T)} \leq \widetilde{M}$, for every $\varepsilon > 0$. Hence, up to a subsequence, it follows that there exists $w \in L^\infty(\Omega_T)$ such that

$$\sigma_\varepsilon(x, u_\varepsilon) \rightharpoonup w \qquad *\text{-weakly in } L^\infty(\Omega_T)\,. \tag{3.6}$$

In order to prove that $w = \sigma_0(u)$, we proceed in a similar way as in Proposition 3.1. For every $\delta > 0$ and every $\eta > 0$ let $\Omega^{\delta,1}_{T,\eta}$, $\Omega^{\delta,2}_{T,\eta}$ and N_δ be the sets constructed in the proof of Proposition 3.1, using the strong convergence of the sequence $\{\tilde{\sigma}(u_\varepsilon)\}$. By (*iv*) and (*v*) of Proposition 3.1 we have that, for ε small enough, $u, u_\varepsilon \geq -C_2(\eta)$, in $\Omega^{\delta,2}_{T,\eta}$. By the continuity of $\sigma(x, \cdot)$, uniformly with respect to x, this implies that $\sigma_\varepsilon(x, u_\varepsilon) \geq 1 - C_4(\eta)$ and $\sigma_\varepsilon(x, u) \geq 1 - C_4(\eta)$, for a suitable $C_4(\eta) > 0$. Hence,

$$|\sigma_\varepsilon(x, u_\varepsilon) - \sigma_\varepsilon(x, u)| \leq C_4(\eta)\,. \tag{3.7}$$

Then, using H1), (3.7) and (2.8), it follows

$$\left|\int_{\Omega_T}[w-\sigma_0(u)]\,\psi\,dx\,dt\right| \leqslant \left|\int_{\Omega_T}[\sigma_\varepsilon(x,u_\varepsilon)-w]\,\psi\,dx\,dt\right| +$$

$$+\left|\int_{\Omega_T}[\sigma_\varepsilon(x,u_\varepsilon)-\sigma_0(u)]\,\psi\,dx\,dt\right| \leqslant \left|\int_{\Omega_T}[\sigma_\varepsilon(x,u_\varepsilon)-w]\,\psi\,dx\,dt\right| +$$

$$+\|\psi\|_{L^\infty}\Bigg[\int_{\Omega^{\delta,1}_{T,\eta}}|\sigma_\varepsilon(x,u_\varepsilon)-\sigma_\varepsilon(x,u)|\,dx\,dt+\int_{\Omega^{\delta,2}_{T,\eta}}\left|\sigma_\varepsilon(x,u_\varepsilon)-\sigma_\varepsilon(x,u)\right|dx\,dt+$$

$$+\int_{N_\delta}|\sigma_\varepsilon(x,u_\varepsilon)-\sigma_\varepsilon(x,u)|\,dx\,dt\Bigg]+\left|\int_{\Omega_T}[\sigma_\varepsilon(x,u)-\sigma_0(u)]\,\psi\,dx\,dt\right| \leqslant$$

$$\leqslant \left|\int_{\Omega_T}[\sigma_\varepsilon(x,u_\varepsilon)-w]\,\psi\,dx\,dt\right| +$$

$$+\|\psi\|_{L^\infty}\left[L\int_{\Omega^{\delta,1}_{T,\eta}}|u_\varepsilon-u|\,dx\,dt+C_4(\eta)\,|\Omega_T|+L\int_{N_\delta}|u_\varepsilon-u|\,dx\,dt\right]+$$

$$+\left|\int_{\Omega_T}[\sigma_\varepsilon(x,u)-\sigma_0(u)]\,\psi\,dx\,dt\right| \leqslant \left|\int_{\Omega_T}[\sigma_\varepsilon(x,u_\varepsilon)-w]\,\psi\,dx\,dt\right| +$$

$$+C\|\psi\|_{L^\infty}\left[\int_{\Omega^{\delta,1}_{T,\eta}}|u_\varepsilon-u|\,dx\,dt+C_4(\eta)+|N_\delta|\right]+\left|\int_{\Omega_T}[\sigma_\varepsilon(x,u)-\sigma_0(u)]\,\psi\,dx\,dt\right|$$

for every test function $\psi\in C^0(\overline{\Omega}_T)$, such that $\psi(x,T)=0$ on Ω and $\psi(x,t)=0$ on $\partial\Omega\times(0,T)$. Taking into account (3.6), (3.5) and Remark 3.2, and letting $\varepsilon\to 0^+$, $\eta\to 0^+$ and finally $\delta\to 0^+$, we obtain that the right hand side in the previous inequality converges to 0. This implies that $w=\sigma_0(u)$ and that the whole sequence (not only a subsequence) converges. So the required result is achieved. □

THEOREM 3.4. *Assume that $\sigma:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ is a measurable function satisfying all the hypotheses in* (2.1) *and such that $\sigma(\cdot,s)$ is Y-periodic. Let $\phi\in H^1(\Omega)\cap L^\infty(\Omega)$ and $\Phi\in H^2(\Omega_T)\cap L^\infty(\Omega_T)$, with $\Delta\Phi=0$ in Ω_T. For every $\varepsilon>0$, let $u_\varepsilon\in L^2(0,T;H^1_0(\Omega))$ be the solution of* (3.1). *Then, the sequence $\{u_\varepsilon\}$, converges weakly in $L^2(0,T;H^1(\Omega))$ to the solution $u\in L^2(0,T;H^1(\Omega))$ of the problem*

$$\begin{cases}\sigma_0'(u)\,\dfrac{\partial u}{\partial t}=\Delta u & \text{in } \Omega_T;\\ u(x,0)=\phi(x) & \text{on } \Omega;\\ u(x,t)=\Phi(x,t) & \text{on } \partial\Omega\times(0,T).\end{cases} \tag{3.8}$$

PROOF. Taking into account the weak formulation of (3.1), we have that, for every $\varepsilon > 0$,

$$\int_{\Omega_T} \sigma_\varepsilon(x, u_\varepsilon) \frac{\partial \psi}{\partial t}(x, t)\, dx\, dt + \int_{\Omega} \sigma_\varepsilon(x, \phi(x))\, \psi(x, 0)\, dx = \int_{\Omega_T} \nabla u_\varepsilon \nabla \psi\, dx\, dt \tag{3.9}$$

for every test function $\psi \in C^1(\Omega_T)$, such that $\psi(x, T) = 0$ on Ω and $\psi(x, t) = 0$ on $\partial\Omega \times (0, T)$, with $u_\varepsilon = \Phi$ on $\partial\Omega \times (0, T)$ in the sense of traces. Taking into account Lemma 3.3, (3.2) and passing to the limit in (3.9) for $\varepsilon \to 0^+$, we obtain

$$\int_{\Omega_T} \sigma_0(u) \frac{\partial \psi}{\partial t}(x, t)\, dx\, dt + \int_{\Omega} \sigma_0(\phi(x))\, \psi(x, 0)\, dx = \int_{\Omega_T} \nabla u \nabla \psi\, dx\, dt$$

which implies that u is a weak solution of (3.8). Hence, the whole sequence converges and the thesis is accomplished. □

REMARK 3.5. In the framework of physical applications, the model we have in mind is given by

$$\sigma(x, s) = \sum_{i=1}^{N} \sigma_i(s)\, \chi_{E_i}(x)$$

where, for every $i = 1, \ldots, N$, $\sigma_i \in \mathrm{Lip}_L(\mathbb{R})$, $\sigma_i(s) \leqslant 1$ for every $s \in \mathbb{R}$, $\sigma_i(s) \equiv 1$ for every $s \geqslant 0$, σ_i' is strictly positive for $s < 0$, $E_i \subset Y$, $\cup E_i = Y$, $\cap E_i = \emptyset$ and χ_{E_i} is the characteristic function of the set E_i, extended by periodicity to the whole space $\mathbb{R}^n$. It follows that σ satisfies all the required assumptions in (2.1), with $\tilde{\sigma}$ defined by the relation $\tilde{\sigma}'(s) = \min \sigma_i'(s)$. Hence, by Theorem 3.4, we obtain the homogenization result, where now the function σ_0 in (3.8) is given by $\sigma_0(s) = \sum_i \sigma_i(s)\, |E_i|$.

REMARK 3.6. Clearly, Theorem 3.4 continues to hold, if we assume that, for every $\varepsilon > 0$, $\sigma_\varepsilon(x, s)$ satisfies all the hypotheses in (2.1) and there exists $\sigma(x, s)$ such that, for every $s \in \mathbb{R}$,

$$\sigma_\varepsilon(\cdot, s) \rightharpoonup \sigma(\cdot, s) \quad * \text{-weakly in } L^\infty(\Omega)$$

when $\varepsilon \to 0^+$. Note that, in this more general setting, the limit saturation can depend on the position.

As an example, we can think of a microstructure of concentric sperical layers of two different alternating materials; note that, also in this case, the effective saturation does not depend on x.

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