Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Elisabetta Barozzi, Eduardo Gonzalez, Umberto Massari

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **14** (2003), n.4, p. 257–277.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2003_9_14_4_257_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2003.

Rend. Mat. Acc. Lincei
s. 9, v. 14:257-277 (2003)

Calcolo delle variazioni. — *The mean curvature of a Lipschitz continuous manifold.* Nota di Elisabetta Barozzi, Eduardo Gonzalez e Umberto Massari, presentata (*) dal Socio M. Miranda.

Abstract. — The paper is devoted to the description of some connections between the mean curvature in a distributional sense and the mean curvature in a variational sense for several classes of non-smooth sets. We prove the existence of the mean curvature measure of ∂E by using a technique introduced in [4] and based on the concept of variational mean curvature. More precisely we prove that, under suitable assumptions, the mean curvature measure of ∂E is the weak limit (in the sense of distributions) of the mean curvatures of a sequence of regular n-dimensional manifolds M_j convergent to ∂E. The manifolds M_j are closely related to the level surfaces of the variational mean curvature H_E of E.

Key words: Calculus of Variations; Geometric Measure Theory; Functions of Bounded Variation; Mean Curvature.

Riassunto. — *La curvatura media di una varietà Lipschitziana*. L'articolo è dedicato allo studio di alcuni legami tra la curvatura media nel senso delle distribuzioni e la curvatura media in senso variazionale di alcune classi di insiemi non regolari. Si dimostra l'esistenza di curvatura media misura per ∂E usando tecniche introdotte in [4] e basate sul concetto di curvatura media variazionale. Più precisamente, si dimostra, sotto opportune ipotesi, che la curvatura media misura della frontiera di E è il limite debole (nel senso delle distribuzioni) delle curvature medie di una successione di varietà n-dimensionali M_j regolari convergenti alla frontiera di E . Le varietà M_j sono legate alle superfici di livello della curvatura media variazionale H_E di E.

0. Introduction

A function $H \in L^1(U)$ (U an open set of $\boldsymbol{R}^{n+1}$) is said to be a **variational mean curvature** of a given set $E \subset U$ if E locally minimizes the functional

$$\mathcal{F}_H(F) = \int_U |D\phi_F| + \int_{U \cap F} H(x)\, dx \tag{0.1}$$

(see § 1).

By computing the first variation of (0.1), it can be easily seen that if H is a variational mean curvature of E, ∂E is a smooth manifold in a neighbourhood of a point $x \in \partial E \cap U$ and H is a continuous function at x, then $H(x)$ is (up a constant factor) the classical mean curvature of ∂E at x. This is the reason why minimizers of (0.1) are called «sets of variational mean curvature H».

It is well known that if H is a variational mean curvature of E and $H \in L^p(U)$ with $p > n+1$, then we have the decomposition

$$\partial E \cap U = \Sigma_r \cup \Sigma_s,$$

(*) Nella seduta del 19 giugno 2003.

where Σ_r (the so-called regular subset) is an n-dimensional $C^{1,\alpha}$ manifold and Σ_s (the so-called singular subset) is a closed subset of $\partial E \cap U$, and

$$\mathcal{H}_k(\Sigma_s) = 0 \quad \forall k > n-7$$

(see [14, 15]).

However, the existence of a variational mean curvature $H \in L^1(U)$ does not imply, in general, any smoothness of ∂E. As a matter of fact, a variational mean curvature $H_E \in L^1(U)$ can be constructed for every set $E \subset U$ of finite perimeter (see [4, 5]). We refer to [13] for a more detailed account.

On the other hand, the mean curvature of non regular manifolds can be defined in a different way by using the methods of Geometric Measure Theory (see, for example, [1]). In particular, assume that $M \subset U$ is an n-dimensional Lipschitz-continuous manifold, and that exists a positive constant K such that

$$\left| \int_M \operatorname{div}_M X d\mathcal{H}_n \right| \leq K \|X\|_\infty \quad \forall X \in C_0^1(U, \boldsymbol{R}^{n+1}). \tag{0.2}$$

By (0.2) and by the Riesz Representation Theorem, it follows the existence of an (n+1)-dimensional vector valued Radon measure on M, which we denote by

$$\vec{\boldsymbol{H}} = (\boldsymbol{H}_1, \dots, \boldsymbol{H}_n, \boldsymbol{H}_{n+1}),$$

such that

$$\int_M \operatorname{div}_M X d\mathcal{H}_n = -\int_M X \bullet d\vec{\boldsymbol{H}} = -\sum_{j=1}^{n+1} \int_M X_j d\boldsymbol{H}_j \quad \forall X \in C_0^1(U, \boldsymbol{R}^{n+1}). \tag{0.3}$$

The measure $\vec{\boldsymbol{H}}$ will be called the **mean curvature measure** of M in U.

An interesting case is when the Radon measure $\vec{\boldsymbol{H}}$ is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}_{n|M}$. Then we have

$$\vec{\boldsymbol{H}} = \vec{H} \cdot \mathcal{H}_{n|M}$$

where the density $\vec{H}: M \to \boldsymbol{R}^{n+1}$ belongs to $[L^1(M)]^{n+1}$. In this case (0.3) becomes

$$\int_M \operatorname{div}_M X \, d\mathcal{H}_n = -\int_M H \bullet X \, d\mathcal{H}_n \quad \forall X \in C_0^1(U, \boldsymbol{R}^{n+1}). \tag{0.4}$$

The connection between the two definitions of mean curvature does not seem to be evident even when (0.4) holds. A variational mean curvature is defined as an element of $L^1(U)$ which is typically discontinuous at points $x \in \partial E \cap U$. Instead mean curvature measures (or more simply density functions) are defined only over the manifold $M = \partial E$.

In this paper, we prove the existence of a mean curvature measure of ∂E by using a technique introduced in [4] and based on the concept of variational mean curvature. More precisely, we prove that, under suitable assumptions, the mean curvature measure of $\partial E \cap U$ is the weak limit (in the sense of measures) of the mean curvatures of a sequence of n-dimensional manifolds M_j convergent to ∂E. The manifolds M_j are closely related to the level surfaces of the variational mean curvature H_E.

The main Theorem is the following:

THEOREM 0.1. *Suppose that $\partial E \cap U$ is locally the graph of a function $f \in C^{1,\alpha}$ that is a weak supersolution of the minimal surface equation or that E is a convex set. Then there exists a* (n + 1)*-dimensional vector valued Radon measure $\vec{\boldsymbol{H}}$ such that*

$$\int_M \operatorname{div}_M X \, d\mathcal{H}_n = -\int_M X \bullet d\vec{\boldsymbol{H}} \quad \forall X \in C_0^1(U, \boldsymbol{R}^{n+1}),$$

where $M = \partial E$.

1. THE VARIATIONAL MEAN CURVATURE

The notion of variational mean curvature H is a generalization of the definition of minimal boundary introduced by E. De Giorgi in the fifties (see [6, 7]), in the context of sets of finite perimeter or Caccioppoli sets (see for example [11, 16]).

We now recall some basic definitions and results that will be used in the sequel.

If $U \subset \boldsymbol{R}^{n+1}$ is an open set and E is a subset of U, we denote by $\int_U |D\phi_E|$ the perimeter of E in U, that is

$$\int_U |D\phi_E| = \sup \left\{ \int_U \operatorname{div} g(x)\, dx,\ g \in C_0^1(U, \boldsymbol{R}^{n+1}),\ \|g\|_\infty \leq 1 \right\}. \tag{1.1}$$

For $H \in L^1(U)$ and $F \subset U$, define

$$\mathcal{F}(F, U) = \int_U |D\phi_F| + \int_F H(x)\, dx. \tag{1.2}$$

A set E is said to have variational mean curvature H in U if

$$\begin{cases} i) & \displaystyle\int_V |D\phi_E| < +\infty \quad \forall V \subset\subset U, \\ ii) & \mathcal{F}(E, V) \leq \mathcal{F}(F, V) \quad \forall V \subset\subset U, \quad \forall F \subset U \\ & \text{such that } (E - F) \cup (F - E) \subset\subset V. \end{cases} \tag{1.3}$$

The next theorem, due to E. De Giorgi [6], U. Massari [14, 15], is probably the most important result concerning the variational mean curvature:

THEOREM 1.1. *If E has variational mean curvature H in U and $H \in L^p(U)$ with $p > n+1$, then*

$$\partial E \cap U = \Sigma_r \cup \Sigma_s, \tag{1.4}$$

where Σ_r (the regular part of ∂E) is a n-dimensional $C^{1,\alpha}$ manifold and Σ_s (the singular part of ∂E) is a closed subset of ∂E such that

$$\mathcal{H}_k(\Sigma_s) = 0 \quad \forall s > n - 7, \tag{1.5}$$

where $\mathcal{H}_k$ *is the Hausdorff measure of dimension* $k \in \boldsymbol{R}$. *In particular, for* $n \leqslant 6$ *we have* $\Sigma_s = \emptyset$.

Theorem 1.1 and decomposition (1.4) cannot be extended to the case $p = n+1$ (see [13]).

Finally, for $H \in L^p(U)$ with $1 \leqslant p < n+1$, no regularity result may be expected. In fact, in [5], Barozzi, Gonzalez and Tamanini have proved that every set E of finite perimeter has a variational mean curvature $H_E \in L^1(U)$. For the critical case $p = n+1$ see [12].

For the reader's convenience, we outline the construction of H_E.

Let $h : \boldsymbol{R}^{n+1} \to \boldsymbol{R}$ be a non negative, measurable function such that $\int_E h(x)\,dx < +\infty$. Moreover suppose that

$$F \subset E, \ \int_F h(x)\,dx = 0 \iff |F| = 0,$$

where $|\cdot|$ denotes the Lebesgue measure in $\boldsymbol{R}^{n+1}$. For $\lambda \geqslant 0$ and $F \subset E$, consider the functional

$$\mathcal{B}_\lambda(F) = \int_{\boldsymbol{R}^{n+1}} |D\phi_F| + \lambda \int_{E-F} h(x)\,dx \,. \tag{1.6}$$

By well known results of Calculus of Variations, for every $\lambda \geqslant 0$ there exists a solution E_λ of the minimum problem

$$\begin{cases} i) & \mathcal{B}_\lambda(F) \to \min\,, \\ ii) & F \in \mathcal{E}_\lambda = \{F,\ F \subset E\}\,. \end{cases} \tag{1.7}$$

Moreover,

$$\begin{aligned} &i)\ \text{if } 0 \leqslant \lambda < \mu \Rightarrow E_\lambda \subset E_\mu \\ &ii)\ \cup\{E_\lambda,\ \lambda > 0\} = E\,. \end{aligned} \tag{1.8}$$

By defining

$$H_E(x) = -\inf\{\lambda h(x),\ x \in E_\lambda,\ \lambda \geqslant 0\} \quad \forall x \in E \tag{1.9}$$

we obtain a function $H_E \colon E \to \boldsymbol{R}$ with the following two properties:

$$\int_E |H_E(x)|\,dx = \int_{\boldsymbol{R}^{n+1}} |D\phi_E| \tag{1.10}$$

$$\int_{\boldsymbol{R}^{n+1}} |D\phi_E| + \int_E H_E(x)\,dx \leqslant \int_{\boldsymbol{R}^{n+1}} |D\phi_F| + \int_F H_E(x)\,dx \ \ \forall F \subset E. \tag{1.11}$$

Arguing in the same way with E replaced by $\boldsymbol{R}^{n+1} - E$, we can define H_E in $\boldsymbol{R}^{n+1} - E$ too. In [4, 5] it is proved that the function H_E obtained above is a variational mean curvature for E in $\boldsymbol{R}^{n+1}$. Moreover, we have

$$\int_{\boldsymbol{R}^{n+1}} |H_E(x)|\,dx = 2 \int_{\boldsymbol{R}^{n+1}} |D\phi_E|\,. \tag{1.12}$$

Whenever E is a bounded set, two interesting choices for the function h in the above construction of H_E (see (1.6)) are given by

$$h(x) = 1 \quad \forall x \in E \tag{1.13}$$

$$h(x) = \text{dist}\,(x, \partial E) \quad \forall x \in E. \tag{1.14}$$

E. Barozzi in [4] has used (1.13) to prove a minimality property of the L^p-norm of H_E. Almgren, Taylor and Wang in [2] have used (1.14) to introduce a variational approach to the motion by the mean curvature. In the second case we remark that $\Sigma_{\lambda, r} \cap \text{Int}\,(E)$ ($\text{Int}\,(E)$ = the interior of E, $\Sigma_{\lambda, r}$ the regular part of ∂E_λ, see (1.4)) is a smooth n-dimensional $C^{2,\alpha}$ manifold with classical mean curvature H_λ given by

$$H_\lambda(x) = \lambda h(x)\, \nu(x) \quad \forall x \in \Sigma_{\lambda, r} \cap \text{Int}\,(E) \tag{1.15}$$

(where $\nu(x)$ is the outer normal vector to $\Sigma_{\lambda, r}$ at x). Moreover, if we assume that E is a convex set, we can use the strong maximum principle to conclude that $E_\lambda \subset\subset \text{Int}\,(E)$. Then $\Sigma_{\lambda, r} \in C^{2,\alpha}$, and we can write (0.4) in the form

$$\int_{M_\lambda} \text{div}_{M_\lambda} X \, d\mathcal{H}_n = -\lambda \int_{M_\lambda} h(x)\, X \bullet \nu d\mathcal{H}_n \quad \forall X \in C_0^1(U, \boldsymbol{R}^{n+1}) \tag{1.16}$$

where $M_\lambda = \partial E_\lambda$.

The main purpose of this paper is to study the behaviour of (1.16) when $\lambda \to +\infty$ or, equivalently, the behaviour of the family of measures

$$\nu_\lambda(A) = \lambda \int_{M_\lambda \cap A} h(x)\, d\mathcal{H}_n, \quad A \subset U. \tag{1.17}$$

EXAMPLE 1.2. Let $E = B_R(0)$ and h given by (1.14). By a straightforward computation we obtain that for $\lambda R^2 > \dfrac{4(n+1)^2}{n+2}$, the unique solution E_λ of the minimum problem (1.7) is the sphere $B_{R_\lambda}(0)$, where

$$R_\lambda = \frac{R}{2} + \sqrt{\frac{R^2}{4} - \frac{n}{\lambda}}\,.$$

In this case

$$H_E(x) = \begin{cases} \dfrac{4(n+1)^2}{(n+2)R^2}(R - |x|) & \text{if } 0 \leq |x| \leq \dfrac{(n+2)\,R}{2(n+1)} \\[2ex] \dfrac{n}{|x|} & \text{if } \dfrac{(n+2)\,R}{2(n+1)} \leq |x| \leq R \\[2ex] 0 & \text{if } |x| > R\,. \end{cases}$$

REMARK 1.3. If h is given by (1.14), then we can estimate the distance between ∂E_λ and ∂E. Precisely, we have:

i)

$$\operatorname{dist}(x, \partial E) \leqslant 2\sqrt{\frac{n+1}{\lambda}} \quad \text{for } x \in \partial E_\lambda \cap \operatorname{Int}(E).$$

In fact, by applying the inequality (see Tamanini [17], formula (1.10))

$$\int_{\partial B} (1-\phi_F)\, d\mathcal{H}_n \leqslant \int_B |D\phi_F| + \frac{n+1}{r}|B-F|, \tag{1.18}$$

(which holds for any ball $B \subset \boldsymbol{R}^{n+1}$ with radius r) with $B = B_{R/2}$, $R = \operatorname{dist}(x, \partial E)$ and $F = E_\lambda$, we obtain

$$\int_{\partial B} (1-\phi_{E_\lambda})\, d\mathcal{H}_n \leqslant \int_B |D\phi_{E_\lambda}| + \frac{2(n+1)}{R}|B-E_\lambda|.$$

On the other hand, from the minimality of E_λ, we have

$$\int_B |D\phi_{E_\lambda}| \leqslant \int_{\partial B}(1-\phi_{E_\lambda})\, d\mathcal{H}_n - \lambda \int_{B-E_\lambda} \operatorname{dist}(z, \partial E)\, dz$$

and therefore

$$\frac{2(n+1)}{R}|B-E_\lambda| \geqslant \lambda \int_{B-E_\lambda} \operatorname{dist}(z, \partial E)\, dz \geqslant \lambda \frac{R}{2}|B-E_\lambda|$$

and the desired inequality follows.

ii) if E satisfies an internal sphere condition (that is, if there exists $R>0$ and, for every point $y \in \partial E$, a ball of radius R such that $B \subset E$ and $\overline{B} \cap \partial E = \{y\}$), if $\lambda R^2 > \dfrac{4(n+1)^2}{n+2}$ and $x \in \partial E_\lambda \cap \operatorname{Int}(E)$, then

$$\operatorname{dist}(x, \partial E) \leqslant \frac{R}{2} - \sqrt{\frac{R^2}{4} - \frac{n}{\lambda}} < \frac{2n}{R\lambda}.$$

In fact, if $E_1 \subset E_2$ and $E_{1,\lambda}$, $E_{2,\lambda}$ are solutions of the minimum problem (1.7) with E_1 and E_2 respectively, then $E_{1,\lambda} \subset E_{2,\lambda}$. It follows that the ball B_{R_λ} of Example 1.2 is contained in E_λ and the desired inequality follows.

Therefore, in this case we obtain

$$\lambda h(x) = \lambda \operatorname{dist}(x, \partial E) < \frac{2n}{R} \quad \forall x \in \partial E_\lambda \cap \operatorname{Int}(E).$$

We conclude this section with some further remarks about the non parametric case.

We assume that $U = \Omega \times \boldsymbol{R}$ (Ω an open subset of $\boldsymbol{R}^n$) and $E = \{x = (y, z) \in \Omega \times \boldsymbol{R},\ z < f(y)\}$, where $f : \Omega \to \boldsymbol{R}$ is a given function. For $f \in C^2(\Omega)$, we set:

$$Tf(y) = \frac{Df(y)}{\sqrt{1+|Df(y)|^2}}, \quad y \in \Omega$$

and

$$H(y,z) = \operatorname{div}(Tf)(y) = \sum_{j=1}^{n} D_j \left(\frac{D_j f(y)}{\sqrt{1+|Df(y)|^2}} \right), \qquad (y,z) \in U. \tag{1.19}$$

It is easy to see that

$$\int_M \operatorname{div}_M X \, d\mathcal{H}_n = -\int_M H X \bullet \nu \, d\mathcal{H}_n \quad \forall X \in C_0^1(U, \boldsymbol{R}^{n+1}). \tag{1.20}$$

Here $M = \partial E \cap U$ and ν is the outer normal to $\partial E \cap U$.

In this case we can write

$$\operatorname{div}_M X = \sum_{j=1}^{n+1} \delta_j X_j \tag{1.21}$$

where δ_j, $j = 1, \dots, n+1$ are the tangential derivatives, that is,

$$\delta_j = D_j - \nu_j \sum_{h=1}^{n+1} \nu_h D_h, \qquad j = 1, \dots, n+1.$$

The function H (given by (1.19)) is a variational mean curvature for E in U.

Sometimes a formula like (1.20) may be true with a given function $H \in L^1(M)$ without the assumption $f \in C^2$. In such a case we shall say that $H \in L^1(M)$ is a weak mean curvature of M.

For example, we can consider a symmetric surface M, *i.e.*,

$$f(y) = g(\varrho), \qquad \varrho = |y| \in (0, R) \tag{1.22}$$

with $g \in C^2(0, R)$. In this case (1.19) becomes

$$H(y,z) = \frac{g''(\varrho)}{(1+g'^2(\varrho))^{3/2}} + \frac{(n-1)\, g'(\varrho)}{\varrho(1+g'^2(\varrho))^{1/2}}, \qquad \varrho \in (0, R). \tag{1.23}$$

Now, denoting by $M_r = M - B_r \times \boldsymbol{R}$, $(0 < r < R,\ B_r = \{y \in \boldsymbol{R}^n\colon\ |y| < r\})$, we obtain

$$\begin{aligned} \int_{M_r} \operatorname{div}_M X \, d\mathcal{H}_n = & -\int_{M_r} H X \bullet \nu \, d\mathcal{H}_n + \\ & + \left(1 - \frac{1}{(1+g'^2(r))^{1/2}}\right) \int_{\partial B_r} \sum_{j=1}^{n} X_j \frac{y_j}{r} \, d\mathcal{H}_{n-1} - \frac{g'(r)}{\sqrt{1+g'^2(r)}} \int_{\partial B_r} X_{n+1} \, d\mathcal{H}_{n-1} \\ & \forall X \in C_0^1(U, \boldsymbol{R}^{n+1}). \end{aligned} \tag{1.24}$$

We study the behaviour of (1.24) when $r \to 0$ with the choice

$$g(\varrho) = c\varrho^\alpha, \qquad c > 0, \qquad \alpha \in (0, 1] \tag{1.25}$$

(a cusp when $\alpha \in (0,1)$, a cone when $\alpha = 1$). Whenever $n \geq 2$, the last two integrals in the right side of (1.24) go to zero as $r \to 0$ and then (1.20) is true with H given by

$$H(y,z) = \frac{c\alpha |y|^{\alpha-2} (\alpha + n - 2 + (n-1)\, c^2 \alpha^2 |y|^{2\alpha-2})}{(1 + c^2 \alpha^2 |y|^{2\alpha-2})^{3/2}}. \tag{1.26}$$

For $n = 1$, from (1.24) when $r \to 0$, we obtain

$$\int_M \operatorname{div}_M X d\mathcal{H}_1 = -\int_M HX \bullet \nu d\mathcal{H}_1 - 2X_2(0, 0) \lim_{r\to 0} \frac{g'(r)}{\sqrt{1 + g'^2(r)}} \tag{1.27}$$

and the value of the limit is

$$\mathcal{L}(\alpha) = \begin{cases} 1 & \text{if } \alpha \in (0, 1) \\ \dfrac{c}{\sqrt{1 + c^2}} & \text{if } \alpha = 1. \end{cases} \tag{1.28}$$

In this case (1.20) fails to be true. In fact, (1.27) implies that the Radon measure which represents the linear functional

$$X \to \int_M \operatorname{div}_M X d\mathcal{H}_1, \qquad X \in C_0^1(U, \boldsymbol{R}^2)$$

has a singular component with respect to the Hausdorff measure $\mathcal{H}_{1|M}$, given by the «Dirac measure»

$$\mu_s = (0, \ -2\,\mathcal{L}(\alpha)\,\delta_{(0,0)}). \tag{1.29}$$

2. Variational mean curvature of a pseudoconvex set

In this section we construct a variational mean curvature of a subgraph E of a Lipschitz continuous function f, by following the method introduced in [4, 5].

Let $A \subset \boldsymbol{R}^n$ be an open bounded set. Let $f : A \to \boldsymbol{R}$ be a Lipschitz continuous function. Let $\Omega \subset A$ be an open set with $\partial\Omega \in C^2$ and mean curvature of $\partial\Omega$ nonnegative. Let $E = \{(y, z) \in \Omega \times \boldsymbol{R} : y \in \Omega, z \leq f(y)\}$ be the subgraph of f. In the following we shall suppose that E is a pseudoconvex set, *i.e.*

$$\int_\Omega \sqrt{1 + |Df|^2} \leq \int_\Omega \sqrt{1 + |Dv|^2} \quad \forall v \in BV(\Omega), \ \operatorname{spt}(v - f) \subset\subset \Omega, \quad v \geq f, \tag{2.1}$$

or, in other words, that f is a weak supersolution of the minimal surface equation, *i.e.*,

$$\int_\Omega Tf \bullet D\phi \, dy \geq 0 \quad \forall \phi \in C_0^1(\Omega), \quad \phi \geq 0. \tag{2.2}$$

For each $\lambda \geq 0$, we define the funcional $\mathcal{B}_\lambda$: $BV(\Omega) \to \boldsymbol{R}$ by setting

$$\mathcal{B}_\lambda(v) = \int_\Omega \sqrt{1 + |Dv|^2} + \frac{\lambda}{2} \int_\Omega (f - v)^2 dy + \int_{\partial\Omega} |f - v| \, d\mathcal{H}_{n-1}. \tag{2.3}$$

Then we can state the following (see for example [10]).

Theorem 2.1. *The functional* $\mathcal{B}_\lambda$ *has a unique minimizer* $u_\lambda \in BV(\Omega)$. *Moreover* $u_\lambda \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ $\forall \alpha \in (0, 1)$ *and* $u_\lambda(y) = f(y)$ $\forall y \in \partial\Omega$.

REMARK 2.2. The function u_λ is a solution of the Euler equation associated to the functional $\mathcal{B}_\lambda$, *i.e.*

$$Mu_\lambda(y) = \operatorname{div}(Tu_\lambda)(y) = -\lambda(f(y) - u_\lambda(y)) \quad \forall y \in \Omega. \tag{2.4}$$

REMARK 2.3. From the inequality $\mathcal{B}_\lambda(u_\lambda) \leqslant \mathcal{B}_\lambda(f)$ and the lower-semicontinuity of the area functional (with respect to the $L^1(\Omega)$-convergence), we obtain

(2.5) *i*)
$$\lim_{\lambda \to +\infty} \lambda \int_\Omega (u_\lambda - f)^2 \, dy = 0$$

(in particular, $u_\lambda \to f$ in $L^2(\Omega)$)

(2.5) *ii*)
$$\lim_{\lambda \to +\infty} \int_\Omega \sqrt{1 + |Du_\lambda|^2} = \int_\Omega \sqrt{1 + |Df|^2}.$$

Moreover,

(2.5) *iii*)
$$Du_\lambda \to Df \text{ in } L^1(\Omega).$$

In fact, from (2.5) *i*) and (2.5) *ii*), we have that Du_λ weakly converges as distributions to Df and that $\{Du_\lambda\}_\lambda$ is bounded in $L^1(\Omega)$, and (2.5) *iii*) follows (see [3, Exercise 1.20]).

PROPOSITION 2.4.

$$0 \leqslant \lambda < \mu \Rightarrow u_\lambda(y) \leqslant u_\mu(y) \leqslant f(y) \quad \text{a.e.} \quad y \in \Omega. \tag{2.6}$$

PROOF. From (2.1) it follows that $u_\mu \wedge f$ is also a minimum for $\mathcal{B}_\mu$. Thus, by the uniqueness of u_μ it follows that $u_\mu \wedge f = u_\mu$, *i.e.* $u_\mu \leqslant f$.

Now, let $v = u_\lambda \wedge u_\mu$, $w = u_\lambda \vee u_\mu$, $G = \{x \in \Omega : u_\lambda(y) > u_\mu(y)\}$. Adding the inequalities

$$\mathcal{B}_\mu(u_\mu) \leqslant \mathcal{B}_\mu(w); \qquad \mathcal{B}_\lambda(u_\lambda) \leqslant \mathcal{B}_\lambda(v),$$

and recalling that

$$\int_\Omega \sqrt{1 + |Dv|^2} + \int_\Omega \sqrt{1 + |Dw|^2} \leqslant \int_\Omega \sqrt{1 + |Du_\lambda|^2} + \int_\Omega \sqrt{1 + |Du_\mu|^2},$$

we obtain

$$\mu \left[\int_G \left((f - u_\mu)^2 - (f - u_\lambda)^2 \right) dy \right] \leqslant \lambda \left[\int_G \left((f - u_\mu)^2 - (f - u_\lambda)^2 \right) dy \right].$$

On the other hand

$$(f - u_\mu)^2 - (f - u_\lambda)^2 = (2f - u_\mu - u_\lambda)(u_\lambda - u_\mu) > 0 \quad \text{in } G.$$

Hence $|G| = 0$, that is $u_\lambda \leqslant u_\mu$ a.e. in Ω.

REMARK 2.5. Suppose now that there exists a function $Mf \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega Tf \bullet D\varphi dy = -\int_\Omega (Mf)\, \varphi dy \quad \forall \varphi \in C_0^1(\Omega), \tag{2.7}$$

i.e. suppose that the distributional divergence of the vector $Tf = \dfrac{Df}{\sqrt{1+|Df|^2}}$ is a function $Mf \in L^1_{\mathrm{loc}}(\Omega)$.

If $Mf \in L^p(\Omega)$, $1 < p \leq +\infty$, then the family

$$\psi_\lambda(y) = \lambda(f(y) - u_\lambda(y)) \quad \forall y \in \Omega \quad \forall \lambda > 0 \tag{2.8}$$

is bounded in $L^p(\Omega)$ and the estimate

$$\|\psi_\lambda\|_p \leq \|Mf\|_p \quad \forall \lambda > 0 \tag{2.9}$$

holds. Moreover we have

$$Mu_\lambda \to Mf \text{ weakly in } L^p(\Omega). \tag{2.10}$$

In fact, multiplying (2.4) by $(f - u_\lambda)^{p-1}$ and integrating by parts, we obtain

$$\lambda \int_\Omega (f - u_\lambda)^p \, dy = \int_\Omega Tu_\lambda \bullet D[(f - u_\lambda)^{p-1}] \, dy .$$

Recalling that

$$(Tf - Tu_\lambda) \bullet (Df - Du_\lambda) \geq 0,$$

from (2.7) and Hölder's inequality, we obtain

$$\lambda \int_\Omega (f - u_\lambda)^p \, dy = \int_\Omega Tu_\lambda \bullet D(f - u_\lambda)^{p-1} \, dy \leq \int_\Omega Tf \bullet D(f - u_\lambda)^{p-1} \, dy =$$

$$= -\int_\Omega Mf (f - u_\lambda)^{p-1} \, dy \leq \|Mf\|_p \, \|f - u_\lambda\|_p^{p-1}$$

and (2.9) follows.

We now prove (2.10). Observe that (2.5) iii) implies

$$\lim_{\lambda \to +\infty} \int_\Omega |Tu_\lambda(y) - Tf(y)| \, dy = 0. \tag{2.11}$$

Then we have, $\forall \varphi \in C^1_0(\Omega)$

$$\int_\Omega Mf \varphi \, dy = -\int_\Omega Tf \bullet D\varphi \, dy = -\lim_{h \to \infty} \int_\Omega Tu_{\lambda_h} \bullet D\varphi \, dy = \lim_{h \to \infty} \int_\Omega Mu_{\lambda_h} \varphi dy,$$

which proves (2.10).

EXAMPLE 2.6. Let $g : [0, 2] \to \boldsymbol{R}$ be the function defined by

$$g(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ -(t-1)^\alpha & \text{if } 1 \leq t \leq 2 \end{cases}$$

where $\alpha \in (1, 2)$ and let $f(y) = g(|y|)$, $y \in \boldsymbol{R}^2$, $|y| \leq 2$. It is easy to see that (2.7) is verified and $Mf \in L^p(B_2)$ if and only if $p(2 - \alpha) < 1$. In particular, if $\alpha > \dfrac{3}{2}$, then we have $Mf \in L^2(B_2)$.

We now proceed to the construction of the variational mean curvature of the set

$$E = \{(y, z) \in \Omega \times \boldsymbol{R}, \; z < f(y)\}.$$

Let

$$E_\lambda = \{(y, z) \in \Omega \times \boldsymbol{R},\ z < u_\lambda(y)\},$$

where u_λ is the unique minimizer of $\mathcal{B}_\lambda$ and define

$$H(y, z) = \begin{cases} -\inf\{\lambda(f(y) - z), (y, z) \in E_\lambda,\ \lambda \geq 0\} & \text{se } (y, z) \in E \\ 0 \quad \text{se } (y, z) \in (\Omega \times \boldsymbol{R}) - E. \end{cases} \tag{2.12}$$

We claim that the function H, just defined, is a variational mean curvature of E in $\Omega \times \boldsymbol{R}$.

From (2.1) and a standard symmetrization argument, it is sufficient to prove

$$\mathcal{F}_H(f) \leq \mathcal{F}_H(v) \quad \forall v \in BV(\Omega) \quad \operatorname{spt}(f - v) \subset\subset \Omega, \quad v \leq f \tag{2.13}$$

where $\mathcal{F}_H \colon BV(\Omega) \to \boldsymbol{R}$ is the functional

$$\mathcal{F}_H(v) = \int_\Omega \sqrt{1 + |Dv|^2} + \int_\Omega \left(\int_{-\infty}^{v(y)} H(y, z)\, dz \right) dy. \tag{2.14}$$

From (2.6) and (2.12), we get that if $0 \leq \lambda < \mu$ and $(y, z) \in E_\mu - E_\lambda$, then

$$-\mu(f(y) - z) \leq H(y, z) \leq -\lambda(f(y) - z). \tag{2.15}$$

For $k \in \boldsymbol{N}$, put

$$\lambda_j = \frac{j}{2^k},\ j = 0, 1, 2, 3, \ldots, \quad u_j = u_{\lambda_j}, \quad E_j = E_{\lambda_j},$$

and define

$$H_k(y, z) = \begin{cases} -\lambda_j(f(y) - z) & \text{if } (y, z) \in E_j - E_{j-1}, j \in \boldsymbol{N} \\ 0 & \text{if } (y, z) \in E_0 \cup [(\Omega \times \boldsymbol{R}) - E)]. \end{cases} \tag{2.16}$$

We prove now the following

THEOREM 2.7. *$H \in L^1(\Omega \times \boldsymbol{R})$ and*

$$\mathcal{F}_H(f) \leq \mathcal{F}_H(v) \quad \forall v \in BV(\Omega) \quad \operatorname{spt}(f - v) \subset\subset \Omega, \quad v \leq f \tag{2.17}$$

(i.e., *H is a variational mean curvature for E). Moreover*

$$\|H\|_{L^1(\Omega \times \boldsymbol{R})} = \int_\Omega \sqrt{1 + |Df|^2} - \int_\Omega \sqrt{1 + |Du_0|^2}. \tag{2.18}$$

PROOF. From (2.15) whith $\lambda = \dfrac{j-1}{2^k}$, $\mu = \dfrac{y}{2^k}$ we obtain

$$-\frac{j}{2^k}(f(y) - z) \leq H(y, z) \leq -\frac{j-1}{2^k}(f(y) - z) \quad \forall (y, z) \in E_j - E_{j-1} \tag{2.19}$$

and therefore

$$\|H\|_{L^1(\Omega \times \boldsymbol{R})} \leq \sum_{j=1}^{\infty} \frac{j}{2^k} \int_\Omega \left(\int_{u_{j-1}}^{u_j} (f(y) - z)\, dz \right) dy \quad \forall k \in \boldsymbol{N}. \tag{2.20}$$

Now, from $\mathcal{B}_{\lambda_{j-1}}(u_{j-1}) \leqslant \mathcal{B}_{\lambda_{j-1}}(u_j)$, we obtain

$$\frac{j-1}{2^k}\int_\Omega\left(\int_{u_{j-1}}^{u_j}(f(y)-z)\,dz\right)dy \leqslant \int_\Omega\sqrt{1+|Du_j|^2}-\int_\Omega\sqrt{1+|Du_{j-1}|^2},$$

and therefore

$$\sum_{j=1}^{\infty}\frac{j}{2^k}\int_\Omega\left(\int_{u_{j-1}}^{u_j}(f(y)-z)\,dz\right)dy =$$

$$= \sum_{j=1}^{\infty}\frac{1}{2^k}\int_\Omega\left(\int_{u_{j-1}}^{u_j}(f(y)-z)\,dz\right)dy + \sum_{j=1}^{\infty}\frac{j-1}{2^k}\int_\Omega\left(\int_{u_{j-1}}^{u_j}(f(y)-z)\,dz\right)dy \leqslant$$

$$\leqslant \frac{1}{2^k}\int_\Omega\left(\int_{u_0}^{f}(f(y)-z)\,dz\right)dy + \sum_{j=1}^{\infty}\left[\int_\Omega\sqrt{1+|Du_j|^2}-\int_\Omega\sqrt{1+|Du_{j-1}|^2}\right] =$$

$$= \frac{1}{2^k}\int_\Omega\left(\int_{u_0}^{f}(f(y)-z)\,dz\right)dy + \left[\int_\Omega\sqrt{1+|Df|^2}-\int_\Omega\sqrt{1+|Du_0|^2}\right].$$

Letting $k \to +\infty$ we obtain

$$\|H\|_{L^1(\Omega\times\mathbf{R})} \leqslant \int_\Omega\sqrt{1+|Df|^2}-\int_\Omega\sqrt{1+|Du_0|^2} \tag{2.21}$$

and therefore $H \in L^1(\Omega\times\mathbf{R})$.

Let

$$\mathcal{F}_k(v) = \mathcal{F}_{H_k}(v) = \int_\Omega\sqrt{1+|Dv|^2} + \int_\Omega\left(\int_{-\infty}^{v}H_k(y,z)\,dz\right)dy\,. \tag{2.22}$$

We now prove that

$$\mathcal{F}_k(u_j) \leqslant \mathcal{F}_k(v) \quad \forall v \in BV(\Omega),\ \operatorname{spt}(u_j - v) \subset\subset \Omega,\ v \leqslant u_j. \tag{2.23}$$

We argue by induction on j. If $j=1$, then (2.23) follows from the inequality $\mathcal{B}_{\lambda_1}(u_1) \leqslant \mathcal{B}_{\lambda_1}(v)$ of Theorem 2.1. Suppose now that (2.23) holds for some j. If $v \in BV(\Omega)$ is such that $\operatorname{spt}(u_{j+1}-v) \subset\subset \Omega$ and $v \leqslant u_{j+1}$, then we can write

$$\mathcal{F}_k(u_j) \leqslant \mathcal{F}_k(u_j \wedge v) \quad \text{(inductive assumption)}$$

$$\mathcal{B}_{\lambda_{j+1}}(u_{j+1}) \leqslant \mathcal{B}_{\lambda_{j+1}}(u_j \vee v) \quad \text{(by Theorem 2.1)}.$$

From these inequalities, we obtain

$$\mathcal{F}_k(u_{j+1}) \leqslant \mathcal{F}_k(v).$$

Indeed

$$\mathcal{F}_k(u_{j+1}) = \int_\Omega \sqrt{1+|Du_{j+1}|^2} + \int_\Omega \left(\int_{-\infty}^{u_{j+1}} H_k(y,z)\,dz \right) dy =$$

$$= \int_\Omega \sqrt{1+|Du_j|^2} + \int_\Omega \left(\int_{-\infty}^{u_j} H_k(y,z)\,dz \right) dy +$$

$$+ \int_\Omega \sqrt{1+|Du_{j+1}|^2} - \int_\Omega \sqrt{1+|Du_j|^2} + \int_\Omega \left(\int_{u_j}^{u_{j+1}} H_k(y,z)\,dz \right) dy \leqslant$$

$$\leqslant \int_\Omega \sqrt{1+|D(u_j \wedge v)|^2} + \int_\Omega \left(\int_{-\infty}^{u_j \wedge v} H_k(y,z)\,dz \right) dy +$$

$$+ \int_\Omega \sqrt{1+|Du_{j+1}|^2} - \int_\Omega \sqrt{1+|Du_j|^2} + \int_\Omega \left(\int_{u_j}^{u_{j+1}} H_k(y,z)\,dz \right) dy =$$

$$= \mathcal{B}_{j+1}(u_{j+1}) + \int_\Omega \sqrt{1+|D(u_j \wedge v)|^2} - \int_\Omega \sqrt{1+|Du_j|^2} +$$

$$+ \int_\Omega \left(\int_{-\infty}^{u_j \wedge v} H_k(y,z)\,dz \right) dy - \frac{j+1}{2^k} \int_\Omega \left(\int_{u_j}^{f} (f(y)-z)\,dz \right) dy \leqslant$$

$$\leqslant \mathcal{B}_{j+1}(u_j \vee v) + \int_\Omega \sqrt{1+|D(u_j \wedge v)|^2} - \int_\Omega \sqrt{1+|Du_j|^2} +$$

$$+ \int_\Omega \left(\int_{-\infty}^{u_j \wedge v} H_k(y,z)\,dz \right) dy - \frac{j+1}{2^k} \int_\Omega \left(\int_{u_j}^{f} (f(y)-z)\,dz \right) dy \leqslant$$

$$\leqslant \int_\Omega \sqrt{1+|Dv|^2} + \frac{j+1}{2^k} \int_\Omega \frac{1}{2}(f-(u_j \vee v))^2\,dy -$$

$$- \frac{j+1}{2^k} \int_\Omega \frac{1}{2}(f-u_j)^2\,dy + \int_\Omega \left(\int_{-\infty}^{u_j \wedge v} H_k(y,z)\,dz \right) dy =$$

$$= \int_\Omega \sqrt{1+|Dv|^2} + \int_\Omega \left(\int_{u_j}^{u_j \vee v} H_k(y,z)\,dz \right) dy + \int_\Omega \left(\int_{-\infty}^{u_j \wedge v} H_k(y,z)\,dz \right) dy =$$

$$= \int_\Omega \sqrt{1+|Dv|^2} + \int_\Omega \left(\int_{-\infty}^{v} H_k(y,z)\,dz \right) dy = \mathcal{F}_k(v).$$

Hence, (2.23) follows.

Now, letting $j \to +\infty$ in (2.23), we obtain

(2.24) $$\mathcal{F}_k(f) \leqslant \mathcal{F}_k(v) \quad \forall v \in BV(\Omega), \quad \operatorname{spt}(f-v) \subset\subset \Omega, \ v \leqslant f$$

Hence H_k is a mean variational curvature for the set E.

From (2.19) it follows that $H_k \xrightarrow{k} H$ in $L^1(\Omega \times \boldsymbol{R})$ and we obtain immediately that also H is a mean variational curvature for E.

Finally, from $\mathcal{F}_H(f) \leqslant \mathcal{F}_H(u_0)$, we obtain

$$\int_\Omega \sqrt{1+|Df|^2} + \int_\Omega \left(\int_{-\infty}^{f} H(y,z)\, dz \right) dy \leqslant \int_\Omega \sqrt{1+|Du_0|^2},$$

i.e.

(2.25) $$\int_\Omega \sqrt{1+|Df|^2} - \int_\Omega \sqrt{1+|Du_0|^2} \leqslant \|H\|_{L^1(\Omega \times \boldsymbol{R})}.$$

By (2.25) and (2.21) we obtain (2.18).

Hence, the proof of Theorem 2.7 is complete.

We can repeat the preceding construction all over again replacing the function $h(y,z) = [f(y) - z] \vee 0$ by an arbitrary measurable function $h: \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ such that $h(y,z) \geqslant 0$ a.e. $(y,z) \in \Omega \times \boldsymbol{R}$, and $\int_F h(x)\, dx = 0 \Rightarrow |F| = 0$. The relevance of the choice $h(y,z) = |f(y) - z|$ becomes clear from the following.

THEOREM 2.8. *In addiction to the hypotheses of Theorem* 2.7 *suppose that f is not a solution of the minimal surface equation in Ω. Then*:

a) $u_\lambda(y) < f(y)\ \forall y \in \Omega\ \forall \lambda \geqslant 0$
b) $0 \leqslant \lambda < \mu \Rightarrow u_\lambda(y) < u_\mu(y)\ \forall y \in \Omega$
c) $u_\lambda \in C^{2,\alpha}(\Omega)\ \forall \lambda \geqslant 0$.

PROOF. The inequalities $u_\lambda(y) \leqslant u_\mu(y) \leqslant f(y)\ \forall y \in \Omega$ were already proved in Proposition 2.4. Now we set $g(p) = \sqrt{1+|p|^2}$, $p \in \boldsymbol{R}^n$ and

$$a_{ij}(y) = \int_0^1 \frac{\partial^2 g}{\partial p_i \partial p_j}(Df(y) + t(Du_\lambda(y) - Df(y)))\, dt.$$

Then we have

$$Tu_\lambda - Tf = \sum_{i,j=1}^n a_{ij} D_j(u_\lambda - f),$$

and accordingly,

(2.26) $$\int_\Omega (Tu_\lambda - Tf) \bullet D\varphi dy = \sum_{i,j=1}^n \int_\Omega a_{ij} D_j(u_\lambda - f)\, D_i \varphi\, dy.$$

Therefore, if $\varphi \in C_0^1(\Omega)$, $\varphi \geqslant 0$, then we have

$$\sum_{i,j=1}^n \int_\Omega a_{ij} D_j(u_\lambda - f)\, D_i \varphi dy \leqslant -\int_\Omega M u_\lambda \varphi dy = \lambda \int_\Omega (f - u_\lambda)\, \varphi dy.$$

We obtain

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_j (u_\lambda - f)\, D_i \varphi dy + \lambda \int_{\Omega} (u_\lambda - f)\, \varphi dy \leqslant 0 \quad \forall \varphi \in C_0^1(\Omega),\ \varphi \geqslant 0. \tag{2.27}$$

Then, from the strong maximum principle applied to $u_\lambda - f$ (see [9, Theorem 8.19]), *a*) follows.

In the same way we can prove that

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_j (u_\lambda - u_\mu)\, D_i \varphi dy = \int_{\Omega} (Tu_\lambda - Tu_\mu) \bullet D\varphi dy = \int_{\Omega} [\lambda(f - u_\lambda) - \mu(f - u_\mu)]\, \varphi dy$$

and observing that

$$\lambda(f - u_\lambda) - \mu(f - u_\mu) = -\lambda(u_\lambda - u_\mu) + (\mu - \lambda)(u_\mu - f) \leqslant -\lambda(u_\lambda - u_\mu),$$

we obtain

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_j (u_\lambda - u_\mu)\, D_i \varphi dy + \lambda \int_{\Omega} (u_\lambda - u_\mu)\, \varphi dy \leqslant 0 \quad \forall \varphi \in C_0^1(\Omega),\ \varphi \geqslant 0. \tag{2.28}$$

Then we can apply the strong maximum principle and *b*) follows.

Statement *c*) is a straightforward consequence of *a*) and the classical regularity theory (see [14, 15]).

REMARK 2.9.
i) If $\lambda_j \uparrow \lambda$, then $u_{\lambda_j} \uparrow u_\lambda$ uniformly.
ii) If $u_0(y) < z < f(y)$, then there exists $\lambda > 0$ such that $u_\lambda(y) = z$ $(y \in \Omega)$.
Then we easily conclude that H is continuous in E.

3. A GRADIENT ESTIMATE

In this section, we prove a global gradient estimate for the family of functions $\{u_\lambda\}_{\lambda \geqslant 0}$ that is independent of λ. Such a gradient bound is obtained in the following two cases:
i) when $f \in C^{1,\alpha}(A)$ and (2.1) is verified;
ii) when f is a concave function.
Each case needs a suitable choice of Ω.
At first, we prove the following

LEMMA 3.1. Let $f \in C^{1,\alpha}(\Omega)$ and the function $|Du_\lambda(y)|$ has a relative maximum at a point $y_0 \in \Omega$. Then

$$|Du_\lambda(y_0)| \leqslant |Df(y_0)| \tag{3.1}$$

PROOF. For simplicity, we set $u = u_\lambda$. Then we can write (2.4) in the form

$$\sum_{i,j=1}^{n} a_{ij}(Du)\, D_{ij} u = -\lambda(f - u), \tag{3.2}$$

where

$$a_{ij}(p) = \frac{(1+|p|^2)\,\delta_{ij} - p_i p_j}{(1+|p|^2)^{3/2}} \quad \forall p \in \boldsymbol{R}^n.$$

The assumption $f \in C^{1,\alpha}(\Omega)$ implies that $u \in C^{3,\alpha}(\Omega)$. Let $w = \frac{1}{2}|Du|^2$. Differentiating equation (3.2) with respect to y_h, multiplying by $D_h u$ and summing with respect to h, we obtain

$$(3.3) \qquad \sum_{i,j=1}^{n} a_{ij} D_{ij} w - \sum_{i,j,h=1}^{n} a_{ij} D_{ih} u D_{jh} u + \sum_{i,j,k=1}^{n} D_{p_k} a_{ij} D_{ij} u D_k w = \\ = -\lambda \left(\sum_{h=1}^{n} D_h f D_h u - \sum_{h=1}^{n} D_h u D_h u \right)$$

Recalling now that

$$D_k w(y_0) = 0$$

$$\sum_{i,j=1}^{n} a_{ij}(Du(y_0))\, D_{ij} w(y_0) \leqslant 0$$

$$\sum_{i,j=1}^{n} a_{ij}(Du(y_0))\, D_{ih} u(y_0)\, D_{jh} u(y_0) \geqslant 0 \quad \forall h = 1, 2, \ldots, n,$$

from (3.3) we deduce

$$|Du(y_0)|^2 \leqslant Df(y_0) \bullet Du(y_0) \leqslant |Df(y_0)||Du(y_0)|$$

and (3.1) is proved.

THEOREM 3.2. *Assume that $\partial\Omega \in C^3$, that the mean curvature of $\partial\Omega$ be non negative* (*take for example $\Omega =$ a sphere*) *and assume that $f \in C^{1,\alpha}(\Omega)$ and* (2.2) *holds. Then there exists a constant $k > 0$* ($k = k(n, \Omega, \|f\|_{C^{1,\alpha}(\partial\Omega)})$) *such that*

$$(3.4) \qquad |Du_\lambda(y)| \leqslant k \quad \forall y \in \Omega, \quad \forall \lambda \geqslant 0.$$

PROOF. By Proposition 2.4, we have

$$(3.5) \qquad u_0(y) \leqslant u_\lambda(y) \leqslant f(y) \quad \forall y \in \Omega \quad \forall \lambda \geqslant 0.$$

Now u_0 is a solution of the minimal surface equation in Ω and then, by Theorem 2.1 of [8], we have

$$(3.6) \qquad |u_0(y_1) - u_0(y_2)| \leqslant k_1 |y_1 - y_2| \quad \forall y_1, y_2 \in \overline{\Omega}$$

where $k_1 = k_1(n, \Omega, \|f\|_{C^{1,\alpha}(\partial\Omega)})$.

Now, if there exists $y_0 \in \Omega$ such that

$$|Du_\lambda(y)| \leqslant |Du_\lambda(y_0)| \quad \forall y \in \Omega,$$

from Lemma 3.1, we obtain

$$|Du_\lambda(y)| \leqslant |Df(y_0)| \leqslant L \quad \forall y \in \Omega.$$

On the other hand, from (3.5) it follows that

$$-k_1\,|y_1-y_2| \leqslant u_0(y_1)-u_0(y_2) \leqslant f(y_1)-f(y_2) \quad \forall y_1 \in \Omega \quad \forall y_2 \in \partial\Omega$$

(because $u_0(y_2)=f(y_2)$).

We deduce that

$$|Du_\lambda| \leqslant \max\{k_1, L\}$$

and Theorem 3.2 is proved.

REMARK 3.3. If f is supposed to be only Lipschitz-continuous, we may no more use the results of [8] to obtain the estimate (3.6) and then it fails to exist (in general) an inferior Lipschitz-continuous barrier. For example, the function $u(y_1, y_2) = \ln\left(\dfrac{2+\sqrt{3}}{\sqrt{y_1^2+y_2^2}+\sqrt{y_1^2+y_2^2-1}}\right)$ is a solution of the minimal surface equation in $\Omega=\{(y_1, y_2)\in \boldsymbol{R}^2,\ y_1>1+y_2^2\}$, $u(1+y_2^2, y_2)$ is Lipschitz-continuous but $|Du|$ is not bounded at all.

Now, let's consider the case *ii*) in which f is a concave function.

Let $y_0 \in A$, $0<\varrho<\frac{1}{4}\operatorname{dist}(y_0, \partial A)$, and L be the Lipschitz constant of f in $B_{4\varrho}(y_0)$. Let

$$v(y)=f(y_0)-4\varrho L+3L|y-y_0| \quad \forall y \in \boldsymbol{R}^n. \tag{3.7}$$

Clearly,

$$v(y) \geqslant f(y) \quad \forall y \in B_{4\varrho}(y_0)-B_{2\varrho}(y_0)$$

$$v(y) \leqslant f(y) \quad \forall y \in B_\varrho(y_0).$$

Now we choose

$$\Omega=\{y\in B_{2\varrho}(y_0): v(y)<f(y)\}. \tag{3.8}$$

Clearly, Ω is an open convex subset of A such that

$$B_\varrho(y_0) \subset \Omega \subset B_{2\varrho}(y_0).$$

Since v is a convex function, we have

$$v(y) \leqslant u_\lambda(y) \leqslant f(y) \quad \forall y \in \Omega. \tag{3.9}$$

We are now ready to state the following

THEOREM 3.5. *Let* Ω *be defined by* (3.8). *We have*

$$|Du_\lambda(y)| \leqslant 3L \quad \forall y \in \Omega. \tag{3.10}$$

PROOF. We have to slightly modify the proof of Lemma 3.1. Indeed, in general $u_\lambda \notin C^3(\Omega)$. We sketch the proof.

Let

$$f_h(y)=\int_{B_{4\varrho}(y_0)} f(y-z)\,\tau_h(z)\,dz,$$

where $\{\tau_h\}_h$ is a standard sequence of nonnegative mollifiers. Then $\{f_h\}_h$ is a sequence of concave functions in C^∞ such that

$$|Df_h(y)| \leqslant L \quad \forall y \in B_{4\varrho}(y_0)$$

and which converges uniformly to f on compact subsets of $B_{4\varrho}(y_0)$. Let

$$v_h(y) = f_h(y_0) - 4\varrho L + 3L|y - y_0|$$

$$\Omega_h = \{y \in B_{2\varrho}(y_0) : v_h(y) < f_h(y)\}$$

and denote by u_λ^h the unique minimizer of

$$\mathcal{B}_\lambda^h(v) = \int_{\Omega_h} \sqrt{1 + |Dv|^2}dy + \frac{\lambda}{2}\int_{\Omega_h}(f_h - v)^2\,dy + \int_{\partial\Omega_h} |f_h - v|\,d\mathcal{H}_{n-1}.$$

We have that $u_\lambda^h \in C^3(\Omega_h)$ and

$$v_h(y) \leqslant u_\lambda^h(y) \leqslant f_h(y) \quad \forall y \in \Omega_h \quad \forall \lambda \geqslant 0.$$

We can then conclude that

$$|Du_\lambda^h(y)| \leqslant 3L \quad \forall y \in \Omega_h \quad \forall \lambda \geqslant 0.$$

If $h \to +\infty$, the last inequality implies (3.10).

4. Mean curvature measures

In this Section, using the Riesz Representation Theorem and the u_λ's functions, we shall define the mean curvature of some classes of manifolds that are the graph of non-smooth functions.

We denote by M_λ and M the graphs of u_λ and f respectively and by div_{M_λ} and div_M the tangential divergence with respect to M_λ and M. For example, if $X \in C_0^1(\Omega \times \times \boldsymbol{R}, \boldsymbol{R}^{n+1})$, then

$$\mathrm{div}_M X = \sum_{h=1}^{n+1} \delta_h X^h = \sum_{h=1}^{n+1} (D_h X^h - \nu_h(\nu \bullet DX^h)),$$

where $\nu = (\nu^1, \dots, \nu^{n+1})$ is the unit normal vector to M:

$$\nu = \frac{(-Df, 1)}{\sqrt{1 + |Df|^2}}.$$

Denoting by

$$\vec{H}_\lambda(y, u_\lambda(y)) = Mu_\lambda(y)\,\nu_\lambda(y, u_\lambda(y)), \quad y \in \Omega \tag{4.1}$$

the following formula of integration by parts holds:

$$\int_{M_\lambda} \mathrm{div}_{M_\lambda} X d\mathcal{H}_n = -\int_{M_\lambda} \vec{H}_\lambda \bullet X d\mathcal{H}_n \quad \forall X \in C_0^1(\Omega \times \boldsymbol{R}, \boldsymbol{R}^{n+1}). \tag{4.2}$$

We state the following

THEOREM 4.1. *$\forall X \in C_0^1(\Omega \times \boldsymbol{R}, \boldsymbol{R}^{n+1})$ we have*

$$\lim_{\lambda \to +\infty} \int_{M_\lambda} \mathrm{div}_{M_\lambda} X d\mathcal{H}_n = \int_M \mathrm{div}_M X d\mathcal{H}_n. \tag{4.3}$$

PROOF. Let $\{\lambda_j\}$ be an increasing sequence with $\lim_{j\to+\infty}\lambda_j = +\infty$. We set $M_j = M_{\lambda_j}$, $u_j = u_{\lambda_j}$, $\nu_j = \nu_{\lambda_j}$. By Remark 2.3, we have

$$\lim_{j\to+\infty}\int_\Omega |Du_j(y) - Df(y)|\,dy = 0. \tag{4.4}$$

Recalling that

$$\int_M \operatorname{div}_M X\,d\mathcal{H}_n = \int_\Omega \sum_{h=1}^{n+1}(D_h X^h - \nu^h(\nu \bullet DX^h))\sqrt{1+|Df|^2}\,dy,$$

$$\int_{M_j} \operatorname{div}_{M_j} X\,d\mathcal{H}_n = \int_\Omega \sum_{h=1}^{n+1}(D_h X^h - \nu_j^h(\nu_j \bullet DX^h))\sqrt{1+|Du_j|^2}\,dy,$$

where $X = (X^1, \dots, X^n, X^{n+1})$, $\nu_j = (\nu_j^1, \dots, \nu_j^n, \nu_j^{n+1})$, we obtain

$$\left|\int_{M_j}\operatorname{div}_{M_j} X\,d\mathcal{H}_n - \int_M \operatorname{div}_M X\,d\mathcal{H}_n\right| \le \int_\Omega \left|\sum_{h=1}^{n+1} D_h X^h\right| \left|\sqrt{1+|Du_j|^2} - \sqrt{1+|Df|^2}\right|dy +$$

$$+\int_\Omega \sum_{h=1}^{n+1}\left|(DX^h \bullet \nu_j)\,\nu_j^h\sqrt{1+|Du_j|^2} - (DX^h \bullet \nu)\,\nu^h\sqrt{1+|Df|^2}\right|dy.$$

Now (4.3) easily follows from (4.4).

THEOREM 4.2. *Suppose that* i) *or* ii) *in the beginning of Section* 3 *holds. Then there exists an* $(n+1)$*-dimensional vector valued Radon measure*

$$\vec{\boldsymbol{H}} = (\boldsymbol{H}_1, \boldsymbol{H}_2, \dots, \boldsymbol{H}_{n+1})$$

such that

$$\int_\Omega \operatorname{div}_M X\,d\mathcal{H}_n = -\int_\Omega X \bullet d\vec{\boldsymbol{H}} \quad \forall X \in C_0^1(\Omega \times \boldsymbol{R}, \boldsymbol{R}^{n+1}). \tag{4.5}$$

Such a measure $\vec{\boldsymbol{H}}$ *will be called the mean curvature measure of* M.

PROOF. We have

$$Mu_\lambda(y) = -\lambda(f(y) - u_\lambda(y)) \le 0 \quad \forall y \in \Omega$$

and therefore (4.1) implies that

$$|\vec{H}_\lambda(y, u_\lambda(y))| = -Mu_\lambda(y) \quad \forall y \in \Omega.$$

By the results of Section 3 we may suppose that

$$|Du_\lambda(y)| \le k \quad \forall y \in \Omega \quad \forall \lambda \ge 0. \tag{4.6}$$

Then (4.2) implies that

$$\left|\int_{M_\lambda} \operatorname{div}_{M_\lambda} X d\mathcal{H}_n\right| = \left|\int_{M_\lambda} \vec{H}_\lambda \bullet X d\mathcal{H}_n\right| \leq \|X\|_\infty \int_\Omega |\vec{H}_\lambda| \sqrt{1 + |Du_\lambda|^2} dy \leq$$

$$\leq \|X\|_\infty \sqrt{1+k^2} \int_\Omega |\vec{H}_\lambda| dy \leq \|X\|_\infty \sqrt{1+k^2} \left|\int_\Omega -Mu_\lambda \, dy\right| =$$

$$= \|X\|_\infty \sqrt{1+k^2} \left|\int_{\partial\Omega} Tu_\lambda \bullet \nu_e \, d\mathcal{H}_{n-1}\right| \leq \sqrt{1+k^2}\, \mathcal{H}_{n-1}(\partial\Omega) \|X\|_\infty ,$$

where ν_e is the outer normal to $\partial\Omega$. Then Theorem 4.1 implies:

$$\left|\int_M \operatorname{div}_M X d\mathcal{H}_n\right| \leq \sqrt{1+k^2}\, \mathcal{H}_{n-1}(\partial\Omega) \|X\|_\infty \quad \forall X \in C_0^1(\Omega \times \boldsymbol{R}, \boldsymbol{R}^{n+1}) . \tag{4.7}$$

Theorem 4.2 is then a consequence of the Riesz Representation Theorem.

REMARK 4.3. From (4.2) and Theorems 4.1, 4.2 we immediately obtain that

$$\int_M X \bullet d\vec{\boldsymbol{H}} = \lim_{\lambda \to +\infty} \int_{M_\lambda} \vec{H}_\lambda \bullet X d\mathcal{H}_n \quad \forall X \in C_0^1(\Omega \times \boldsymbol{R}, \boldsymbol{R}^{n+1}). \tag{4.8}$$

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Pervenuta il 14 marzo 2003,
in forma definitiva il 12 aprile 2003.

E. Barozzi:
Dipartimento di Matematica
Università degli Studi di Trento
Via Sommarive, 14 - 38050 POVO TN
barozzi@science.unitn.it

E. Gonzalez:
Dipartimento di Metodi e Modelli Matematici
Università degli Studi di Padova
Via Belzoni, 7 - 35131 PADOVA
gonzalez@dmsa.unipd.it

U. Massari:
Dipartimento di Matematica
Università degli Studi di Ferrara
Via Machiavelli, 35 - 44100 FERRARA
massari@dm.unife.it