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# SOME LIOUVILLE THEOREMS FOR PDE PROBLEMS IN PERIODIC MEDIA

Abstract. — Liouville problems in periodic media (*i.e.* the study of properties of global solutions to PDE) arise both in homogenization and dynamical systems. We discuss some recent results for minimal surfaces and free boundaries.

Key words: Periodic Media; Liouville; Homogenization.

For second order PDE's, specially those of geometric type, like minimal surfaces and free boundary problems, the classification of global solutions is often linked to the local regularity properties of solutions of the equation under consideration.

The most classical example, is perhaps, the classification of minimal cones, and its consequences on the local regularity of minimal surfaces. Similar considerations apply to the regularity theory of free boundaries,singular perturbation problems and one may also argue that the regularity theory of fully non linear equations, depends in some sense of this classification.

In fact, blow up arguments, that reduce the problem of local behavior of solutions to a Liouville type problem, often have the advantage of freeing the discussion of the need to consider the effect of lower order, less important aspects of the problem, and reveal the essential features of the solution.

Although of a more complex nature, another instance of this connection between global and local, arises in the context of homogenization theory.

In that case, we are given a highly oscillatory equation, at a very small (epsilon) scale, and as the scale of the oscillations, epsilon, goes to zero, one tries to determine if the solutions of the epsilon problem, will necessarily converge to a solution of some now translation invariant problem (equation, or set of inequalities).

Perhaps the most classical example is to consider an equation in divergence form (for instance from the calculus of variations)

$$\begin{cases} D_i F_i\left(\nabla u,\ \dfrac{X}{\varepsilon}\right) \\ F_i \text{ periodic in } X \end{cases}$$

and ask if the limit of solutions, $u_\varepsilon$, as epsilon goes to zero, will converge to a solution, $u_0$, of an equation that does not depend on $x$, anymore.

A classical way to approach this problem, is to make the ansatz that the solution, $u_\varepsilon$, will consist of the limiting solution, $u_0$, plus an oscillatory term, $w_\varepsilon$, of period epsilon, whose gradient will compensate for the oscillatory character of the equation. It is natural then to dilate the solution $u_\varepsilon$, by epsilon, homogeneously of degree one:

$$\overline{u}_\varepsilon = \frac{1}{\varepsilon} u_\varepsilon(\varepsilon X)$$

and try to classify the limiting configuration. Since $u_0$ is expected to be smooth, $\overline{u}_0$ will become under such dialations, a linear funtion, $p$ and $w_\varepsilon$ an associated periodic term $w$ of order one.

This periodic term is called a corrector, $p + w$ is a «plane like solution» of the global periodic problem, and its average energy gives us the «effective» energy functional that the limiting solution must minimize. In case the oscillation depends on the independent variables, the theory is extensively developed. Much less so, when our solution consists of a surface (a minimal surface or a free boundary) embedded in a periodic configuration.

This is because periodicity is lost when the surface under consideration must be asymptotically an irrational plane.

Perhaps the most influential results in this case are those due to Moser.

In this context, we would like to discuss issues of existence of global solutions to several problems in periodic media that share common ideas and techniques:

- Plane like minimal surfaces.
- Phase transition problems where the transition takes place on a given strip.
- Free boundary problems of both stationary and of traveling wave type (arising for instance in flame propagation, or the geometry of a capillary drop sitting on a non homogeneous plane).

The common themes of all theses problems are:

- Techniques based on elementary geometric properties of the solutions (uniform density properties of surfaces and sets).
- Regularity of constrained problems.
- Appropriate choice of solutions to obtain the desired invariance (supersolution type methods are used).

We first discuss Moser's problem of constructing **plane like minimal surfaces in periodic media:**

Given a generalized area functional $F(X, \nu)$, periodic in $X$, and a plane $P$, in $R^n$, find a global $F$-minimal surface, $S$, that remains at a fixed distance from $P$.

This problem arises in dynamical systems (foliations of the torus by minimal surfaces), and in homogenization (shape of a capillary drop in a periodic media).

Main Theorem (in collaboration with R de la Llave, CPAM 2001).

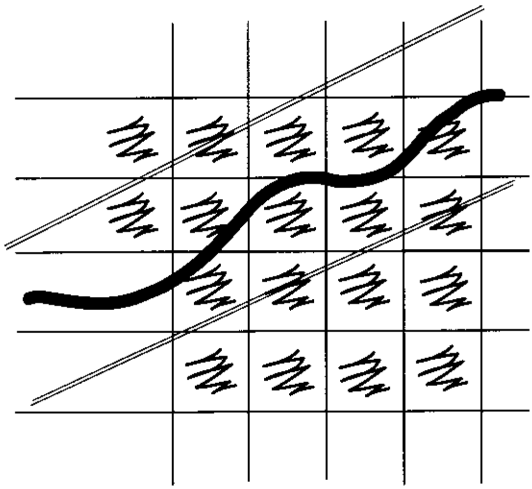

Fig. 1 – Minimal surface stays inside a «strip» in a periodic media.

1. *Indeed such a surface exists.*
2. *The distance to the plain depends only on the ellipticity of F.*
3. *As we slide the plain, the corresponding surfaces laminate the torus.*
4. *The effective area is convex.*

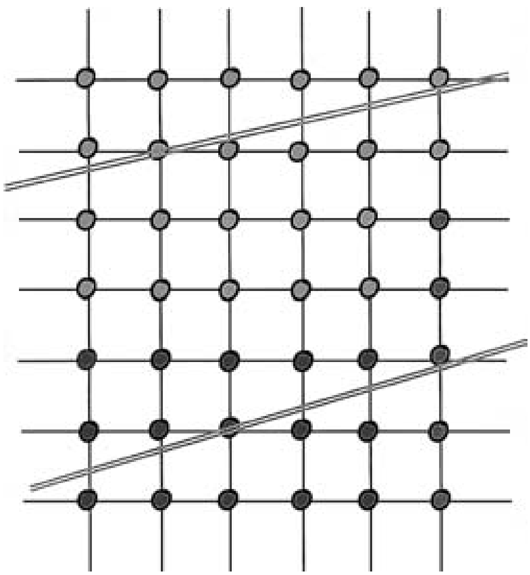

Fig. 2. – A mimimal array of spins, with interphase inside a spin.

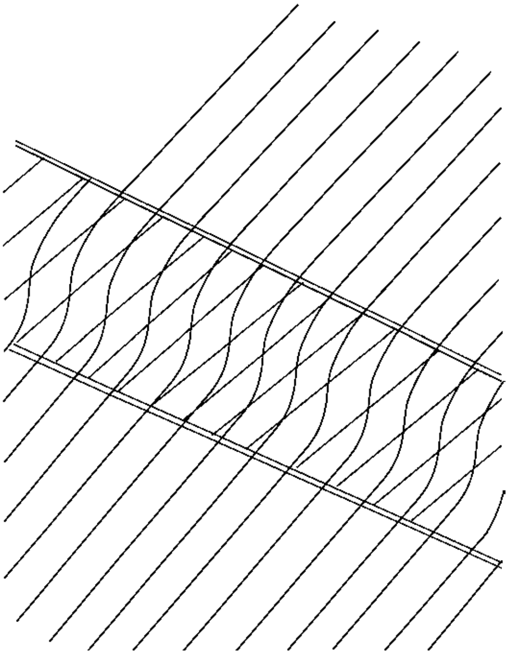

Fig. 3. – $u$ is asymptotically $\pm 1$ and $u = 0$ stays in a strip.

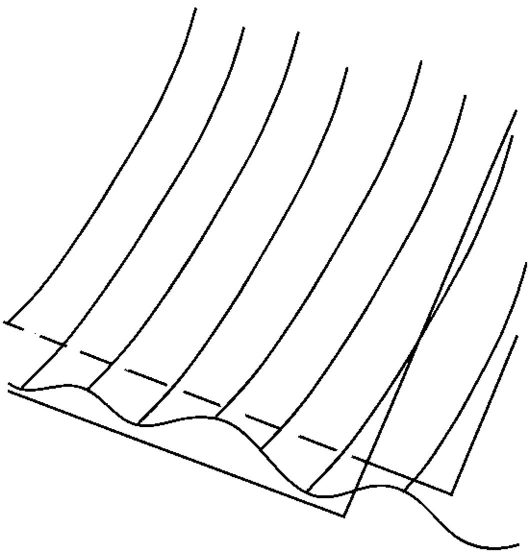

Fig. 4. – $\Delta u = 0$ for $u > 0$; $u_\nu = f(x, \nu)$ periodic in $x$.

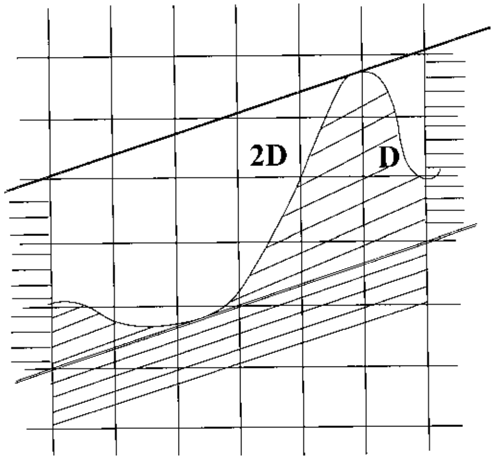


Fig. 5. – Constrained (double obstacle) minimal surface.

For geodesics in the plane, this is due to Morse, for geodesics in $R^3$, this is not true. This shows that codimension one is essential, and will show up in our construction. There is related work by Bangert.

**Phase transition** (in collaboration with R de la Llave):

The typical, simplest, problem would be, given the lattice, $Z^n$, we have arrays of «spins», *i.e.*, functions, $u$, that take values 1 or $-1$.

We define an energy, based on near neighborhood interactions:

$$E = \sum_{\|x_i - x_j\| \leq 1} |u(x_i - x_j)| .$$

Given a hyperplane $P$, find a local minimizer, in $R$, that changes phase at a finite distance from $P$.

(The theorem is true under very general interaction rules, as long as they preserve order in a strict way (maximum principle).)

A bridge between the two problems is the construction of **solutions to periodic Landau Ginzburg type equations** (E. Valdinocci):

Given a hyperplane, $P$, we can construct a solution of the periodic Landau Ginzburg equation, div $u$ a local minimizer of

$$\int (\nabla u)^T a_{ij}(x) \nabla u + F(u, x)$$

with periodic $X$ dependance whose zero level surface stays at a finite distance from $P$. Finally, for **free boundary problems**, we have a **stationary**, and a **dynamic** version.

In the stationary, the free boundary problem under consideration would be: (work in collaboration with Ki Ahm Lee) Given a hyperplane, $P$, in $R$, find a non negative function $u$, harmonic in $\Omega = \{X/u > 0\}$ that satisfies, along the free boundary, $\partial\Omega$,

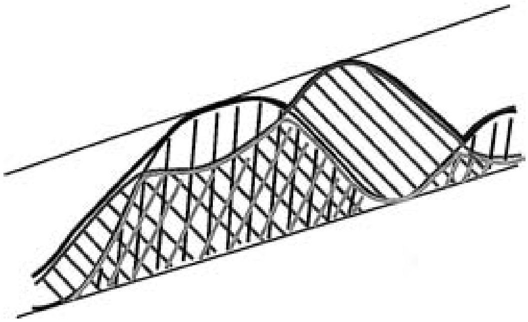

Fig. 6. – $F$-Area $(\partial(D_1 \cup D_2))$ + $F$-Area $(\partial(D_1 \cap D)2))$ $\leq$ $F$-Area $(\partial D_1)$ + $F$-Area $(\partial D_2)$.

the extra condition:

$$u_\nu = f(X)\,, \qquad f \text{ periodic}$$

and that stays at a finite distance from $P$.

This problem is related to flame propagation, or the contact angle condition when a drop sits on a plane with periodic structure.

At infinity, in $\Omega$, $u$ has linear behavior, that supposedly describes the effective free boundary condition (contact angle).

Finally, the evolution case corresponds to global traveling wave solution to the associated singular perturbation problem,and its free boundary limit This is work in collaboration with K.A. Lee and A. Mellet and in the construction, we use ideas from the work of Berestycki and Hamel.

Given the singular perturbation equation

$$\Delta u - u_t = q\left(\frac{X}{\varepsilon}\right) \nabla u \to \beta_\delta\left(u,\, \frac{X}{\varepsilon}\right)$$

there exists a solution, unique up to time translation, trapped between the two «traveling wave» type profiles:

$$[1 - e^{-\langle \nu,\, X\rangle + B_i}]\,.$$

We would like now to discuss the main steps in the construction of plane like minimal surfaces:
general area functional is of the form

$$\int_S F(x,\, \nu)\, dH^{n-1}$$

where $F(x, \nu)$, when extended in $\nu$, as a homogeneous function of degree one, becomes a convex cone, bounded by above and below by multiples of $|\nu|$;

$$A|\nu| \leq F(x,\, \nu) \leq B|\nu|$$

$A$ and $B$ are the ellipticity constants referred to above

We will work in the context of boundaries of sets of finite perimeter: this has the effect of ordering the surfaces, by inclusion of the sets of which they are boundaries, and allow us to treat them almost as graphs.

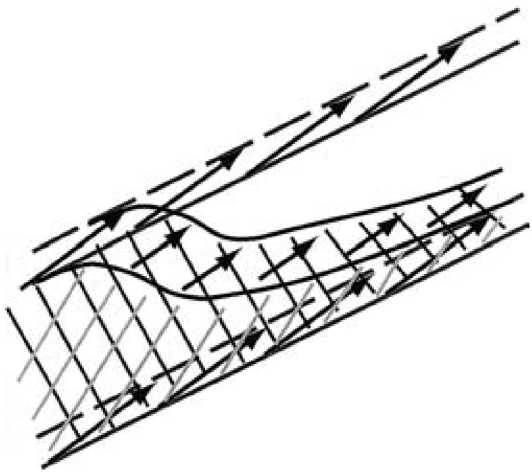

Fig. 7. – The integer translation by $v$ raises the strip, thus, $D + \vec{v} \supset D$.

$F(x, \nu)$ will be uniformly elliptic and periodic in $X$. It could be, for instance the differential of area induced by a Riemann metric on the torus.

$$g_{i,j}(X)\, \nu_i \nu_j.$$

The main steps in our construction are the following: When the slope of the plane under consideration is rational,one expects a periodic surface.

Thus, given a rational plane, we construct a periodic surface, $S = \partial D$, of minimal «area» among those trapped between two parallel planes (we solve a double obstacle problem).

Next, we note that if two different sets, $D_1$ and $D_2$, have boundaries that are minimizers, so do the intersection, $D_1 \cap D_2$ and the union, $D_1 \cup D_2$.

We thus can talk of the minimal minimizer, *i.e.*, the boundary, $S$ of the intersection of all minimizers, (the least supersolution). This boundary has an important monotonicity property, the Birkhoff property, that establishes that if we translate the whole configuration by an integer vector $\tau$ that «raises» the strip constraining $S$, $D$ is then contained in its own translation.

This is due to the fact that $D_\tau \cup D$ is admissible for the translated strip, and $D\tau \cap D$ is admissible for the original strip, so they must coincide with $D_\tau$ and $D$, respectively.

We contend now, that if we can find just one «clean» period, in the «middle» of the strip, contained in $\mathcal{C}D$, all integer translations $\vec{v}$, as above, will map this period inside $\mathcal{C}D$:

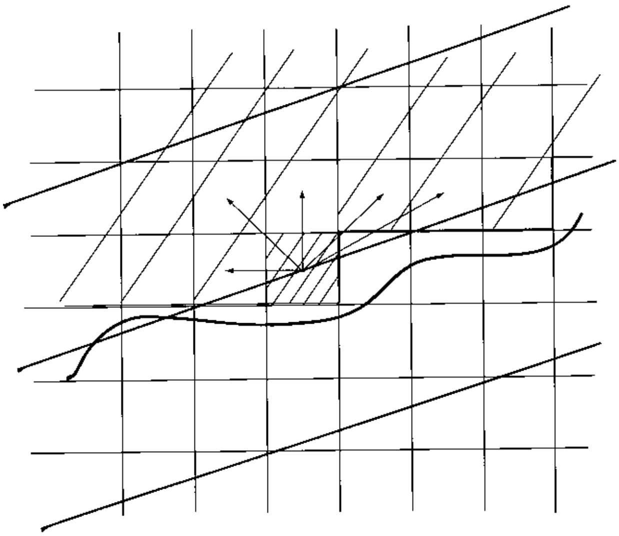

Fig. 8. – The $\vec{v}$ translations, map $Q$ into $\mathcal{C}D$.

and thus $D$ stays away from the upper constraint. That will automatically make it an unconstrained minimizer, since we may then translate it up one unit, and the translated surface would be a minimizer that does not touch neither the top nor the bottom.

The remaining fact, that there is at least one clean cube, is the combination of two properties. The uniform density of minimal surfaces:

If a ball, $B$, is centered on the surface, $S$, then the area of $S \cap B$ is proportional to the area of $B$.

And a simple counting argument: how many cubes can touch $S$?

If we take a strip with base a large cube of sides $L$, and height, $M$, such that $M$ cubes are stacked on top of each other, since the bottom of the strip is always a comparison surface, the total area of $S$ must be less that $C_1 L^{n-1}$, where $C_1$ is a constant that depends only on ellipticity.

On the other hand, if $S$ touches all cubes in the central third of the strip, the total area of $S$, is at least, $C_2 L^{n-1} M$, where $C$, depends only on the area estimate above, that in turns, depends itself on ellipticity. We have a contradiction, if $L$ is large enough (depending only on ellipticity and dimension).

This shows that some cube in the middle strip must be disjoint with $S$. But if such a cube is contained in $D$, the bottom third of the strip will be contained in $D$. This is impossible, because then, we could lower $S$ by one unit, contradicting minimality.

Thus the top third of the strip is contained in $\mathcal{C}D$. Still, $S$ may touch the bottom of the strip, but now, we can translate it one unit up, without touching the top. Thus, this translation is still a constrained minimizer, but it stays away from the top and bottom of the strip. Thus it is a local, unconstrained minimizer.

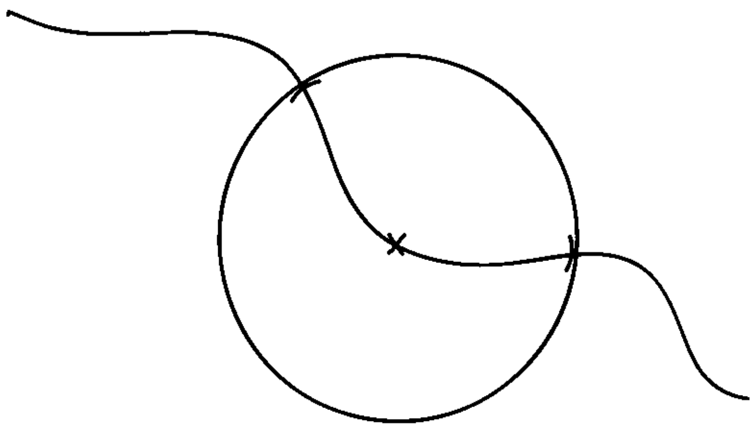

Fig. 9. – $C_2 \times$ Number of intersected cubes $\leq F$-Area $S \leq F$-Area of Bottom $\leq C_1 L^{n-1}$.

To close, two interesting directions we are pursuing

**Homogenization limits:** To show that actually the effective area or free boundary condition defined by these global solutions is the one that determines the limiting configuration in a homogenization process is a highly non trivial matter mainly because of the very week continuity properties of the effective parameters. In the case of free boundary problems, like flame propagation, or of the contact angle condition for a capillary drop, the limits can be completely different if one looks at solutions constructed variationally (the limit is in this case trivial) or as least supersolution. The limit in this case is very complex, it satisfies free boundary inequalities, and is the relevant one for evolution problems.

The second direction concerns the fact that our construction does not really require the existence of a dense family of «rational» foliations.

For instance, an interesting case is the following media in $R$:

We have the standard lattice, $Z$

In the horizontal directions we have invariance under integer translations, but translations in the vertical direction are coupled with a horizontal deformation given by the transformation

$$(x, y, z) \to (x', y', z') \text{ with } z^1 = z + 1 \,, \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In this case, invariant surfaces are of the form:

$$S_1 = \{x_3 = C\}\,, \qquad S_2 = \{x_1 = A\lambda^{x_3}\}\,, \qquad S_3 = \{x_2 = B\lambda^{-x_3}\}$$

(with $\lambda$, $-\lambda^{-1}$, the eigenvalues of the matrix $M$, above and $x_1$, $x_2$ coordinates on the eigenvectors.

We then construct (in collaboration with A. Candel and R. de la Llave) «plane like minimal surfaces» around each such «plane» by basically the same method. The main difference is that periodic approximations must be substituted by monotone solutions of «Dirichlet like » problems in increasing domains.

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