BERNARD DACOROGNA, NICOLA FUSCO, LUC TARTAR

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Accademia Nazionale dei Lincei

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ON THE SOLVABILITY OF THE EQUATION $\text{div} \; u = f$ IN $L^1$ AND IN $C^0$

Abstract. — We show that the equation $\text{div} \; u = f$ has, in general, no Lipschitz (respectively $W^{1,1}$) solution if $f$ is $C^0$ (respectively $L^1$).

Key words: Divergence; Lorentz spaces; Sobolev imbedding theorem.

1. Introduction

Consider a bounded open set $\Omega \subset \mathbb{R}^n$ and a vector field $u : \Omega \rightarrow \mathbb{R}^n$. Define the linear operator $L : X \rightarrow Y$ by

$$Lu(x) = \text{div} \; u = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}.$$ 

Usually this operator is coupled with some boundary conditions but we will be concerned here only with a local problem of regularity. It is well known that this operator is onto (as a direct consequence of classical regularity results on Laplace equation) in the following cases

$$X = C^{k+1,\alpha}, \; Y = C^{k,\alpha} \quad \text{with} \; k \geq 0 \; \text{and} \; 0 < \alpha < 1$$

$$X = W^{k+1,p}, \; Y = W^{k,p} \quad \text{with} \; k \geq 0 \; \text{and} \; 1 < p < \infty.$$ 

The aim of this report is to show that this operator is not onto when

$$X = C^1 \; \text{(or} \; W^{1,\infty}), \; Y = C^0$$

$$X = W^{1,1}, \; Y = L^1.$$ 

It may seem that this follows at once from the known counterexamples for Laplace equation; this is not the case because the equation $\text{div} \; u = f$ has other solutions than the one of the form $u = \text{grad} \; v$.

After having solved the problem we have learnt that both questions have already been studied by several authors. The result concerning $C^0$ and $L^\infty$ have been proved (to the best of our knowledge) by Preiss [6], Mc Mullen [4] and have been announced by Bourgain and Brézis in [2], who also mention the case of $L^1$. 
2. The $L^1$ Case: A First Approach

Let

$$\psi(x_1, x_2) = x_1 x_2 V(|x|)$$

then

$$\psi_{x_1 x_1} = \frac{x_1^3 x_2}{|x|^2} V''(|x|) + \frac{x_1 x_2}{|x|^3} (2x_1^2 + 3x_2^2) V'(|x|)$$

$$\psi_{x_2 x_2} = \frac{x_2^3 x_1}{|x|^2} V''(|x|) + \frac{x_1 x_2}{|x|^3} (2x_2^2 + 3x_1^2) V'(|x|)$$

$$\psi_{x_1 x_2} = \frac{x_1^2 x_2^2}{|x|^2} V''(|x|) + \frac{x_1^4 + x_1^2 x_2^2 + x_2^4}{|x|^3} V'(|x|) + V(|x|).$$

Choosing $\Omega = \{ x \in \mathbb{R}^2 : |x| < 1/2 \}$ and for $0 < \alpha < 1,$

$$V(r) = \log r^\alpha$$

we get that

$$\psi_{x_1 x_1}, \psi_{x_2 x_2} \in C^0(\Omega), \quad \psi_{x_1 x_2} \notin L^\infty(\Omega).$$

Let $\eta \in C_0^\infty(\Omega)$ and $\eta \equiv 1$ near $|x| = 0.$ Define for $N$ an integer

$$\psi_N(x) = \eta(x) x_1 x_2 V\left(\frac{1}{N} + |x|\right).$$

Observe that $\psi_N \in C_0^\infty(\Omega)$ and there exists a constant $c_1$ independent of $N$ so that

$$|\psi_{N x_1 x_1}|_{L^\infty} + |\psi_{N x_2 x_2}|_{L^\infty} \leq c_1.$$

We have furthermore, for $u = (u^1, u^2) \in W^{1,1}(\Omega; \mathbb{R}^2),$ that

$$\int\int_{\Omega} (u_{x_1}^1 + u_{x_2}^2) \psi_{N x_3 x_2}^N dx_1 dx_2 = \int\int_{\Omega} (u_{x_2}^1 \psi_{N x_1 x_1}^N + u_{x_1}^2 \psi_{N x_2 x_2}^N) dx_1 dx_2$$

and thus there exists a constant $c_2 = c_2(c_1, |u|_{W^{1,1}})$ independent of $N$ so that

$$\left| \int\int_{\Omega} \text{div} u \psi_{N x_1 x_2}^N dx_1 dx_2 \right| \leq c_2.$$
– However if we choose
\[ f(x) = \frac{1}{|x|^2 \log |x|^{\alpha + 1}} \]
we get that \( f \in L^1(\Omega) \) and, for \( \psi^N \) as above, we get by Fatou lemma that
\[
\lim_{N \to \infty} \left| \int_{\Omega} f \psi^N_{x_1,x_2} \, dx_1 \, dx_2 \right| = \infty.
\]
The combination of the above facts leads to the desired conclusion.

3. The \( L^1 \) case: A second approach

We start by recalling the definition of Lorentz spaces.
Let \( u : \Omega \subset \mathbb{R}^n \to \mathbb{R} \) (\( \Omega \) a bounded open set) be measurable; we then define the distribution function by
\[
\lambda(s) = \text{meas} \{ x \in \Omega : |u(x)| \geq s \}
\]
and the decreasing rearrangement of \( u \) by
\[
u^*(t) = \inf \{ s : \lambda(s) < t \}, \quad t \in [0, |\Omega|].
\]
If \( 1 \leq p, q < \infty \) we define the Lorentz space \( L^{p,q}(\Omega) \) to be the space of \( u \) such that
\[
|u|_{L^{p,q}} = \left( \frac{|\Omega|}{\int_0^{|\Omega|} \nu^*(t)^q \, t^{\frac{q}{p}-1} \, dt} \right)^{1/q} < \infty
\]
and if \( q = \infty \)
\[
|u|_{L^{p,\infty}} = \text{ess sup} \left[ u^*(t)^{\frac{1}{p}} \right] < \infty.
\]
In particular \( L^{p,p} \) can be identified with \( L^p \).
We now give an example that will be used below.

**Proposition 1.** Let \( \Omega = \{ x \in \mathbb{R}^n : 0 < |x| < 1/2 \} \). Let \( \eta \in C^\infty(0, 1/2) \) be such that
\[
\eta(t) = \begin{cases} 1 & \text{near } t = 0 \\ 0 & \text{near } t = 1/2. \end{cases}
\]
Let
\[ V(r) = -\int_{1/2}^{r} \frac{\varrho^{1-n}}{\log \varrho} \, d\varrho, \quad 0 < r < 1/2 \]
\[ \varphi(x) = \eta(|x|) \, V(|x|) \]
(note that, when \( n = 2 \), \( V(|x|) = \log |\log |x|| - \log \log 2 \)). Call
\[ f(x) = \Delta \varphi(x) = \begin{cases} (|x|^2 \log |x|)^{-1} & \text{near } |x| = 0 \\ 0 & \text{near } |x| = 1/2. \end{cases} \]
Then \( f \in L^1(\Omega) \) and \( \varphi \) solves, in the sense of distributions,
\[
\begin{cases}
\Delta \varphi = f & \text{in } \Omega \\
\varphi = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(1)

Note however that \( \nabla \varphi \notin L^{\frac{n}{n-1}, 1}(\Omega) \) and that, in the case \( n = 2 \), \( \varphi \notin L^{\infty}(\Omega) \).

Remark 2. More refined examples show that solutions of (1) have their gradients in \( L^{\frac{n}{n-1}, 1}(\Omega) \) but not in \( L^{\frac{n}{n-1}, q}(\Omega) \) for every \( q < \infty \).

Proof. Clearly \( f \in L^1(\Omega) \) and \( \varphi \notin L^{\infty}(\Omega) \) when \( n = 2 \). We therefore only check that \( \nabla \varphi \notin L^{\frac{n}{n-1}, 1}(\Omega) \). We have
\[
\nabla \varphi(x) = \left[ \eta(|x|) \, V'(|x|) + \eta'(|x|) \, V(|x|) \right] \frac{x}{|x|}
\]
and hence the result will follow if we can show that \( \psi \notin L^{\frac{n}{n-1}, 1}(0, r_0) \), for \( r_0 > 0 \) sufficiently small, where \( (\omega_n \text{ denoting the measure of the unit ball}) \)
\[
\psi(t) = V'\left(\left(\frac{t}{\omega_n}\right)^{1/n}\right) = \frac{\left(\frac{t}{\omega_n}\right)^{\frac{1-n}{n}}}{\frac{1}{n} \log \left(\frac{t}{\omega_n}\right)}.\]

We therefore have
\[
|\psi|_{L^{\frac{n}{n-1}, 1}} = \int_0^{r_0} |\psi(t)| t^{-\frac{1}{n}} \, dt = n(\omega_n)^{\frac{n-1}{n}} \int_0^{r_0} \frac{dt}{t(\log t - \log \omega_n)} = \infty. \quad \Box
\]

The combination of the preceding counterexample and the following proposition gives the result for the \( L^1 \) case.

Proposition 3. Let \( \Omega \subset \mathbb{R}^n \) be the unit ball and let \( u \in W^{1,1}(\Omega; \mathbb{R}^n) \). Then there exists a solution, in the sense of distributions, of
\[
\begin{cases}
\Delta \varphi = \text{div } u & \text{in } \Omega \\
\varphi = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(2)
Furthermore $\nabla \varphi \in L^{\frac{n}{n-1}}(\Omega)$ and hence in particular, when $n = 2$, $\varphi$ is continuous.

**Proof.** We just sketch the main ingredients of the proof.

– The first fact is that a more refined version of the Sobolev imbedding theorem gives that $u \in W^{1,1}$ implies $u \in L^{\frac{n}{n-1},1}$, cf. [9].

– Using the Green function $G = G(x, y)$ (cf. [3]) and applying the divergence theorem we can write the solution in terms of singular integrals, namely

$$
\varphi(x) = \int_{\Omega} \text{div } u(y) \, G(x, y) \, dy = - \int_{\Omega} \langle u(y); \nabla G(x, y) \rangle \, dy.
$$

– The estimate on the gradient can be obtained as follows. Let $Tu = \nabla \varphi = (\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n})$. Standard results on singular integrals (cf. [7, Theorem 3, Chapter II, p. 39]), show that for every $1 < p < \infty$ we can find a constant $c_p$ such that

$$
|Tu|_{L^p} \leq c_p |u|_{L^p}.
$$

– Since $u \in L^{\frac{n}{n-1},1}$ we can use Marcinkiewicz interpolation theorem (see Theorem 5.3.2 in [1, p. 113] or Theorem 3.15 of Chapter V in [8, p. 197]) to find a constant $c_{n/(n-1)}$ such that

$$
|\nabla \varphi|_{L^{\frac{n}{n-1},1}} = |Tu|_{L^{\frac{n}{n-1},1}} \leq c_{n/(n-1)} |u|_{L^{\frac{n}{n-1},1}}.
$$

The result then follows. □

**Remark 4.** It is interesting to compare the two arguments that have been used in this section and in the preceding one.

The second method only uses the fact that $W^{1,1} \subset L^{\frac{n}{n-1},1}$ and shows that not all functions of $L^1$ are divergences of functions in $L^{\frac{n}{n-1},1}$. It essentially uses the convolution by the elementary solution of the Laplacian, which has a singularity of the form $r^{2-n}$ (or $\log r$ if $n = 2$). One easily generalizes this fact. Note first that if $a$ has derivatives that belong to $L^{n,\infty}$ (so that $a \in \text{BMO}$) then $\text{div } u \ast a = \sum u^j \ast a_{x_j}$ (after truncation) is continuous. However $f \ast a$ cannot be continuous for all $f \in L^1$ unless $a$ is bounded.

The first method uses a larger class of functions $a$ (the $\psi_{x_j,x_0}$ of the counterexample), those that satisfy $a_{x_j} \in W^{-1,\infty}$ for all $j$ (note that since $W^{1,1}_0$ is dense in $L^{\frac{n}{n-1},1}$ we have $L^{n,\infty} \subset W^{-1,\infty}$). Indeed if $f = \text{div } u$ with $u \in W^{1,1}_0$ then $\langle f; a \rangle = - \sum \langle u^j; a_{x_j} \rangle$ is well defined.

4. The continuous case

We recall an example due to Ornstein [5] (Mc Mullen uses the more abstract version of Ornstein theorem to prove his result).
Let $N \in \mathbb{N}$ and $\Omega = (0, 1)^2$ then there exists $\psi^N = \psi^N(x_1, x_2) \in C_0^\infty(\Omega)$ such that

$$
|\psi_{x_1}^N|_{L^1} + |\psi_{x_2}^N|_{L^1} = 1
$$

$$
N(|\psi_{x_1}^N|_{L^1} + |\psi_{x_2}^N|_{L^1}) \leq |\psi_{x_1}^N|_{L^1}.
$$

Note that, for $u = (u^1, u^2) \in W^{1, \infty}(\Omega; \mathbb{R}^2)$,

$$
\iint_{\Omega} (u^1_{x_1} + u^2_{x_2}) \psi_{x_1}^N dx_1 dx_2 = \iint_{\Omega} (u^1_{x_2} \psi_{x_1}^N + u^2_{x_1} \psi_{x_2}^N) dx_1 dx_2
$$

and hence

$$
\left| \iint_{\Omega} \text{div} u \psi_{x_1}^N dx_1 dx_2 \right| \leq |u|_{W^{1, \infty}}.
$$

Since $\lim_{N \to \infty} |\psi_{x_1}^N|_{L^1} = \infty$, using Banach-Steinhaus we can find $f \in C^0$ such that

$$
\lim_{N \to \infty} \left| \iint_{\Omega} f \psi_{x_1}^N dx_1 dx_2 \right| = \infty.
$$

Combining the above facts we have even shown that there is $f \in C^0$ such that no vector field $u \in W^{1, \infty}(\Omega; \mathbb{R}^2)$ can satisfy $\text{div} u = f$.

Of course this result immediately extends to higher dimensions.

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**References**


B. Dacorogna:
Institut de mathématiques, EPFL
1015 LAUSANNE (Svizzera)
bernard.dacorogna@dma.epfl.ch

N. Fusco:
Dipartimento di Matematica e Applicazioni «R. Caccioppoli»
Università degli Studi di Napoli «Federico II»
Complesso Universitario Monte S. Angelo
Via Cintia - 80126 NAPOLI
n.fusco@unina.it

L. Tartar:
Department of Mathematical Sciences
Carnegie Mellon University
PITTSBURGH, PA 15213 (USA)
tartar@andrew.cmu.edu