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ON THE SOLVABILITY OF THE EQUATION $\operatorname{div} u = f$ IN L^1 AND IN C^0

Abstract. — We show that the equation $\operatorname{div} u = f$ has, in general, no Lipschitz (respectively $W^{1,1}$) solution if f is C^0 (respectively L^1).

Key words: Divergence; Lorentz spaces; Sobolev imbedding theorem.

1. Introduction

Consider a bounded open set $\Omega \subset \mathbb{R}^n$ and a vector field $u : \Omega \to \mathbb{R}^n$. Define the linear operator $L : X \to Y$ by

$$Lu(x) = \operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}\,.$$

Usually this operator is coupled with some boundary conditions but we will be concerned here only with a local problem of regularity. It is well known that this operator is onto (as a direct consequence of classical regularity results on Laplace equation) in the following cases

$$X = C^{k+1,\alpha},\; Y = C^{k,\alpha} \text{ with } k \geq 0 \text{ and } 0 < \alpha < 1$$

$$X = W^{k+1,p},\; Y = W^{k,p} \text{ with } k \geq 0 \text{ and } 1 < p < \infty\,.$$

The aim of this report is to show that this operator is *not* onto when

$$X = C^1 \;(\text{or } W^{1,\infty}),\; Y = C^0$$

$$X = W^{1,1},\; Y = L^1.$$

It may seem that this follows at once from the known counterexamples for Laplace equation; this is not the case because the equation $\operatorname{div} u = f$ has other solutions than the one of the form $u = \operatorname{grad} v$.

After having solved the problem we have learnt that both questions have already been studied by several authors. The result concerning C^0 and L^∞ have been proved (to the best of our knowledge) by Preiss [6], Mc Mullen [4] and have been announced by Bourgain and Brézis in [2], who also mention the case of L^1.

2. The L^1 case: a first approach

Let

$$\psi(x_1, x_2) = x_1 x_2 V(|x|)$$

then

$$\psi_{x_1 x_1} = \frac{x_1^3 x_2}{|x|^2} V''(|x|) + \frac{x_1 x_2}{|x|^3}(2x_1^2 + 3x_2^2)\, V'(|x|)$$

$$\psi_{x_2 x_2} = \frac{x_2^3 x_1}{|x|^2} V''(|x|) + \frac{x_1 x_2}{|x|^3}(2x_2^2 + 3x_1^2)\, V'(|x|)$$

$$\psi_{x_1 x_2} = \frac{x_1^2 x_2^2}{|x|^2} V''(|x|) + \frac{x_1^4 + x_1^2 x_2^2 + x_2^4}{|x|^3} V'(|x|) + V(|x|).$$

Choosing $\Omega = \{x \in \mathbb{R}^2 \colon |x| < 1/2\}$ and for $0 < \alpha < 1$,

$$V(r) = |\log r|^\alpha$$

we get that

$$\psi_{x_1 x_1}, \psi_{x_2 x_2} \in C^0(\Omega), \quad \psi_{x_1 x_2} \notin L^\infty(\Omega).$$

– Let $\eta \in C_0^\infty(\Omega)$ and $\eta \equiv 1$ near $|x| = 0$. Define for N an integer

$$\psi^N(x) = \eta(x)\, x_1 x_2 V\left(\frac{1}{N} + |x|\right).$$

Observe that $\psi^N \in C_0^\infty(\Omega)$ and there exists a constant c_1 independent of N so that

$$|\psi^N_{x_1 x_1}|_{L^\infty} + |\psi^N_{x_2 x_2}|_{L^\infty} \leq c_1.$$

– We have furthermore, for $u = (u^1, u^2) \in W^{1,1}(\Omega; \mathbb{R}^2)$, that

$$\iint_\Omega (u^1_{x_1} + u^2_{x_2})\, \psi^N_{x_1 x_2}\, dx_1\, dx_2 = \iint_\Omega (u^1_{x_2} \psi^N_{x_1 x_1} + u^2_{x_1} \psi^N_{x_2 x_2})\, dx_1\, dx_2$$

and thus there exists a constant $c_2 = c_2(c_1, |u|_{W^{1,1}})$ independent of N so that

$$\left| \iint_\Omega \operatorname{div} u \psi^N_{x_1 x_2}\, dx_1\, dx_2 \right| \leq c_2.$$

– However if we choose

$$f(x) = \frac{1}{|x|^2 \, |\log |x||^{\alpha+1}}$$

we get that $f \in L^1(\Omega)$ and, for ψ^N as above, we get by Fatou lemma that

$$\lim_{N \to \infty} \left| \iint_{\Omega} f \psi^N_{x_1 x_2} \, dx_1 \, dx_2 \right| = \infty .$$

The combination of the above facts leads to the desired conclusion.

3. The L^1 case: a second approach

We start by recalling the definition of Lorentz spaces.

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ (Ω a bounded open set) be measurable; we then define the distribution function by

$$\lambda(s) = \operatorname{meas}\{x \in \Omega : |u(x)| \geq s\}$$

and the decreasing rearrangement of u by

$$u^*(t) = \inf\{s : \lambda(s) < t\}, \quad t \in [0, |\Omega|] .$$

If $1 \leq p,\ q < \infty$ we define the Lorentz space $L^{p,q}(\Omega)$ to be the space of u such that

$$|u|_{L^{p,q}} = \left(\int_0^{|\Omega|} u^*(t)^q \, t^{\frac{q}{p} - 1} \, dt \right)^{1/q} < \infty$$

and if $q = \infty$

$$|u|_{L^{p,\infty}} = \operatorname{ess\,sup}\left[u^*(t) \, t^{\frac{1}{p}} \right] < \infty .$$

In particular $L^{p,p}$ can be identified with L^p.

We now give an example that will be used below.

Proposition 1. *Let $\Omega = \{x \in \mathbb{R}^n : 0 < |x| < 1/2\}$. Let $\eta \in C^\infty(0, 1/2)$ be such that*

$$\eta(t) = \begin{cases} 1 & \textit{near } t = 0 \\ 0 & \textit{near } t = 1/2 . \end{cases}$$

Let

$$V(r) = -\int_{1/2}^{r} \frac{\varrho^{1-n}}{\log \varrho} d\varrho, \quad 0 < r < 1/2$$

$$\varphi(x) = \eta(|x|)\, V(|x|)$$

(*note that, when* $n = 2$, $V(|x|) = \log |\log |x|| - \log \log 2$). *Call*

$$f(x) = \Delta\varphi(x) = \begin{cases} (|x|^n \log^2 |x|)^{-1} & \textit{near } |x| = 0 \\ 0 & \textit{near } |x| = 1/2 \,. \end{cases}$$

Then $f \in L^1(\Omega)$ *and* φ *solves, in the sense of distributions,*

$$\begin{cases} \Delta\varphi = f & \textit{in } \Omega \\ \varphi = 0 & \textit{on } \partial\Omega \,. \end{cases} \tag{1}$$

Note however that $\nabla\varphi \notin L^{\frac{n}{n-1},1}(\Omega)$ *and that, in the case* $n = 2$, $\varphi \notin L^\infty(\Omega)$.

Remark 2. *More refined examples show that solutions of* (1) *have their gradients in* $L^{\frac{n}{n-1},\infty}(\Omega)$ *but not in* $L^{\frac{n}{n-1},q}(\Omega)$ *for every* $q < \infty$.

Proof. Clearly $f \in L^1(\Omega)$ and $\varphi \notin L^\infty(\Omega)$ when $n = 2$. We therefore only check that $\nabla\varphi \notin L^{\frac{n}{n-1},1}(\Omega)$. We have

$$\nabla\varphi(x) = [\eta(|x|)\, V'(|x|) + \eta'(|x|)\, V(|x|)] \frac{x}{|x|}$$

and hence the result will follow if we can show that $\psi \notin L^{\frac{n}{n-1},1}(0, r_0)$, for $r_0 > 0$ sufficiently small, where (ω_n denoting the measure of the unit ball)

$$\psi(t) = V'\left(\left(\frac{t}{\omega_n}\right)^{1/n}\right) = \frac{\left(\frac{t}{\omega_n}\right)^{\frac{1-n}{n}}}{\frac{1}{n}\log\left(\frac{t}{\omega_n}\right)} \,.$$

We therefore have

$$|\psi|_{L^{\frac{n}{n-1},1}} \equiv \int_0^{r_0} |\psi(t)|\, t^{-\frac{1}{n}}\, dt = n(\omega_n)^{\frac{n-1}{n}} \int_0^{r_0} \frac{dt}{t(\log t - \log \omega_n)} = \infty \,. \qquad \square$$

The combination of the preceding counterexample and the following proposition gives the result for the L^1 case.

Proposition 3. *Let* $\Omega \subset \mathbb{R}^n$ *be the unit ball and let* $u \in W^{1,1}(\Omega; \mathbb{R}^n)$. *Then there exists a solution, in the sense of distributions, of*

$$\begin{cases} \Delta\varphi = \operatorname{div} u & \textit{in } \Omega \\ \varphi = 0 & \textit{on } \partial\Omega \,. \end{cases} \tag{2}$$

Furthermore $\nabla\varphi \in L^{\frac{n}{n-1},1}(\Omega)$ *and hence in particular, when* $n = 2$, φ *is continuous.*

PROOF. We just sketch the main ingredients of the proof.

– The first fact is that a more refined version of the Sobolev imbedding theorem gives that $u \in W^{1,1}$ implies $u \in L^{\frac{n}{n-1},1}$, cf. [9].

– Using the Green function $G = G(x, y)$ (cf. [3]) and applying the divergence theorem we can write the solution in terms of singular integrals, namely

$$\varphi(x) = \int_\Omega \operatorname{div} u(y)\, G(x, y)\, dy = -\int_\Omega \langle u(y); \nabla_y G(x, y)\rangle\, dy .$$

– The estimate on the gradient can be obtained as follows. Let $Tu = \nabla\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \dots, \frac{\partial\varphi}{\partial x_n}\right)$. Standard results on singular integrals (cf. [7, Theorem 3, Chapter II, p. 39]), show that for every $1 < p < \infty$ we can find a constant c_p such that

$$|Tu|_{L^p} \leqslant c_p\, |u|_{L^p}.$$

– Since $u \in L^{\frac{n}{n-1},1}$ we can use Marcinkiewicz interpolation theorem (see Theorem 5.3.2 in [1, p. 113] or Theorem 3.15 of Chapter V in [8, p. 197]) to find a constant $c'_{n/(n-1)}$ such that

$$|\nabla\varphi|_{L^{\frac{n}{n-1},1}} = |Tu|_{L^{\frac{n}{n-1},1}} \leqslant c'_{\frac{n}{n-1}}\, |u|_{L^{\frac{n}{n-1},1}}.$$

The result then follows. □

REMARK 4. *It is interesting to compare the two arguments that have been used in this section and in the preceding one.*

The second method only uses the fact that $W^{1,1} \subset L^{\frac{n}{n-1},1}$ *and shows that not all functions of* L^1 *are divergences of functions in* $L^{\frac{n}{n-1},1}$. *It essentially uses the convolution by the elementary solution of the Laplacian, which has a singularity of the form* r^{2-n} (*or* $\log r$ *if* $n = 2$). *One easily generalizes this fact. Note first that if a has derivatives that belong to* $L^{n,\infty}$ (*so that* $a \in BMO$) *then* $\operatorname{div} u * a = \sum u^j * a_{x_j}$ (*after truncation*) *is continuous. However* $f * a$ *cannot be continuous for all* $f \in L^1$ *unless a is bounded.*

The first method uses a larger class of functions a (*the* $\psi_{x_1x_2}$ *of the counterexample*), *those that satisfy* $a_{x_j} \in W^{-1,\infty}$ *for all j* (*note that since* $W_0^{1,1}$ *is dense in* $L^{\frac{n}{n-1},1}$ *we have* $L^{n,\infty} \subset W^{-1,\infty}$). *Indeed if* $f = \operatorname{div} u$ *with* $u \in W_0^{1,1}$ *then* $\langle f; a\rangle = -\sum\langle u^j; a_{x_j}\rangle$ *is well defined.*

4. THE CONTINUOUS CASE

We recall an example due to Ornstein [5] (Mc Mullen uses the more abstract version of Ornstein theorem to prove his result).

Let $N \in \mathbb{N}$ and $\Omega = (0, 1)^2$ then there exists $\psi^N = \psi^N(x_1, x_2) \in C_0^\infty(\Omega)$ such that

$$|\psi^N_{x_1 x_1}|_{L^1} + |\psi^N_{x_2 x_2}|_{L^1} = 1$$

$$N(|\psi^N_{x_1 x_1}|_{L^1} + |\psi^N_{x_2 x_2}|_{L^1}) \leqslant |\psi^N_{x_1 x_2}|_{L^1}.$$

Note that, for $u = (u^1, u^2) \in W^{1,\infty}(\Omega; \mathbb{R}^2)$,

$$\iint_\Omega (u^1_{x_1} + u^2_{x_2})\, \psi^N_{x_1 x_2}\, dx_1\, dx_2 = \iint_\Omega (u^1_{x_2}\, \psi^N_{x_1 x_1} + u^2_{x_1}\, \psi^N_{x_2 x_2})\, dx_1\, dx_2$$

and hence

$$\left| \iint_\Omega \operatorname{div} u \psi^N_{x_1 x_2}\, dx_1\, dx_2 \right| \leqslant |u|_{W^{1,\infty}}.$$

Since $\lim_{N \to \infty} |\psi^N_{x_1 x_2}|_{L^1} = \infty$, using Banach-Steinhaus we can find $f \in C^0$ such that

$$\lim_{N \to \infty} \left| \iint_\Omega f \psi^N_{x_1 x_2}\, dx_1\, dx_2 \right| = \infty.$$

Combining the above facts we have even shown that there is $f \in C^0$ such that no vector field $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ can satisfy $\operatorname{div} u = f$.

Of course this result immediately extends to higher dimensions.

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