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NONLINEAR EQUATIONS ON CARNOT GROUPS AND CURVATURE PROBLEMS FOR CR MANIFOLDS

ABSTRACT. — We give a short overview of sub-Laplacians on Carnot groups starting from a result by Caccioppoli dated 1934. Then we show that sub-Laplacians on Carnot groups of step one arise in studying curvature problems for CR manifolds. We restrict our presentation to the cases of the Webster-Tanaka curvature problem for the CR sphere and of the Levi-curvature equation for strictly pseudoconvex functions.

KEY WORDS: Hypoellipticity; Carnot groups; Sub-Laplacians; Webster-Tanaka curvature; Levi-curvature.

1. SOME *ANTE LITTERAM* RESULTS

In 1934 Caccioppoli proved the following theorem [3, pp. 52-53].

(C) Let u be a locally integrable real function in the open set $\Omega \subset \mathbb{R}^2$ and assume

$$\int_\Omega u \Delta \varphi = 0, \qquad \forall \varphi \in C_0^\infty(\Omega) \tag{1.1}$$

where Δ denotes the Laplace operator. Then there exists a harmonic function v in Ω such that

$$u(x) = v(x) \quad \text{a.e. in } \Omega .$$

In the modern language of distribution theory, Caccioppoli's result can be stated as follows: if $u \in L^1_{\text{loc}}(\Omega)$ satisfies the equation $\Delta u = 0$ in Ω in the weak sense of distributions then $u \in C^\infty(\Omega)$. Caccioppoli's proof is quite simple. By suitably choosing the test function φ in (1.1) he showed that the mean value of u on the sphere $S_r(x)$, *i.e.*

$$u_r(x) := \fint_{S_r(x))} u ds ,$$

is independent of r for almost every $x \in \Omega$ and $0 < r < \text{dist}(x, \Omega)$. As a consequence, by a previous old result by E.E. Levi [1], u is a.e. equal to a harmonic function in Ω. It is not difficult to recognize that Caccioppoli's argument also works in every dimension $N \geq 2$. However in 1940 a different proof was given by H. Weyl who (rediscovered and) extended Theorem **(C)** to $\mathbb{R}^3$ [24, Lemma 2].

Caccioppoli-Weyl's regularity theorem for weak solutions to the Laplace equation

[1] E.E. LEVI, *Opere*. Edited by Unione Mat. It., Cremonese, 1956, 180-186.

can be considered as an *ante litteram* hypoellipticity result. Indeed, in 1940, L. Schwartz introduced the following general definition: a linear partial differential operator L with smooth coefficients is *hypoelliptic* in Ω if every distributional solution to the equation $Lu = f$ in the open set $\Omega \subset \mathbb{R}^N$ is of class C^∞ when $f \in C^\infty(\Omega)$.

Caccioppoli's proof of Theorem **(C)**, together with now classical devices of distribution theory, could be used to recognize the hypoellipticity of Δ in the general sense of Schwartz. That proof essentially rests on the following two facts.

(C1) There exists a fundamental solution with pole at $x = 0$ for Δ, *i.e.* there exists a function $\Gamma \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $-\Delta\Gamma = \delta$ (the Dirac measure);

(C2) Δ is invariant w.r. to left translations of the Euclidean group and commutes with the Euclidean dilations

$$\delta_\lambda(x) = \lambda x \,.$$

From **(C1)** and the first part of **(C2)** it follows that

$$\Gamma(x, y) := \Gamma(-y + x)$$

is a fundamental solution for Δ with pole at $x = y$ and of class C^∞ in the open set $U = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N \colon x \neq y\}$.

It is now well known that the existence of a fundamental solution of class $C^\infty(U)$ is equivalent to the hypoellipticity for every linear partial differential operator with smooth coefficients. Then, the following existence theorem, proved by Kolmogorov in 1934 (see [13]), can be considered as another *ante litteram* hypoellipticity result. In studying diffusion phenomena from a probabilistic point of view, Kolmogorov showed that the probability density of a system with $2n$ degrees of freedom satisfies a linear second order ultra-parabolic equation

$$Ku = 0 \qquad \text{in } \mathbb{R}^{2n} \times \mathbb{R} \,,$$

where $\mathbb{R}^{2n}$ is the phase-space of the system. A prototype for K is the following operator

$$K = \sum_{j=1}^{n} \partial_{x_j}^2 + \sum_{j=1}^{n} x_j \partial_{y_j} - \partial_t \tag{1.2}$$

$$= \Delta_x + \langle x, \nabla_y \rangle - \partial_t, \tag{1.3}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ denote the velocity and the position vectors of the system, respectively. The operator K is very degenerate: its second order part only contains derivatives with respect to the variables $x_1, \ldots, x_n$. Nevertheless, Kolmogorov constructed an explicit fundamental solution Γ_K for K smooth out of its poles [15]. As a consequence: K is hypoelliptic.

From the explicit expression of Γ_K one realizes that

$$\Gamma_K(z, \zeta) = G(\zeta^{-1} \circ z) \,, \qquad z = (x, y, t),\ \zeta = (\xi, \eta, \tau) \in \mathbb{R}^{2n+1},$$

where $\circ$ is a composition law making $\mathcal{K} = (\mathbb{R}^{2n+1}, \circ)$ a Lie group, and

$$G : \mathbb{R}^{2n+1} \setminus \{0\} \to \mathbb{R}$$

is a C^∞ function. Moreover, it is easy to check that K is invariant w.r. to left translations on $\mathcal{K}$ and commutes with the following dilations:

$$\delta_\lambda(z) = (\lambda x, \lambda^3 y, \lambda^2 t), \qquad \lambda > 0 .$$

These dilations are also automorphisms of $\mathcal{K}$. Then $\mathcal{K}$ is a *homogeneous Lie group*, and the Kolmogorov's operator K satisfies conditions **(C1)** and **(C2)** with Γ and the Euclidean group replaced by Γ_K and $\mathcal{K}$, respectively.

2. Hörmander operator, Carnot groups and sub-Laplacians

In addition to the properties showed in Section 1, the Kolmogorov's operator (1.2) has another key feature: it can be written as

$$K = \sum_{j=1}^{n} X_j^2 + Y$$

where $X_j = \partial_{x_j}$ and $Y = \sum_{k=1}^{n} x_k \partial_{y_k} - \partial_t$.

The vector fields $X_1, \ldots, X_n$ and Y generate the Lie algebra of $\mathcal{K}$ and satisfy the following condition

$$\operatorname{rank} \operatorname{Lie}(X_1, \ldots, X_n, Y)(z) = 2n+1 \qquad \forall z \in \mathbb{R}^{2n+1}.$$

Inspired by the previous arguments Hörmander introduced in 1967 the class of linear second order PDO's

$$L = \sum_{j=1}^{m} X_j^2 + Y \tag{2.1}$$

and proved the following celebrated theorem.

(H) Let $X_1, \ldots, X_m$ and Y be smooth vectors fields, *i.e.* linear first order PDO's with smooth coefficients in the open set $\Omega \subset \mathbb{R}^N$. Suppose

$$\operatorname{rank} \operatorname{Lie}(X_1, \ldots, X_m, Y)(x) = N, \qquad \forall x \in \Omega . \tag{2.2}$$

Then the operator (2.1) is hypoelliptic in Ω [11, Theorem 1.1].

Hörmander's work precedes a long series of papers, by many authors, dealing with the operators (2.1). The most remarkable contributions to these studies have been given by Folland, Rothschild and Stein who developed and applied to (2.1) Harmonic and Functional Analysis on Carnot groups.

A Lie group $\mathbb{G} = (\mathbb{R}^N, \circ)$ is a Carnot group if the following properties **(G1)** and **(G2)** hold.

(G1) $\mathbb{R}^N$ can be split as $\mathbb{R}^N = \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_r}$ and the dilations $(\delta_\lambda)_{\lambda>0}$,

$$\begin{aligned}\delta_\lambda(x) &= \delta_\lambda(x^{(1)}, x^{(2)}, \ldots, x^{(r)})\\ &= (\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)}), \qquad x^{(j)} \in \mathbb{R}^{N_j}\end{aligned}$$

are automorphisms of $\mathbb{G}$.

(G2) The Lie algebra $\mathbf{g}$ of $\mathbb{G}$ is generated by the left invariant vector fields $X_1, \ldots, X_{N_1}$ satisfying

$$X_j(0) = \partial_{x_j} \qquad j = 1, \ldots, N_1 .$$

The natural numbers r and

$$Q = N_1 + 2N_2 + \ldots + rN_r$$

are respectively called the *step* and the *homogeneous* dimension of $\mathbb{G}$. The *generators* of $\mathbb{G}$ are the vector fields $X_1, \ldots, X_{N_1}$. The second order operator

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{N_1} X_j^2$$

is called the canonical sub-Laplacian of $\mathbb{G}$. Some of its basic properties are listed below.

(L1) $\Delta_{\mathbb{G}}$ is left translations invariant on $\mathbb{G}$ and commutes with the dilation in **(G1)**.

(L2) $\Delta_{\mathbb{G}}$ is hypoelliptic since

$$\text{rank Lie}\,(X_1, \ldots, X_{N_1})(x) = \dim \mathbf{g} = \mathrm{N}$$

for every $x \in \mathbb{R}^N$.

(L3) The characteristic form of $\Delta_{\mathbb{G}}$

$$q(x, \xi) := \sum_{j=1}^{N_1} \langle X_j(x), \xi \rangle^2$$

is nonnegative definite at any point $x \in \mathbb{R}^N$.

If $N_1 < N$, which is equivalent to $r \geqslant 2$, for every $x \in \mathbb{R}^N$ there exists $\xi \neq 0$ for which $q(x, \xi) = 0$. Then $\Delta_{\mathbb{G}}$ is *not elliptic* at any point.

A striking and deep analogy between $\Delta_{\mathbb{G}}$ and the classical Laplace operator is the structure of their fundamental solutions. If the homogeneous dimension Q of $\mathbb{G}$ is $\geqslant 3$, there exists a homogeneous norm $|\,.\,|$ on $\mathbb{G}$ such that

$$\Gamma_{\Delta_{\mathbb{G}}}(x) = |x|^{-Q+2} \tag{2.3}$$

where $\Gamma_{\Delta_{\mathbb{G}}}$ denotes the fundamental solution of $\Delta_{\mathbb{G}}$ with pole at $x = 0$ (see [6] and [7]). We call *homogeneous norm* on $\mathbb{G}$ a function $x \to |x|$ continuous in $\mathbb{R}^N$ and of class C^∞ outside the origin and such that

- $\delta_\lambda(x) = \lambda |x|$
- $|x^{-1}| = |x| > 0 \ \forall x \neq 0$.

We close this section by recalling a celebrated result of Rothschild and Stein, enlightening the crucial role played by sub-Laplacians in studying second order PDO's sum of squares of vector fields: every operator $L = \sum_{j=1}^m X_j^2$ satisfying the Hörmander's rank condition (2.2) can be *lifted* to an operator $\widehat{L}$ as close as we want to a sub-Laplacian. We refer to [21] for the precise statement of this theorem.

3. The Folland-Stein space and the critical exponent

The *intrinsic gradient* (or *horizontal* gradient) on a Carnot group $\mathbb{G}$ is the vector valued operator

$$\nabla_{\mathbb{G}} = (X_1, \ldots, X_{N_1})$$

where $X_1, \ldots, X_{N_1}$ are the generators of $\mathbb{G}$. Since $\Gamma = \Gamma_{\Delta_{\mathbb{G}}}$ is the fundamental solution for $\Delta_{\mathbb{G}}$, the following representation formula holds:

$$u(x) = \int_{\mathbb{R}^N} \nabla_{\mathbb{G}} \Gamma(x^{-1} \circ y) \cdot \nabla_{\mathbb{G}} u(y)\, dy, \qquad u \in C_0^\infty(\mathbb{R}^N).$$

Thus, if $Q \geq 3$ using (2.3) we get

$$|u(x)| \leq c_Q \int_{\mathbb{R}^N} |x^{-1} \circ y|^{-Q+1} |\nabla_{\mathbb{G}} u(y)|\, dy .$$

From this inequality, by using standard Real Analysis devices, the Folland-Stein's embedding inequality follows:

$$\|u\|_{L^{2^*}(\mathbb{R}^N)} \leq C \|\nabla_{\mathbb{G}} u\|_{L^2(\mathbb{R}^N)}, \tag{3.1}$$

where $2^* = \dfrac{Q}{Q-2}$. Then, if Ω is an open subset of $\mathbb{R}^N$, $u \to \|\nabla_{\mathbb{G}} u\|_{L^2(\Omega)}$ is a norm in $C_0^\infty(\Omega)$. The Folland-Stein's space $S_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to this norm. Inequality (3.1) implies the continuity of the embedding

$$S_0^1(\Omega) \subset L^{2^*}(\Omega).$$

On the other hand, an easy rescaling argument shows that the only exponent p for which the continuous embedding $S_0^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ holds is $p = 2^*$.

The space $S_0^1(\Omega)$ is also continuously embedded in the classical Sobolev space $W_0^{\frac{1}{r}}(\Omega)$, where r is the step of $\mathbb{G}$ (see [6]). As a consequence, from the classical compact embedding theorem for Sobolev spaces, the embedding $S_0^1(\Omega) \subset L^p(\Omega)$ is compact if $1 \leq p < 2^*$ and Ω is bounded. Then $2^* = \dfrac{2Q}{Q-2}$ has to be considered the *critical exponent* for the Folland-Stein space.

4. The Heisenberg group and its Laplacian

The Heisenberg group $\mathbb{H}_n$ is the Lie group $(\mathbb{R}^{2n+1}, \circ)$ whose composition law is defined as follows

$$(z, t) \circ (z', t') = (z + z', t + t' + 2\Im\langle z, z' \rangle). \tag{4.1}$$

Hereafter we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and use

$$\xi = (z, t) = (z_1, \ldots, z_n, t) = (x_1, y_1, \ldots, x_n, y_n, t)$$

to denote the points of $\mathbb{H}_n$. In (4.1), $\langle z, z' \rangle$ stands for the usual Hermitian inner product in $\mathbb{C}^n$:

$$\langle z, z' \rangle = \sum_{j=1}^n z_j \bar{z}_j' .$$

The dilation

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t) \tag{4.2}$$

is an automorphism of $\mathbb{H}_n$ and the vector fields

$$X_j = \partial_{x_j} + 2 y_j \partial_t, \qquad Y_j = \partial_{y_j} - 2 x_j \partial_t$$

are left translation invariant in $\mathbb{H}_n$. One straightforwardly recognizes that the following commutation relations hold:

$$[X_j, Y_j] = -4\partial_t \tag{4.3}$$

and

$$[X_j, X_k] = [Y_j, Y_k] = [X_j, Y_k] = 0, \quad \forall j \neq k \tag{4.4}$$

(4.3) is the canonical commutation relation between momentum and position in Quantum Mechanics. For this reason $\mathbb{H}_n$ is called *Heisenberg group*. From (4.3) and (4.4) it follows that the Lie algebra of $\mathbb{H}_n$ is the vector space

$$\operatorname{span}\{X_1, \ldots, X_n. Y_1, \ldots, Y_n, \partial_t\},$$

whose dimension obviously is $2n+1$. Since $X_j(0) = \partial_{x_j}$ and $Y_j(0) = \partial_{y_j}$, $j = 1, \ldots n$, according to the definitions given in Section 2, $\mathbb{H}_n$ is a Carnot group of Step 2. Its generators are the vector fields $X_1, \ldots, X_n$, $Y_1, \ldots Y_n$.

$\mathbb{H}_n$ has homogeneous dimension

$$Q = 2n + 2$$

(see (4.2)) and its sub-Laplacian is

$$\Delta_{\mathbb{H}_n} = \sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

$\Delta_{\mathbb{H}_n}$ is also called the Kohn Laplacian on $\mathbb{H}_n$. The Folland-Stein inequality embedding in the Heisenberg group takes the form

$$\|u\|_{L^{2^*}(\mathbb{H}_n)} \leqslant C\|\nabla_{\mathbb{H}_n} u\|_{L^2(\mathbb{H}_n)}, \tag{4.5}$$

where $2^* = \dfrac{2Q}{Q-2} = 2 + \dfrac{2}{n}$. The best constant C in this inequality was determined by Jerison and Lee who also proved that the equality is reached iff

$$u(\xi) = \lambda^{\frac{Q-2}{2}} u_0(\delta_\lambda(\xi_0 \circ \xi)), \quad \lambda > 0, \quad \xi_0 \in \mathbb{H}^n, \tag{4.6}$$

where

$$u_0(\xi) = u_0(z, t) = c_0(t^2 + (1 + |z|^2)^2)^{\frac{2-Q}{4}} \tag{4.7}$$

and $c_0 > 0$ is a suitable constant [12, 13].

The Heisenberg group plays a fundamental role in several curvature problems for *CR* manifolds. Among the most important ones is the CR-Yamabe problem, which was completely solved by Jerison and Lee [13], and by Gamara [8] and Yacoub [9]. Very recently, the CR analogue of the scalar curvature problem for the standard Euclidean sphere has been studied.

5. Webster-Tanaka curvature problem for the *CR* sphere

Let us consider the sphere of $\mathbb{C}^{n+1}$

$$\mathbb{S}^{2n+1} = \{z \in \mathbb{C}^{n+1} \colon |z|^2 = 1\},$$

and let

$$\theta_0 = i(\bar{\partial} - \partial)\,\varrho, \quad \varrho(z) = |z|^2 - 1$$

its standard *contact form*. The Webster-Tanaka curvature problem for the *CR* manifold $(\mathbb{S}^{2n+1}, \theta_0)$ can be stated as follows.

(WT) Given a smooth function $\overline{K}$ on $\mathbb{S}^{2n+1}$ find a new contact form

$$\theta = v^{\frac{2}{n}} \theta_0$$

such that $(\mathbb{S}^{2n+1}, \theta_0)$ has Webster-Tanaka curvature equal to $\overline{K}$. The Riemannian analogue of this problem is the *scalar curvature* problem for the standard Euclidean sphere, also quoted in literature as *Nirenberg's problem*, see *e.g.* [10]. The Cayley map

$$F : \mathbb{S}^{2n+1} \setminus \{(0, \dots, 0, -1)\} \to \mathbb{H}_n,$$

$$F(z_1, \dots, z_n, z_{n+1}) = \left(\frac{z_1}{1 + z_{n+1}}, \dots, \frac{z_n}{1 + z_{n+1}}, \frac{\Im_m(z_{n+1})}{|1 + z_{n+1}|^2} \right)$$

transforms **(WT)** in the following semilinear problem

$$\begin{cases} -\Delta_{\mathbb{H}_n} u = K u^{\frac{Q+2}{Q-2}} \\ u > 0, \qquad u \in \mathcal{S}_0^1(\mathbb{H}_n) \end{cases} \tag{5.1}$$

where $K = \overline{K} \circ F^{-1}$. We explicitly notice that the exponent $\dfrac{Q+2}{Q-2} = 1 + \dfrac{2}{n} = 2^* - 1$ is *critical*.

We also remark that the solutions of (5.1) are the critical points of the functional

$$J(u) = \frac{1}{2} \int_{\mathbb{H}_n} |\nabla_{|H_n} u|^2 - \frac{1}{2^*} \int_{\mathbb{H}_n} R u^{2^*}, \qquad u \in \mathcal{S}_0^1(\mathbb{H}_n).$$

Very few results are known for problem (5.1). For $K = 1$ its solutions are the function u in (4.6), (4.7), the Jerison and Lee extremals of the Folland-Stein embedding (4.5).

By using the abstract perturbation method introduced by Ambrosetti and Badiale in [1], very recently Malchiodi and Uguzzoni, have proved the following perturbation result: problem (5.1) has (at least) a solution if $K = 1 + \varepsilon h$, with $\varepsilon > 0$ small enough and h a Morse function satisfying suitable conditions, see [20, Theorem 1].

6. Pseudoconvex domains and Levi-curvature

We begin this section by recalling some basic definitions from the theory of several complex variables. Let us consider in $\mathbb{C}^{n+1}$ the domain

$$D = \{z \in \mathbb{C}^{n+1} : f(z) < 0\},$$

where $f : \mathbb{C}^{n+1} \to \mathbb{R}$ is a C^2 function satisfying

$$\partial_p f \neq 0 \quad \text{if} \quad f(p) = 0.$$

Then

$$bD = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$$

is a real manifold of dimension $2n+1$. Its *complex* tangent space at a point $p \in bD$ is

$$T_p^{\mathbb{C}}(bD) = \{h \in \mathbb{C}^{n+1} : \langle h, \bar{\partial}_p f\rangle = 0\}$$

where $\langle , \rangle$ denotes the usual Hermitian inner product in $\mathbb{C}^{n+1}$ and

$$\bar{\partial}_p f = (f_{\bar{1}}, \dots, f_{\overline{n+1}}), \qquad f_{\bar{k}} = \frac{\partial f}{\partial \bar{z}_k}.$$

Obviously $T_p^{\mathbb{C}}$ is a *complex* vector space of dimension n. However, for what follows it is crucial to interpret $T_p^{\mathbb{C}}$ as a *real* vector space of dimension

$$\dim_{\mathbb{R}} T_p^{\mathbb{C}}(bD) = 2n.$$

We want to stress the loss of a *real* dimension in passing from bD to $T_p^{\mathbb{C}}$.

The Levi form of the function f at a point $p \in bD$ is the restriction to $T_p^{\mathbb{C}}$ of the complex Hessian form

$$L_p(f, \xi) = \sum_{j,k=1}^{n} \frac{\partial f}{\partial z_j \partial \bar{z} z_k}(z)\, \xi_j \bar{\xi}_k, \qquad \xi \in T_z^{\mathbb{C}}(bD).$$

We recall that the domain D is called *strictly pseudoconvex* (or *strictly Levi-pseudoconvex*) at p if

$$L_p(f, \xi) > 0 \qquad \forall\, \xi \in T_p^{\mathbb{C}}(bD) \setminus \{0\}.$$

If this inequality holds at any point $p \in bD$ then D is called *strictly pseudoconvex*.

This notions are independent of the defining function f and are invariant with respect to biholomorphic changes of complex coordinates. Obviously, a strictly convex domain of $\mathbb{C}^{n+1}$ also is strictly pseudoconvex even though the viceversa is not true. However, up to a suitable biholomorphic change of complex coordinates, strict convexity and strict pseudoconvexity are equivalent. More precisely: D is strictly pseudoconvex at a point p iff there exists an open neighborhood Ω of p and a biholomorphic function

$$F : \Omega \cap D \to \mathbb{C}^{n+1}$$

such that $F(\Omega \cap D)$ is strictly convex. We stress that the term *strictly* in the previous assertion cannot be removed, see [14].

We would like to close these preliminaries by recalling that the notion of pseudoconvexity was introduced by E.E. Levi in studying the *domains of holomorphy* of $\mathbb{C}^{n+1}$. Actually, E.E. Levi proved the following results.

(L1) If D is a domain of holomorphy then

$$L_p(f, \xi) \geqslant 0 \qquad \forall \xi \in T_p^{\mathbb{C}}(M) \setminus \{0\}$$

(see [17]).

(L2) If $D \subset \mathbb{C}^2$ is strictly pseudoconvex then every point $p \in bD$ has a neighborhood Ω such that $\Omega \cap D$ is a domain of holomorphy (see [18]).

By using the notion of Levi form, it is quite natural to introduce a notion of curvature which can be seen as the pseudoconvex counterpart of the classical Gauss-curvature.

Let $p \in bD$. We call *normalized complex Hessian* of f at p the following $(n+1) \times \times (n+1)$ Hermitian matrix

$$\mathcal{H}_p(f) := \left(\frac{1}{|\partial_p f|} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(p) \right)_{j,\, k = 1, \dots, n+1}.$$

If $\mathcal{B} = \{h_1, \dots, h_n\}$ is an orthonormal basis of $T_p^{\mathbb{C}}(bD)$ we put

$$L_p^{\mathcal{B}} = \left(\langle \mathcal{H}_p(f)\, h_j,\, h_k \rangle \right)_{j,\, k = 1, \dots, n}.$$

$L_p^{\mathcal{B}}$ is an $n \times n$ Hermitian matrix whose eigenvalues $\lambda_1, \dots, \lambda_n$ are real and independent of $\mathcal{B}$ and of the defining function f. We call *total Levi-curvature* of bD at p the real number

$$K_p(bD) := \prod_{j=1}^{n} \lambda_j.$$

This definition is implicitly contained in a paper by Bedford and Gaveau [2] and it has been explicitly given in a recent note by Lascialfari and Montanari [16]. Definitions of Levi-curvature related to the present one were given by Slodkowski and Tomassini in [22] and [23].

EXAMPLE 5.1. Let us consider the ball

$$D_R = \{z \in \mathbb{C}^{n+1} \colon |z|^2 = R^2\}, \qquad R > 0.$$

A defining function for D_R is

$$f(z) = |z|^2 - R^2$$

Since $f_{\bar{k}} = z_k$ and $f_{j\bar{k}} = \delta_{jk}$ for every $p \in bD$ one has $\mathcal{H}_p f = \dfrac{1}{R} I_{n+1}$. Then, if $\mathcal{B}$ is an orthonormal basis of $T_p^{\mathbb{C}}(bD)$,

$$L_p^{\mathcal{B}} = \frac{1}{R} I_n$$

(we have denoted by I_k the identity matrix of dimension k). As a consequence, all the eigenvalues of $L_p^{\mathcal{B}}$ are equal to $\dfrac{1}{R}$ and

$$K_p(bD_R) = \left(\frac{1}{R} \right)^n.$$

For a general domain D with defining function f, the total Levi-curvature of bD at a point p can be computed as

$$K_p(bD) = -\frac{1}{|\partial_p f|^{n+2}} \det \begin{bmatrix} 0 & f_{\bar 1} & \cdots & f_{\overline{n+1}} \\ f_1 & f_{1,\bar 1} & \cdots & f_{1,\overline{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n+1} & f_{n+1,\bar 1} & \cdots & f_{n+1,\overline{n+1}} \end{bmatrix}$$

where $f_{j,\bar k}$ stands for $\dfrac{\partial^2 f}{\partial z_j \partial \bar z_k}(p)$.

The proof of this identity follows from linear algebra arguments.

7. Levi-curvature equation

Let Ω be an open subset of $\mathbb{R}^{2n+1}$ and

$$u : \Omega \to \mathbb{R}$$

be a C^2 function. Let

$$\Gamma(u) = \{(x, y) \in \Omega \times \mathbb{R} : y > u(x)\}$$

and

$$\gamma(u) = \{(x, y) \in \Omega \times \mathbb{R} : y = u(x)\}$$

the *epigraph* and the *graph* of u, respectively. We say that u is strictly pseudoconvex if $\Gamma(u)$ is strictly pseudoconvex at every points of $\gamma(u)$. Given a function $K : \Omega \times \mathbb{R} \to \mathbb{R}$, the graph of u has Levi-curvature $K(x, u(x))$ at a point $(x, u(x))$ iff u satisfies a second order PDE of the following type

$$\mathcal{L} u = \sum_{j,k=1}^{2n} a_{j,k}(Du, D^2 u)\, X_j X_k u = H(x, u(x)) \tag{7.1}$$

where $(a_{j,k})_{j,k=1,\ldots,2n}$ is a $2n \times 2n$ matrix with real entries and the X_j's are nonlinear first order partial differential operators:

$$X_j u = \sum_{k=1}^{2n+1} a_j^{(k)}(Du)\, \partial_{x_k} u\,.$$

The *curvature function* H depends on the prescribed function K and on the first derivative of u, see [19]. We also notice that the vector fields

$$X_j = \sum_{k=1}^{2n+1} a_j^{(k)}(Du)\, \partial_{x_k}, \qquad j = 1, \ldots, 2n\,,$$

form a *real* basis of the complex tangent space to $\gamma(u)$. We shall call (7.1) *Levi-curvature equation* in the open set $\Omega \subset \mathbb{R}^{2n+1}$. It is fully nonlinear, and also totally degenerate since at any point of Ω it involves derivatives along only $2n$ linearly independent directions. A redeeming feature of (7.1) is the CR structure that the graph of u inherits from $\mathbb{C}^{n+1}$ when u is strictly pseudoconvex. Indeed, in this hypothesis the following propositions hold, see [19].

(P1) The matrix $(a_{j,k}(Du, D^2 u))_{j,k=1,\ldots,2n}$ is locally uniformly positive definite.

(P2) At any point $x \in \Omega$, the linear space

$$\operatorname{span}\{X_j, [X_j, X_k] : j, k = 1, \dots, 2n\} \tag{7.2}$$

has dimension $2n+1$.

Then, if u is strictly pseudoconvex the Levi-curvature equation (7.1) is *elliptic* along $2n$ variable directions in $\mathbb{R}^{2n+1}$. This follows from **(P1)**. From **(P2)** one recovers the *missing direction* by commutation. Property (7.2) in (P2) can be seen as a *Hörmander's rank condition of step two* which, in the case of linear smooth vector fields, implies hypoellipticity.

This kind of analogy is not just formal. Indeed, the following remarkable theorem holds.

THEOREM 7.1. *Let $u \in C^{2+\alpha}(\Omega)$ be a strictly pseudoconvex solution to the Levi-curvature equation* (7.1). *Suppose the curvature function H of class C^∞. Then $u \in C^\infty(\Omega)$.*

For the Levi-curvature equation in $\mathbb{R}^3$ this theorem was proved by Citti [4]. In $\mathbb{R}^{2n+1}$, for every $n \geq 2$, it has been recently announced by Lascialfari and Montanari in [16].

In $\mathbb{R}^3$, corresponding to the case $n = 1$, a stronger and deeper result holds, see [5].

THEOREM 7.2. *Let Ω be an open subset of $\mathbb{R}^3$ and $u : \Omega \to \mathbb{R}$ be a viscosity solution to the Levi-curvature equation. Assume the curvature function H is of class C^∞ and everywhere different from zero. Then $u \in C^\infty(\Omega)$.*

We would like to notice that the condition $H \neq 0$ in the case $n = 1$ is equivalent to the pseudoconvexity of u.

It is an open important problem the extension of Theorem 7.2 to the Levi-curvature equation in $\mathbb{R}^{2n+1}$ for arbitrary $n \geq 2$.

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