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CACCIOPPOLI ESTIMATES AND VERY WEAK SOLUTIONS OF ELLIPTIC EQUATIONS

ABSTRACT. — Caccioppoli estimates are instrumental in virtually all analytic aspects of the theory of partial differential equations, linear and nonlinear. And there is always something new to add to these estimates. We emphasize the fundamental role of the natural domain of definition of a given differential operator and the associated weak solutions. However, we depart from this usual setting (energy estimates) and move into the realm of the so-called *very weak solutions* where important new applications lie. We carry out this task deliberately with a restricted generality in interest of readability, and we hope it pays off handsomely in mathematical insights.

KEY WORDS: Caccioppoli inequality; Very weak solutions; Elliptic equations.

INTRODUCTION

One might possibly say that R. Caccioppoli was one of the finest analysts of his time. To whom we should compare him: C.B. Morrey, J. Leray, J. Schauder, L. Nirenberg, L. Bers, A. Lavrentiev, E. De Giorgi? With this impressive list of names, we now face truly fascinating questions, related not so much to the well known results of Caccioppoli as to the possible generalizations, their substance and universality. There are many admirers of Caccioppoli's ideas, but few know the full complexity and advantage behind his estimates for elliptic partial differential equations, PDEs in short.

These estimates are worth quoting extensively, for they are a rich resource for further exploration in both linear and nonlinear equations (existence, uniqueness and regularity). Among all a priori estimates in PDEs the Caccioppoli estimates are the ones that most strongly emphasize the reverse form of the Poincarè - Sobolev inequality. Indeed, they have been widely noted as

$$\int_{\Omega} |\varphi(x)\, \nabla u(x)|^p \, \mathrm{d}x \ll \int_{\Omega} |u(x)\, \nabla \varphi(x)|^p \, \mathrm{d}x \tag{1}$$

where u is a solution to a given PDE and φ is an arbitrary test function of class $C_0^\infty(\Omega)$. Here and subsequently, the notation $\ll$ is being used for inequalities in which the implied constant does not depend on functions involved therein. Our goal is to present these estimates together with various generalizations as sensitively as possible, to capture their fundamental role in analysis.

There is such a thing as the natural Sobolev exponent $1 < p < \infty$ for the derivatives of solutions to a given PDE, see the next section for details. It is relatively simple to derive Caccioppoli type estimates with this natural exponent, nothing deeper than integration by parts. In this natural setting we refer to them as the energy estimates. They are largely responsible for establishing the regularity properties of the solutions, just to mention the property of higher integrability of the gradient. In this category of useful implications we must certainly include the reverse Hölder inequalities:

$$\left(\fint_B |\nabla u|^p \right)^{\frac{1}{p}} \leqslant C_p \fint_{2B} |\nabla u| \,, \tag{2}$$

where the integral averages are taken over all concentric balls $B \subset 2B \subset \Omega$. The phenomenon of selfimprovement is that we obtain from (2) a new set of inequalities

$$\left(\fint_B |\nabla u|^s \right)^{\frac{1}{s}} \leqslant C_s \fint_{2B} |\nabla u| \tag{3}$$

with some exponent $s > p$ and constant C_s, depending only on p, C_p and the dimension n. This fact was first brought to light in the celebrated paper by F.W. Gehring [23] and then profitably extended by many researchers [11, 10, 18, 21, 25, 22, 66, 56, 24].

Almost half of a century has gone to the development of Caccioppoli's ideas, but only recently we became fully aware of the estimates with exponents below the natural one. They became essential ingredients to nonlinear PDEs [34, 35, 45]. Thus we subscribe to these estimates here, leading to what we call the theory of very weak solutions. The general underlying advantage is that when we lower the exponent for the gradient of the solution we, automatically, lower the exponent for the test function in the right hand side of (1), say

$$\int_\Omega |\varphi(x)\, \nabla u(x)\,|^s \mathrm{d}x \ll \int_\Omega |u(x)\, \nabla \varphi(x)\,|^s \mathrm{d}x \tag{4}$$

for some $1 \leqslant s < p$, as needed for a specific problem. Considering the matter in all its bearings mention should be made of the removability of singularities [35, 41, 43], integrability of the Jacobian determinant [15, 27, 44, 55, 57, 28, 12, 29, 49], the study of equations with measure data [30, 20, 46, 47, 17], and much more [61, 59], … . Chiefly, the questions we are going to discuss here originated from our studies of the geometric function theory [40] and the calculus of variations [63].

For many interesting details concerning life of *Renato Caccioppoli* we refer to [1, 32].

1. Examples of the natural setting

The natural domain of definition of a given PDE is usually determined from an interpretation of its solutions, sometimes as an analytic description of a geometric object, sometimes as a mathematical model of certain physical or mechanical entity. It is not necessary to formulate a rigorous definition of the natural domain in this rather early stage of development. Precise treatment would be dry and would work against the ideas about to be discussed here. Throughout this text the term natural domain, or natural setting, of a given PDE pertains to the situation when its eloquent interpretation is clear from the context. Of course for every PDE (linear or nonlinear) the natural setting ought to be in a complete function space. That is why we must work with weakly differentiable functions whose derivatives possess certain degree of integrability. Orlicz-Sobolev classes will also emerge, as the so-called weak domains. It is illuminating and rewarding to begin with some classical examples.

1.1. *The Harmonic Equation*:

$$\operatorname{div} \nabla u = \operatorname{div} F, \qquad \text{in } \Omega \subset \mathbb{R}^n. \tag{1.1}$$

The well known and perhaps most natural is the setting in the Sobolev space $W^{1,2}(\Omega)$, where the given vector field $F = (F^1, \ldots, F^n)$ lies in $L^2(\Omega, \mathbb{R}^n)$. Such solutions are stationary points of the energy functional

$$\mathcal{E}[u] = \int_\Omega |\nabla u|^2 + 2\langle F, \nabla u\rangle, \tag{1.2}$$

usually subjected to the boundary condition: $u \in u_0 + W_0^{1,2}(\Omega)$ with given Dirichlet data $u_0 \in W^{1,2}(\Omega)$.

A prototype for many nonlinear PDEs is furnished by

1.2. *The p-Harmonic Equation*:

$$\operatorname{div} |\nabla u|^{p-2} \nabla u = \operatorname{div} F, \qquad \text{in } \Omega \subset \mathbb{R}^n. \tag{1.3}$$

The p-harmonic equation is naturally defined for $u \in W^{1,p}(\Omega)$, $1 < p < \infty$, where $F \in L^q(\Omega, \mathbb{R}^n)$, and q is Hölder conjugate to p; that is, $p + q = pq$. The reason for this setting is that this equation arises naturally from the energy integral

$$\mathcal{E}[u] = \int_\Omega |\nabla u|^p + p\langle F, \nabla u\rangle. \tag{1.4}$$

Recent developments of the theory of quasiconformal mappings [40] rely on parallel advances in the study of even more general equations

1.3. *A-harmonic Equation*:

$$\operatorname{div} \boldsymbol{A}(x, \nabla u) = \operatorname{div} F \tag{1.5}$$

where $\boldsymbol{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies, among other conditions, the following ellipticity bounds

$$\langle \boldsymbol{A}(x, \xi), \xi\rangle \stackrel{\text{def}}{=} |\xi|_A^p \geqslant \varrho(x)\, |\xi|^p \tag{1.6}$$

$$\langle \boldsymbol{A}(x, \xi), \zeta\rangle \leqslant |\xi|_A^{p-1}\, |\zeta|_A, \qquad 1 < p < \infty \tag{1.7}$$

with some measurable function $0 < \varrho(x) < \infty$ and all $\xi, \zeta \in \mathbb{R}^n$. In general $\varrho(x)$ need not be constant; consequently, the uniform ellipticity is lost when $\varrho(x)$ approaches zero, a legitimate reason for calling $\varrho(x)$ the *distortion coefficient*. The natural setting for such degenerate elliptic PDEs is in the space of functions having finite energy

$$\mathcal{E}[u] = \int_\Omega \langle A(x, \nabla u), \nabla u \rangle \, dx = \int_\Omega |\nabla u|_A^p \, dx < \infty . \tag{1.8}$$

This does not necessarily mean that $|\nabla u| \in L^p(\Omega)$, as the distortion coefficient $\varrho = \varrho(x)$ may assume arbitrarily small values. Also L^p-integrability of the gradient is insufficient for the energy to be finite, as ϱ may approach infinity.

Degenerate elliptic PDEs have begun to emerge primarily as a consequence of studies in the geometric function theory and parallel advances in nonlinear elasticity. The category of functions that those theories consider consists of mappings $f = (f^1, \ldots, f^n)\colon \Omega \to \mathbb{R}^n$ whose differential matrix $Df(x) = \left[\dfrac{\partial f^i}{\partial x_j} \right]$ has finite distortion

1.4. *The Distortion Inequality*:

$$|Df(x)|^n \leqslant K(x) \det Df(x) \tag{1.9}$$

where $1 \leqslant K(x) < \infty$ is a given measurable (outer distortion) function. The regularity and the geometric properties of these mappings are issues of fundamental importance [11, 40]. To get the subject of the ground one must integrate the Jacobian determinant $J(x, f) = \det Df(x)$. Then, in quest of the Caccioppoli type estimates, we are faced with rather delicate problems regarding integration by parts

$$\int_\Omega \varphi(x)\, J(x, f) \, dx = \int_\Omega \varphi \, df^1 \wedge \ldots \wedge df^n = - \int_\Omega f^1 \, d\varphi \wedge df^2 \wedge \ldots \wedge df^n \tag{1.10}$$

against an arbitrary test function $\varphi \in C_0^\infty(\Omega)$. The relevance of the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ as a natural domain in which one looks for the solutions of the distortion inequality (1.9) is evident. Note, on the side, that such solutions are continuous [26]. However, somewhat weaker and more natural initial regularity assumptions suffice if the distortion function $K = K(x)$ is exponentially integrable; that is, whenever

$$\int_\Omega e^{\gamma K(x)} \, dx < \infty , \qquad \text{for some } \gamma > 0 . \tag{1.11}$$

On the analogy of the finite energy solutions to the second order PDEs we obtain a viable theory of deformations having finite volume integral

$$\int_\Omega J(x, f) \, dx < \infty \tag{1.12}$$

see [36-38, 40, 51, 52], and the references given there. The role of this assumption as a natural one can be further motivated by the fact that (1.12) always holds for weakly differentiable homeomorphisms, a case intensively studied in nonlinear elasticity.

Let us close our list of examples of the natural domains with

1.5. *The Complex Beltrami System*:

$$\frac{\partial F}{\partial \bar z} = \mu(z) \frac{\partial F}{\partial z} , \qquad \text{in } \Omega \subset \mathbb{C} \tag{1.13}$$

for the mapping $F = (F^1, F^2, \ldots, F^n) \colon \Omega \to \mathbb{C}^n$. Here, the Beltrami coefficient $\mu = \mu(z)$ is a measurable function valued in the space $\mathbb{C}^{n \times n}$ of $n \times n$ matrices with complex entries, such that

$$|\mu(z)\,\xi| \leqslant k|\xi|\,, \qquad \text{with } k < 1\,, \qquad \text{for all } \xi \in \mathbb{C}^n.$$

Note that the planar mappings of bounded distortion are governed by this linear equation, where $n = 1$. The natural domain for (1.13) remains the same; that is, the Sobolev space $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C}^n)$. One feature that makes the equations of type (1.13) special ones is that they belong to the same homotopy class of elliptic PDEs as the uncoupled system of the Cauchy-Riemann equations,

$$\frac{\partial F}{\partial \bar z} = \left(\frac{\partial F^1}{\partial \bar z}, \ldots, \frac{\partial F^n}{\partial \bar z} \right) = 0 \quad \in \mathbb{C}^n. \tag{1.14}$$

Indeed, the following one parameter family of elliptic operators

$$\frac{\partial}{\partial \bar z} - t\mu(z)\,\frac{\partial}{\partial z}\,, \qquad 0 \leqslant t \leqslant 1 \tag{1.15}$$

defines a path between (1.14) and (1.13). In spite of this homotopy equivalence the equations at (1.13) exhibit different geometric properties. For instance, the unique continuation property fails (even for Lipschitz solutions) when k is far from zero, see [50]. This remains in sharp contrast with the holomorphic solutions of the Cauchy-Riemann system, $k = 0$.

2. Caccioppoli estimates in the natural setting

We shall use the examples of the previous section to briefly review the Caccioppoli estimates for homogeneous equations in their natural setting. Hopefully these examples contain sufficient degree of generality to grasp a more global perspective. Perhaps the most familiar is the Caccioppoli estimate for the harmonic equation, $\operatorname{div} \nabla u = 0$. Integration by parts yields

$$\|\varphi \nabla u\|_2 \ll \|u \nabla \varphi\|_2\,, \qquad \varphi \in C_0^\infty(\Omega) \tag{2.1}$$

We refer the reader to [14] for more general results. However, the failure of this estimate for exponents $s \neq 2$ is less noted in the literature. With the aid of singular integrals we inevitably obtain additional term in the right hand side

$$\|\varphi \nabla u\|_s \ll \|u \nabla \varphi\|_s + \|\nabla u \otimes \nabla \varphi\|_{\frac{ns}{n+s}}\,, \tag{2.2}$$

for all $s \geqslant \dfrac{n}{n-1}$.

Similarly, for the p-harmonic equation, $\operatorname{div} |\nabla u|^{p-2} \nabla u = 0$, $1 < p < \infty$, the analogue of (2.1) reads as

$$\|\varphi \nabla u\|_p \ll \|u \nabla \varphi\|_p\,, \qquad \varphi \in C_0^\infty(\Omega)\,. \tag{2.3}$$

Here as well, the natural exponent p cannot be replaced by any other exponent. We still have an estimate like (2.2), but only with s sufficiently close to p, see [13].

More generally, the Caccioppoli estimates in the natural setting for the degenerate

A-harmonic equation take the form

$$\|\varphi\,|\nabla u|_A\|_p \ll \|u\,|\nabla\varphi|_A\|_p\,, \qquad \varphi\in C_0^\infty(\Omega) \tag{2.4}$$

for every finite energy solution. As before (because the equation is of second order) there are no analogous estimates beyond the natural exponent p.

Matters are quite different for the first order systems of PDEs. First we take on stage the distortion inequality (1.9) with $1 \leqslant K(x) \leqslant K$ a.e.

In the natural Sobolev class of mappings $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ we have

$$\|\varphi Df\|_n \leqslant nK\|f\otimes\nabla\varphi\|_n\,, \qquad \varphi\in C_0^\infty(\Omega)\,. \tag{2.5}$$

One of the major recent advances in the geometric function theory [35], [40, 41] are the Caccioppoli inequalities with exponents slightly below and slightly above the dimension n. Precise statements are given latter on. It is the case of unbounded distortion that drives us into truly new investigations. Let us assume that the distortion function $K = K(x)$ is exponentially integrable; that is,

$$\int_\Omega e^{\gamma K(x)}\,\mathrm{d}x < \infty\,. \tag{2.6}$$

If $\gamma = \gamma(n)$ is sufficiently large then the solutions of the distortion inequality (1.9) (those having finite volume integral) belong to the Sobolev class $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, and we have

$$\|\varphi Df\|_n \ll \|Kf\otimes\nabla\varphi\|_n\,, \qquad \varphi\in C_0^\infty(\Omega)\,. \tag{2.7}$$

Analogous estimates beyond this natural setting will be discussed a little further on.

We shall now look more closely at the elliptic system (1.13). The first question we wish to address concerns the class of elliptic operators in which the complex dimension of the target space equals 1. There are two components in this class, each represented by one of the two Cauchy-Riemann operators

$$\frac{\partial}{\partial\bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{2.8}$$

The point of a distinction between these two components is that the solutions to a system represented by $\dfrac{\partial}{\partial\bar z}$ are orientation preserving (Jacobian is nonnegative), whereas those which solve a system represented by $\dfrac{\partial}{\partial z}$ are orientation reversing. The fundamental link between these two components is established via a singular integral operator

$$S : L^p(\mathbb{C}) \to L^p(\mathbb{C})\,, \qquad 1 < p < \infty\,. \tag{2.9}$$

This operator, which we dignify with the name Beurling-Ahlfors transform, is completely characterized by the identity

$$S\circ\frac{\partial}{\partial\bar z} = \frac{\partial}{\partial z} : W^{1,p}(\mathbb{C}) \to L^p(\mathbb{C})\,. \tag{2.10}$$

The relevance of this operator to the elliptic PDEs has been evident to researchers for about fifty years [2, 7, 9, 10]. Much creative effort has gone into establishing its p-

norms, [33, 3-5, 58]

$$S_p \stackrel{\text{def}}{=} \|S : L^p(\mathbb{C}) \to L^p(\mathbb{C})\|, \qquad 1 < p < \infty . \tag{2.11}$$

Following [33] we recall one of the outstanding problems in this field

CONJECTURE 2.1.

$$S_p = \begin{cases} p-1 & \text{if } p \geq 2 \\ \dfrac{1}{p-1} & 1 < p \leq 2 . \end{cases} \tag{2.12}$$

Up to now we know that S_p is at most twice as large as that conjectured in (2.12), see [58].

The operator S can also be considered on the space $L^p(\mathbb{C}, \mathbb{C}^n)$, which we define by the rule $S(F_{\bar{z}}) = F_z$. A general fact, attributed to Marcinkiewicz and Zygmund, is that the norm of this extension remains the same, see [19] for further comments and references. In this way we arrive at the estimate

$$\int |\varphi|^s \, |DF|^s \ll \int |\nabla\varphi|^s \, |F|^s, \qquad \varphi \in C_0^\infty(\Omega) \tag{2.13}$$

for the full range of the exponents $s \in (q_k, p_k)$ in which the implied constant is independent of n. The so-called critical exponents q_k and p_k are Hölder conjugate numbers determined by the equation

$$S_{q_k} = S_{p_k} = \frac{1}{k}, \qquad q_k < 2 < p_k . \tag{2.14}$$

Although we have not yet been able to verify Conjecture 2.1, the full range of the exponents for the Caccioppoli estimate at (2.13) has already been found in case $n = 1$ [2]. Precisely, (2.13) holds for all s satisfying

$$1 + k < s < 1 + \frac{1}{k} . \tag{2.15}$$

In [50] the authors encounter related questions with $n \geq 1, 2, \ldots$.

3. REDUCTION TO THE FIRST ORDER SYSTEMS

It is quite relevant here to bring up the second order equations again. As an example, we begin with the linear elliptic system of Piccinini and Spagnolo [60] in two variables

$$\frac{\partial}{\partial x}\left[A(x,y)\,\frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial y}\left[B(x,y)\,\frac{\partial U}{\partial y}\right] = 0 \tag{3.1}$$

for a vector field $U = (u^1, u^2, \ldots, u^n) \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^n)$. The coefficients $A = A(x,y)$ and $B = B(x,y)$ are measurable functions valued in the space $\mathbb{R}^{n \times n}$ of $n \times n$ matrices.

Following the lead of the Cauchy-Riemann equations we write (3.1) as

$$\begin{cases} A(x, y)\,\dfrac{\partial U}{\partial x} = \dfrac{\partial V}{\partial y} \\[2mm] B(x, y)\,\dfrac{\partial U}{\partial y} = -\,\dfrac{\partial V}{\partial x} \end{cases} \tag{3.2}$$

where $V = (v^1, v^2, \ldots, v^n)$ is a vector field in $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. This procedure results in the first order elliptic system of type (1.13) for the complex field $F = U + iV \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C}^n)$. It shows that the Caccioppoli estimates will remain valid for the second order systems (3.1), provided that we express them in terms of both U and its conjugate field V. Precisely, we have

$$\int_\Omega |\varphi|^s(|U_x| + |U_y| + |V_x| + |V_y|)^s \ll \int_\Omega |\nabla\varphi|^s(|U| + |V|)^s \tag{3.3}$$

in the same range of the exponents; that is, for $s \in (q_k, p_k)$.

This simple observation is at the heart of many more examples [47]. To every solution of an elliptic PDE

$$\operatorname{div} \boldsymbol{A}(\mathrm{x})\,\nabla \mathrm{u} = 0 \tag{3.4}$$

$$K^{-1}(x)\,|\xi|^2 \leqslant \langle \boldsymbol{A}(x)\,\xi, \xi\rangle \leqslant K(x)\,|\xi|^2 \tag{3.5}$$

there corresponds a quasiharmonic field $\mathcal{F} = [B, E]$, a pair of vector fields $E = \nabla u$ and $B = \boldsymbol{A}(x)\,\nabla u$, with $\operatorname{div} B = 0$ and $\operatorname{curl} E = 0$. They are coupled by a distortion inequality

$$|B|^2 + |E|^2 \leqslant (K + K^{-1})\langle B, E\rangle \tag{3.6}$$

where $K = K(x) \geqslant 1$ can be constant (uniformly elliptic case) or a measurable function, finite almost everywhere. In many respects this reduction of the second order equation to the first order system seems to be an excellent generalization of the familiar Beltrami system in the complex plane; far more geometric than (3.4). Continuing this analogy we have established dimension free Caccioppoli type estimates [47] for the exponents s satisfying:

$$q \stackrel{\mathrm{def}}{=} \frac{14K - 12}{7K - 5} < s < \frac{14K - 12}{7K - 7} \stackrel{\mathrm{def}}{=} p \tag{3.7}$$

provided $1 \leqslant K(x) < K$ a.e. These estimates yield the following

THEOREM 3.1 (dimension free regularity). *Every very weak solution to the equation* (3.4) *in the Sobolev class* $W^{1,q}_{\mathrm{loc}}(\Omega)$ *actually belongs to* $W^{1,p}_{\mathrm{loc}}(\Omega)$.

4. VERY WEAK SOLUTIONS

We have used the term very weak solution of a given PDE to describe a solution which is less regular (at least a priori) than those in the natural space. What we want to achieve here is the extent to which the very weak solutions are in fact the natural ones. This idea is best visualized in case of the p-harmonic equation [45]. Recall that a p-harmonic function $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ is a local minimum of the energy integral

$$\mathcal{E}[u] = \int_\Omega |\nabla u(x)|^p \, dx, \qquad 1 < p < \infty. \tag{4.1}$$

This means that $\mathcal{E}[u+\varphi] \geq \mathcal{E}[u]$ for every test function $\varphi \in C_0^\infty(\mathbb{R}^n)$. Equivalently,

$$\int_{\mathbb{R}^n} (|\nabla u + \nabla\varphi|^p - |\nabla u|^p) \geq 0, \qquad \varphi \in C_0^\infty(\Omega). \tag{4.2}$$

This seemingly insignificant change of definition of the variational inequality has for reaching implications. First notice the this new inequality at (4.2) makes sense for all $u \in W^{1,s}_{\mathrm{loc}}(\Omega)$, with $s \geq \max\{1, p-1\}$; simply because the integrand is dominated by

$$|\nabla\varphi| \big(|\nabla u|^{p-1} + |\nabla\varphi|^{p-1} \big) \in L^1(\Omega).$$

A new class of functions, referred to as weak minima, has emerged in the calculus of variations and PDEs. The Euler-Lagrange equation for the weak-minima

$$\int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla\varphi \rangle = 0, \qquad \varphi \in C_0^\infty(\Omega) \tag{4.3}$$

is still valid, where we notice that $|\nabla u|^{p-2}\nabla u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.

While substantial progress has been made, the very weak solutions for nonlinear PDEs are still in their infancy. Their development strongly depends on the Caccioppoli type estimates below the natural setting.

To explain the points of difficulties that occur beyond the natural domain we return to the nonhomogeneus p-harmonic equation (1.3) in the entire space $\mathbb{R}^n$. Let us begin with its integral form

$$\int_{\mathbb{R}^n} \langle |\nabla u|^{p-2} \nabla u, \nabla\varphi \rangle = \int_{\mathbb{R}^n} \langle F, \nabla\varphi \rangle, \qquad \varphi \in C_0^\infty(\mathbb{R}^n). \tag{4.4}$$

This time we assume that the given vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ belongs to $L^{\lambda q}(\mathbb{R}^n, \mathbb{R}^n)$, for some $\lambda > \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$. We must look for the solutions u whose gradient lies in $L^{\lambda p}(\mathbb{R}^n, \mathbb{R}^n)$, this is the only choice. Unluckily, the spaces $L^{\lambda q}(\mathbb{R}^n, \mathbb{R}^n)$ and $L^{\lambda p}(\mathbb{R}^n, \mathbb{R}^n)$ are no longer dual to each other. That is why the existence of such solutions cannot be deduced with the aid of the Browder-Minty theory of monotone operators. As for the local estimates, the reader will observe that $|\nabla u|^{p-2}\nabla u \in L^{\lambda q}(\mathbb{R}^n, \mathbb{R}^n)$. This makes it legitimate to test the identity (4.4) with functions φ whose gradient lies in the dual space $L^s(\mathbb{R}^n, \mathbb{R}^n)$, where $s = \frac{\lambda q}{\lambda q - 1}$.

Unfortunately for the case below the natural setting, when $\lambda < 1$ the Sobolev exponent λp is smaller than s; that is $\lambda p < \frac{\lambda q}{\lambda q - 1}$. The usual test functions $\varphi(x) = \eta(x)[u(x) - \text{const}]$, and their more sophisticated variants that involve powers and truncations of u, are no longer allowed; simply because $\nabla\varphi$ is essentially proportional to ∇u. It is natural to try the expression $|\nabla u|^{\lambda p - p}\nabla u$ in place of $\nabla\varphi$ in order to end up with the integral of $|\nabla u|^{\lambda p}$ in the left hand side of (4.4). But formally this operation is inadmissible since $|\nabla u|^{\lambda p - p}\nabla u$ fails to be a gradient field. In this situation one might ask: how far is $|\nabla u|^{\varepsilon}\nabla u$ from a gradient field in the metric of $L^s(\mathbb{R}^n, \mathbb{R}^n)$, $\varepsilon = (\lambda - 1)p$?

4.1. *Projection onto Gradient Fields.*

Given $s > 0$ and $\varepsilon > \frac{1}{s} - 1$, the closest point projection of $|\nabla v|^{\varepsilon}\nabla v$ into the space of gradient fields comes to the stage. Consider the inequality

$$\inf_{\varphi \in W^{1,s}(\mathbb{R}^n)} \left\| |\nabla v|^{\varepsilon}\nabla v - \nabla\varphi \right\|_s \leqslant k_s(n, \varepsilon)\|\nabla v\|_{s+\varepsilon s}^{1+\varepsilon} \tag{4.5}$$

for every $v \in W^{1, s+\varepsilon s}(\mathbb{R}^n)$, where $0 \leqslant k_s(n, \varepsilon) \leqslant 1$. This inequality is trivial with $k_s(n, \varepsilon) = 1$. On the other hand, $k_s(n, 0) = 0$. More radical attempts, through an interpolation argument [45], reveal that $k_s(n, \varepsilon) < 1$ for ε sufficiently small, positive or negative, see also [65]. In light of these facts a search for the smallest constant $k_s(n, \varepsilon)$ in (4.5) becomes quite appealing.

CONJECTURE 4.1. *The projection estimates at* (4.5) *hold with a constant* $k_s(n, \varepsilon) < 1$ *for all parameters* $s > 1$ *and* $\varepsilon > \frac{1}{s} - 1$.

The case of positive values of ε has already been solved in [39] by arguments which rely on $C^{1,\alpha}$-regularity of the p-harmonic functions, see [34] for the seminal ideas. Thus the problem remains unsolved only when ε is away from zero on the negative side.

Returning to the p-harmonic equation, we take the minimizer $\varphi \in W^{1,s}(\mathbb{R}^n)$ in (4.5) to serve as a test function at (4.4), $s = \dfrac{\lambda q}{\lambda q - 1}$ and $\varepsilon = (\lambda - 1)p$. This yields

$$\int_{\mathbb{R}^n} \langle |\nabla u|^{p-2}\nabla u, \quad |\nabla u|^{\lambda p - p}\nabla u - \nabla\varphi\rangle \leqslant k_s(n, \varepsilon)\int_{\mathbb{R}^n} |\nabla u|^{\lambda p}$$

Once $k_s(n, \varepsilon) < 1$, by virtue of the equation (4.4), we conclude with the desired estimate

$$\|\nabla u\|_{\lambda p}^{p-1} \leqslant \frac{1 + k_s(n, \varepsilon)}{1 - k_s(n, \varepsilon)}\|F\|_{\lambda q} \tag{4.6}$$

4.2. *A Study of Nonlinear PDEs via Conjugate Fields.*

A reduction of a second order equation to the first order system can be made via exterior algebra. We illustrate this idea by considering the p-harmonic equation

$$\mathrm{d}^*\left(|\mathrm{d}u|^{p-2}\,\mathrm{d}u\right) = 0 \tag{4.7}$$

where d^* is the Hodge star codifferential defined on 1-forms. Actually slightly more general setting will provide us with a better framework for the discussion. In this greater generalities, we view u as a differential form of arbitrary degree $0 \leqslant l < n$. The famous Poincaré Lemma states that there is an $(l+1)$-form v such that

$$|\mathrm{d}u|^{p-2}\,\mathrm{d}u = \mathrm{d}^* v \tag{4.8}$$

where v is a differential form of degree $l+1$. Equivalently,

$$|\mathrm{d}^* v|^{q-2}\,\mathrm{d}^* v = \mathrm{d}u\,.$$

Thanks to the identity $dd = 0$, this yields the so-called Hodge dual equation for v

$$\mathrm{d}(|\mathrm{d}^* v|^{q-2}\,\mathrm{d}^* v) = 0\,. \tag{4.9}$$

We refer to the pair (u, v) as (p, q)-conjugate fields, see [35, 48, 42] and [38] for further developments and applications.

Caccioppoli's estimates beyond the natural exponent can be formulated for this system of first order PDEs as follows

$$\int_\Omega |\varphi|^{\lambda pq}\big(|\mathrm{d}u|^p + |\mathrm{d}^* v|^q\big)^\lambda \ll \int_\Omega \big(|u|^p|\varphi|^q|\nabla\varphi|^p + |v|^q|\varphi|^p|\nabla\varphi|^q\big)^\lambda \tag{4.10}$$

for all $\varphi \in C_0^\infty(\Omega)$, and $\lambda > \lambda(n, p)$, where $\max\left\{\frac{1}{p}, \frac{1}{q}\right\} \leqslant \lambda(n, p) < 1$. We conjecture that (4.10) holds for all $\lambda > \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

4.3. *The p-Harmonic Transform.*

Returning to the scalar case, the p-harmonic equation $\operatorname{div} |\nabla u|^{p-2}\nabla u = \operatorname{div} F$ gives rise to a nonlinear operator

$$\mathrm{H}_p\colon L^q(\mathbb{R}^n, \mathbb{R}^n) \to L^q(\mathbb{R}^n, \mathbb{R}^n)$$

which carries a given vector field $F \in L^q(\mathbb{R}^n, \mathbb{R}^n)$ into $|\nabla u|^{p-2}\nabla u \in L^q(\mathbb{R}^n, \mathbb{R}^n)$. Of course, the space $L^q(\mathbb{R}^n, \mathbb{R}^n)$ serves us with the natural domain of the definition of H_p. However, it is of great interest to know whether H_p extends continuously beyond this domain. Because of this, the pressing question is:

QUESTION 4.2. *For what exponents $s > 1$ we have the estimate*

$$\|\mathrm{H}_p F - \mathrm{H}_p G\|_s \ll \|F - G\|_s^\alpha \big(\|F\|_s + \|G\|_s\big)^{1-\alpha}, \tag{4.11}$$

with $F, G \in L^q(\mathbb{R}^n, \mathbb{R}^n) \cap L^s(\mathbb{R}^n, \mathbb{R}^n)$, and some $0 < \alpha = \alpha(n, p, s) \leqslant 1$?

This is certainly true if $s = q$, the natural setting. In the linear case ($p = 2$) the operator H_2 reduces to the second order Riesz transforms $\mathrm{H}_2 F = \mathrm{R}\langle \mathrm{R}, F\rangle$, where $\mathrm{R} = (R_1, \ldots, R_n)$. Therefore, in this case (4.11) holds with all $1 < s < \infty$.

4.4. *The Grand Lebesgue Norms.*

Trying to determine how far we can go beyond the natural setting [30] without losing continuity of H_p have led us to the spaces slightly below $L^q(\mathbb{R}^n, \mathbb{R}^n)$. The so-called grand Lebesgue space $L^{q)}(\mathbb{R}^n, \mathbb{R}^n)$, [44, 45, 62] is worth recording. By the definition, it consists of vector fields $F \in \bigcap_{1 \leq s < q} L^s(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\|F\|_{q)} = \sup_{1 \leq s < q} \left[(q-s) \int_{\mathbb{R}^n} |F|^s \right]^{\frac{1}{s}} < \infty . \tag{4.12}$$

The grand Sobolev space $W^{1,p)}(\mathbb{R}^n)$ is furnished with the seminorm

$$\|u\|_{1,p)} = \|\nabla u\|_{p)} . \tag{4.13}$$

These spaces together with several extensions were profitably employed in PDEs [47]. Hölder continuity of the operator $\mathrm{H}_p \colon L^{q)}(\mathbb{R}^n, \mathbb{R}^n) \to L^{q)}(\mathbb{R}^n, \mathbb{R}^n)$ established in [30], follows from the estimate

$$\|\mathrm{H}_p F - \mathrm{H}_p G\|_{q)} \ll \|F - G\|_{q)}^{\alpha} \left(\|F\|_{q)} + \|G\|_{q)} \right)^{1-\alpha} \tag{4.14}$$

with $0 < \alpha = \alpha(p) \leq 1$.

5. What do we gain from weak estimates?

There are many examples that illustrate the role of the very weak solutions in PDEs, geometric analysis and some areas of applied mathematics. Let us sketch some of them.

5.1. *Measure in the right hand side.*

Throughout this section Ω stands for a bounded regular domain in $\mathbb{R}^n$, $n \geq 2$. For example, Lipschitz domains are regular. Let μ be a Radon measure on Ω. The Dirichlet problem

$$\begin{cases} \operatorname{div} \nabla v = \mu \\ v = 0 \quad \text{on } \partial\Omega \end{cases} \tag{5.1}$$

can be solved explicitly by means of the Green's function for Ω:

$$\nabla v = \int_{\Omega} \nabla_x G(x, y) \, \mathrm{d}\mu(y).$$

Elementary analysis of this integral shows that the vector field $F = \nabla v$ belongs to the grand Lebesgue space $L^{q)}(\Omega, \mathbb{R}^n)$, with $q = \dfrac{n}{n-1}$. Precisely, we have

$$\|F\|_{q)} \ll \int_{\Omega} |\mathrm{d}\mu| . \tag{5.2}$$

If, moreover, μ happens to be absolutely continuous with respect to the Lebesgue measure, then

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Omega} |F|^{q-\varepsilon} = 0 . \tag{5.3}$$

This latter condition characterizes the closure of $L^q(\Omega, \mathbb{R}^n)$ in $L^{q)}(\Omega, \mathbb{R}^n)$.

It is at this stage where the Caccioppoli estimates with exponents at (3.7) come to play the critical role in establishing the existence and uniqueness of the solution of a general elliptic linear equation

$$\operatorname{div} \boldsymbol{A}(x)\, \nabla u = \mu\,, \qquad u \in W_0^{1,q)}(\Omega),\ q = \frac{n}{n-1}\,. \tag{5.4}$$

Here we assume that the distortion function at (3.5) satisfies

$$1 \leqslant K(x) \leqslant K < \frac{7n-12}{n-2}\,. \tag{5.5}$$

We thus infer from (5.2) that such solutions are uniformly controlled by the total variation of μ;

$$\|\nabla u\|_{\frac{n}{n-1})} \ll \fint_\Omega |\mathrm{d}\mu|\,. \tag{5.6}$$

Viewing the situation in relation to the nonlinear PDEs we turn next to the Dirichlet problem for the p-harmonic equation

$$\begin{cases} \operatorname{div} |\nabla u|^{p-2}\, \nabla u = \mu \\ u = 0 \quad \text{on} \quad \partial\Omega\,. \end{cases} \tag{5.7}$$

By virtue of the earlier estimates for the Laplace equation we may express μ as divergence of a vector field $F \in L^{q)}(\Omega, \mathbb{R}^n)$, $q = \dfrac{n}{n-1}$. The Hölder conjugate of this exponent is exactly the dimension n of the domain $\Omega \subset \mathbb{R}^n$. That is why we can handle the case $p = n$, directly by (4.14), see [30].

THEOREM 5.1. *For each Radon measure μ on Ω, the n-harmonic equation*

$$\operatorname{div} |\nabla u|^{n-2}\, \nabla u = \mu \tag{5.8}$$

admits exactly one solution $u \in W_0^{1,n)}(\Omega)$.

As a particular consequence of the estimates slightly below the dimension [45] we also conclude that the very weak solutions of (5.8) in the Sobolev class $W_0^{1,n-\varepsilon}(\Omega)$ actually belong to $W_0^{1,n)}(\Omega)$ and, therefore, are unique as well. Moreover, if μ is absolutely continuous with respect to the Lebesgue measure, then

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega |\nabla u|^{n-\varepsilon} = 0 \tag{5.9}$$

There are of course many more possible spaces in which the n-harmonic equation (5.8) admits unique solution [17, 6, 8, 20, 53, 16], but such spaces lay beyond the range of this survey.

In much the same way the positive answer to the Question 4.2 would enable us to solve (uniquely) the p-harmonic Dirichlet problem (5.7). To this effect one needs only establish (4.11) with s slightly below $\dfrac{n}{n-1}$.

The Dirichlet problem explains rather convincingly why we must look for the Caccioppoli estimates much below the natural exponent. There is an important place for such estimates in the theory of nonlinear PDEs, though they remain yet to be fully understood.

5.2. *Removability of singularities.*

Rather than discussing this theme in a general context it is perhaps worthwhile to illustrate how the very weak estimates imply removability of singularities for K-quasiregular mappings. All that is needed for our discussion is the following extension of the Caccioppoli estimate at (2.5):

$$\|\varphi Df\|_s \ll \|f \otimes \nabla\varphi\|_s, \tag{5.10}$$

for exponents s in a range $q(n, K) \leq s < n$, see [35] and [41].

THEOREM 5.2. *Let E be a closed subset of a domain $\Omega \subset \mathbb{R}^n$ of Hausdorff dimension* $\dim E < n - s$. *Then every bounded K-quasiregular map $f : \Omega \backslash E \to \mathbb{R}^n$ extends to a quasiregular mapping of Ω.*

The proof of this runs as follows. Under the above hypotheses the s-capacity of E is zero. On the other hand (5.10) gives us the estimates

$$\|\varphi Df\|_s \ll \|f \otimes \nabla\varphi\|_s \ll \|\nabla\varphi\|_s$$

for every $\varphi \in C_0^\infty(\Omega \backslash E)$, where the implied constants do not depend on the particular test function φ. Clearly this remains true for all $\varphi \in C_0^\infty(\Omega)$; after all, E has zero s-capacity and zero Lebesgue measure. We thus infer from it that $f \in W^{1,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Moreover, f satisfies the distortion inequality

$$|Df(x)|^n \leq K \det Df(x)$$

almost everywhere in the entire domain Ω. All that we need to here is the higher integrability of the differential of the solutions in the Sobolev class $W^{1,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Thus f lies in the natural space $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, as desired.

New phenomena [31] seem to suggest that Caccioppoli type estimates with Sobolev exponent smaller than the dimension cannot be used so effectively for mappings between manifolds. However, their analytic and geometric consequences have yet to be explored.

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