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ON SOME RECENT DEVELOPMENTS OF THE THEORY OF SETS OF FINITE PERIMETER

ABSTRACT. — In this paper we describe some recent progress on the theory of sets of finite perimeter, currents, and rectifiability in metric spaces. We discuss the relation between intrinsic and extrinsic theories for rectifiability.

KEY WORDS: Sets of finite perimeter; *BV* functions; Currents; Rectifiability; Plateau problem.

1. INTRODUCTION

In my talk I will describe some recent progress on the theory of sets of finite perimeter and on its natural higher codimension extension, the theory of currents. I will also try to illustrate the link and the continuity, that I feel quite strongly, between the pioneering work of Caccioppoli and De Giorgi and the present work of many mathematicians, including me.

The summary of the talk is the following:

Part 1. We review the classical theory of sets of finite perimeters and currents.

Part 2. We recall the «metric» theory of De Giorgi.

Part 3. We describe some recent work on intrinsic theories of rectifiability and of sets of finite perimeter.

2. THE PLATEAU PROBLEM

Let $\Gamma = \Gamma^k \subset \boldsymbol{R}^n$ be an embedded C^1 surface without boundary. The Plateau problem is

$$\min \{ \text{Area}(M) : \partial M = \Gamma \},$$

with suitable definitions of $(k+1)$-dimensional surface area and boundary.

- Parametric methods work only in particular situations: $k = 0$ (curves), $k = 1$ (conformal parametrizations), graphs, see for instance [16].
- Therefore it was natural to look for general and non-parametric definitions of surface area and boundary.

3. CACCIOPPOLI SETS

Caccioppoli sets, or sets of finite perimeter, can be characterized by the following definition.

DEFINITION 3.1. *E is a set of finite perimeter if there exist polyhedral sets E_h such that $E_h \to E$ locally in measure and*

$$\sup_{h \in \mathbf{N}} \text{Area}\,(\partial E_h) < +\infty\,.$$

It turns out that for any set of finite perimeter a weak Gauss-Green formula holds, namely

$$\int_E \operatorname{div} g\, dx = -\int_{\mathbf{R}^n} \langle g, \nu_E \rangle\, d\mu_E \qquad \forall g \in C_c^\infty(\mathbf{R}^n, \mathbf{R}^n).$$

In modern language $\nu_E \mu_E$ is the polar decomposition of $D\chi_E$, the derivative of χ_E (the characteristic function of E) in the sense of distributions.

The project of Caccioppoli was not limited to codimension 1 surfaces. In the more general setting a weak form of Stokes theorem arises.

4. DE GIORGI'S REDUCED BOUNDARY AND RECTIFIABILITY

DEFINITION 4.1 (Reduced boundary). *Let $\mathcal{F}E$ be the set of all points $x \in \operatorname{spt} \mu_E$ such that*

$$\exists\ \bar\nu_E(x) := \lim_{r \to 0^+} \frac{1}{\mu_E(B_r(x))} \int_{B_r(x)} \nu_E(y)\, d\mu_E(y)$$

and $|\bar\nu_E(x)| = 1$.

De Giorgi showed in [8] the the blow-up of E at any point $x \in \mathcal{F}E$ is the halfspace orthogonal to $\bar\nu_E(x)$ and containing $\bar\nu_E(x)$. Using this information he proved that $\mathcal{F}E$ is rectifiable, *i.e.*

$$(*) \qquad \mu_E\left(\mathcal{F}E \setminus \bigcup_{i=0}^{\infty} \Gamma_i\right) = 0$$

for suitable C^1 embedded hypersurfaces Γ_i.

Three years later Federer recognized in [11] the role of the Hausdorff measure $\mathcal{H}^{n-1}$ and showed that one can replace $\mathcal{H}^{n-1}$ by μ_E in (*).

REMARKS. (1) Notice that in (*) one can consider, instead of C^1 hypersurfaces Γ_i, either:

(*a*) level sets of C^1 functions without critical points (by the implicit function theorem)

(*b*) Lipschitz images of subsets of $\mathbf{R}^{n-1}$ (by Whitney's extension theorem).

Condition (*a*) is intrinsic, while condition (*b*) has an extrinsic nature.

(2) No perimeter measure lives out of $\mathcal{F}E$ and

$$\bar\nu_E = \nu_E \qquad \mu_E\text{-a.e. in } \mathbf{R}^n,$$

by Besicovitch differentiation theorem. Therefore the weak Gauss-Green formula becomes much closer to the classical one:

$$\int_E \operatorname{div} g \, dx = -\int_{\mathcal{F}E} \langle g, \nu_E \rangle \, d\mathcal{H}^{n-1} \qquad \forall g \in C_c^\infty(\boldsymbol{R}^n, \boldsymbol{R}^n).$$

The only difference is in the way normal and boundary are understood (topological versus measure-theoretic).

5. The Federer-Fleming theory of currents

- k-dimensional currents are defined by duality with $\mathcal{D}^k(\boldsymbol{R}^n)$, the space of smooth k-forms with compact support (this idea goes back to De Rham).
- The pull-back and the exterior derivative operators on forms induce a push-forward operator and a boundary operator on currents, namely

$$f_\# T(\omega) := T(f^\# \omega) \quad \text{whenever } f : \boldsymbol{R}^n \to \boldsymbol{R}^m$$

for any $\omega \in \mathcal{D}^k(\boldsymbol{R}^m)$ and

$$\partial T(\omega) := T(d\omega) \quad \text{for any } \omega \in \mathcal{D}^{k-1}(\boldsymbol{R}^n).$$

- Inside the (large) class of k-currents one can single out the rectifiable ones, associated to the integration on a k-rectifiable set M of an integer multiplicity function θ, *i.e.*

$$T(\omega) := \int_M \theta(x) \langle \tau(x), \omega(x) \rangle \, d\mathcal{H}^k(x).$$

Here τ is a simple unit k-vector providing an orientation of the (approximate) tangent space to M and ω is thought as a k-covector field.
- This dual representation provides the notion of *mass* $M(T)$ of a current T. For rectifiable currents T as above it reduces to $\int_M \theta | \, d\mathcal{H}^k$.

The main results of the theory of currents are [10]:

Boundary rectifiability Theorem. *If T is rectifiable and ∂T has finite mass, then ∂T is rectifiable.*

Closure and compactness Theorem. *If T_h are rectifiable and the sequence $M(T_h) + M(\partial T_h)$ is bounded, then T_h has converging subsequences (in the natural dual topology) and any limit point is still rectifiable.*

Polyedral approximation Theorem. *If T and its boundary have finite mass, then T can be approximated by polyhedral currents T_h keeping $M(T_h)$ and $M(\partial T_h)$ bounded.*

The theory provides general existence and partial regularity results for the non-parametric Plateau problem and has by now a large class of applications which go much beyond the Plateau problem, for instance:

- geometric evolution problems (*e.g.* the mean curvature flow in any dimension and codimension);
- variational problems involving vector-valued maps;
- description of singularities and energy concentration effects on lower dimensional sets;
- regularity theory (monotonicity, ε-regularity theorems).

6. The metric theory of De Giorgi

Let (X, d) be a metric space. De Giorgi's idea (see [9]) is to define k-currents in X by looking to the duality with metric «k-forms», *i.e. formal* expressions

$$f_0\, df_1 \wedge \ldots \wedge df_k$$

with f_i Lipschitz, $0 \leq i \leq k$, and f_0 bounded.

- In this general setting $df_1 \wedge \ldots \wedge df_k$ has no pointwise meaning.

This approach has some analogies with Cheeger's recent work [7] on Rademacher theorem in a general metric setting.

- De Giorgi's aim was to cover non-oriented objects (*e.g.* varifolds) as well.
- The push-forward operator and the boundary operator are naturally defined in this setting:

$$\varphi_{\#} T(f_0\, df_1 \wedge \ldots \wedge df_k) := T(f_0 \circ \varphi\, df_1 \circ \varphi \wedge \ldots \wedge df_k \circ \varphi)$$

for $\varphi \in \mathrm{Lip}\,(X, Y)$ and

$$\partial T(f_0\, df_1 \wedge \ldots \wedge df_{k-1}) := T(1\ df_0 \wedge \ldots \wedge df_{k-1}).$$

- The mass $\|T\|$ is the least measure μ in X satisfying

$$|T(f_0\, df_1 \wedge \ldots \wedge df_k)| \leq \prod_{i=1}^{k} \mathrm{Lip}\,(f_i) \int_X f_0\, d\mu\, .$$

In a Euclidean or Riemannian setting this definition is consistent with the Federer-Fleming one.

- Rectifiable currents are simply defined as $\varphi_{\#} T_\theta$, where $\varphi \in \mathrm{Lip}\,(\boldsymbol{R}^k, X)$, $\theta \in L^1(\boldsymbol{R}^k, \boldsymbol{Z})$ and T_θ is the canonical k-current in $\boldsymbol{R}^k$ given by

$$T_\theta(f_0\, df_1 \wedge \ldots \wedge df_k) := \int_{\boldsymbol{R}^k} \theta(x)\ f_0(x) \det \nabla f(x)\ dx\, .$$

These definitions appear in the paper [9] entitled *Problema di Plateau generale e funzionali geodetici*. In the same paper De Giorgi raises the question of finding conditions on the metric space X and S ensuring the existence of solutions to the generalized Plateau problem

$$(*) \qquad \min\left\{\|T\|(X) : \partial T = S\right\}$$

in the class of rectifiable currents.

The answer is given in two joint papers with B. Kirchheim [3, 4].

THEOREM. *The closure theorem and the boundary rectifiability theorem for rectifiable currents hold in any complete metric space.*

COROLLARY. *If* (X, d) *is compact, problem* (*) *has a solution, provided the class* $\{T : \partial T = S\}$ *is not empty.*

At this stage one might ask what are the reasons (besides the undoubtable beauty of De Giorgi's definitions) for developing such a general theory. Here are some answers:

- The theory provides a natural language to deal with Lipschiz manifolds, manifolds with singularities, ...
- Some features (the essential ones) of the classical Federer-Fleming theory become more transparent in this general framework. We obtain new results even within the classical theory (*e.g.* rectifiability criteria for k-currents based on slicing or on Lipschitz projections on $\boldsymbol{R}^{k+1}$).
- We provide an existence theorem for the k-dimensional Plateau problem in infinite dimensional Banach spaces (even the Hilbert case was open). In order to overcome the lack of local compactness of the ambient space the proof uses Gromov-Hausdorff convergence, the Ekeland-Phelps variational principle and, in an essential way, the validity of the closure theorem in general metric spaces.

On the negative side, we find important examples of metric spaces where De Giorgi's theory of rectifiable currents does not apply simply because the class of rectifiable currents is very poor. The simplest of these examples is the Heisenberg group H_n.

We recall that $H_n \sim \boldsymbol{C}^n \times \boldsymbol{R}$ is a Lie group with group law (the generic point $x \in H_n$ has coordinates (z, t))

$$(z, t) \oplus (z', t') = \left(z + z', t + t' + 2 \sum_{i=1}^{n} \operatorname{Im}(z_i \bar{z}'_i) \right).$$

Besides the family of translations, there is a family of dilations $\delta_r : H_n \to H_n$ commuting with the group law, defined by

$$\delta_r(z, t) := (rz, r^2 t).$$

An example of left invariant and homogeneous metric is the Korányi metric:

$$d(x, y) := \|y^{-1}x\| \quad \text{where} \quad \|(z, t)\| := \sqrt[4]{|z|^4 + t^2}.$$

The Heisenberg group has topological dimension $2n + 1$ but metric dimension $2n + 2$, due to the behaviour of the distance in the «vertical» directions.

H_n is the simplest example of stratified nilpotent (step 2) Lie group, associated via the exponential map to the nilpotent Lie algebra generated by the $2n$ vector fields

$$X_i := (1, 0, 2y_i), \quad Y_i := (0, 1, -2x_i), \quad i = 1, \ldots, n$$

whose commutators $[X_i, Y_i]$ is $(0, 0, 4)$.

The following result has been proved in [3].

THEOREM. *If* (X, d) *is the Heisenberg group* H_1 *endowed with any left invariant homogeneous metric, then any k-dimensional rectifiable current is identically* 0 *for* $k = 2, 3, 4$.

The underlying reason is that any homogeneous group homeomorphism $L: \mathbf{R}^k \to \to H_1$ is not injective. The Area formula [20, 17] and Pansu's differentiability theorem yield that

$$\mathcal{H}^k(\varphi(A)) = 0 \quad \forall \varphi \in \mathrm{Lip}(A, H_1).$$

7. Upper gradients and intrinsic Sobolev and BV spaces

Here we need more than the metric structure, condidering a metric measure space (X, d, μ).

- (Heinonen-Koskela) Let $u: X \to \mathbf{R}$ be a continuuous function. We say that $g: X \to [0, +\infty]$ is an upper gradient of u if

$$|u(x) - u(y)| \leq \int_\gamma g$$

for any Lipschitz curve $\gamma: [0, 1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$.

- If some upper gradient is in L^p, $p > 1$, we recover a «minimal» upper gradient by minimizing $\int |g|^p \, d\mu$ in the class of upper gradients. For locally Lipschitz functions the minimal upper gradient is the modulus of the gradient in the Euclidean case (or Riemannian case) and is the modulus of the horizontal gradient when E is a Carnot group (in this case μ is the Haar measure).
- We can define the Sobolev space ($p > 1$) or the BV space ($p = 1$) by looking at all maps u such that there exists a sequence of locally Lipschitz functions u_h converging to u in L^1_{loc} and satisfying

$$(*) \qquad \sup_{h \in N} \int_X g_h^p \, d\mu(x) < +\infty .$$

- This definition is consistent with the Euclidean (or Riemannian) theory, with the BV theory in Carnot groups [15] and with the theory of BV functions in weighted measure spaces [5]. For Sobolev spaces, there is consistency with the Folland-Stein theory of horizontal Sobolev spaces.
- By looking at the local minimal energy in (*) one can define a total variation (or perimeter measure). Many properties of BV functions (*e.g.* the coarea formula) are still true in this setting [18].

8. Sets of finite perimeter

Let E be a set with finite perimeter in X and let μ_E be its perimeter measure. In view of De Giorgi's rectifiability theorem in Euclidean spaces, the following two problems are natural:

PROBLEM 1. Can we say that μ_E lives on a «codimension 1» set?

PROBLEM 2. If this is the case, is this set «intrinsically rectifiable»?

In order to give a general answer to Problem 1 we make the following two assumptions:

(A1) (*Ahlfors regularity*) There exist $n \in (1, \infty)$ and constants $c_1 \geq c_2 > 0$ such that

$$c_1 r^n \geq \mu(B_r(x)) \geq c_2 r^n \quad \forall x \in X, \ r \in (0, 1).$$

(A2) (*Poincaré inequality*) There exist $c_P \geq 0$ and $\lambda \geq 1$ such that

$$\int_{B_r(x)} |u(y) - u_r| \, d\mu(y) \leq c_P r \int_{B_{\lambda r}(x)} g \, d\mu$$

for any locally Lipschitz function u and any upper gradient g.

The following result has been proved by the author in [1].

THEOREM. *Under assumptions* (A1), (A2) *the essential boundary*

$$\Sigma := \left\{ x : \liminf_{r \to 0^+} \frac{\mu(B_r(x) \cap E) \wedge \mu(B_r(x) \setminus E)}{\mu(B_r(x))} > 0 \right\}$$

has finite $\mathcal{H}^{n-1}$-measure and μ_E is representable as $\theta \mathcal{H}^{n-1} \llcorner \Sigma$ for a suitable function θ. In addition

$$(*) \qquad \limsup_{r \to 0^+} \frac{\mu_E(B_{2r}(x))}{\mu_E(B_r(x))} < +\infty$$

for μ_E-a.e. $x \in X$.

The classical proof of De Giorgi can not be adapted to this situation, due to the lack of an homogeneous structure and to the failure of Besicovitch differentiation theorem for general measures in a general metric space. We use some ideas about lower semicontinuity and quasi-minimality coming from the theory of minimal surfaces. Notice also that Assumption A1 can be replaced by a more general doubling condition, after giving a good definition of a codimension 1 Hausdorff measure in this setting [2].

When an homogeneous structure (dilations, translations) is present, we can initiate the De Giorgi blow-up procedure and try to give an answer to Problem 2. This is the case for Carnot groups. It turns out that the asymptotic doubling condition (*) implies that the one can differentiate with respect to the perimeter measure μ_E.

9. INTRINSIC RECTIFIABILITY IN CARNOT GROUPS

Let $\boldsymbol{G}$ be a stratified nilpotent Carnot group, with homogenous dimension Q.

DEFINITION 9.1. *In $\boldsymbol{G}$ a C^1 surface is defined as the level set of a function with continuous and nonzero horizontal gradient.*

We say that $S \subset \boldsymbol{G}$ is $\boldsymbol{G}$-rectifiable if there exist countably many C^1 surfaces Γ_i such that

$$\mathcal{H}^{Q-1}\Big(S \setminus \bigcup_i \Gamma_i\Big) = 0 .$$

In an analogus fashion, one can define the reduced boundary $\mathcal{F}E$ of a set of $\boldsymbol{G}$-finite perimeter E using the horizontal distributional derivative of χ_E.

The following result is due to Franchi, Serapioni and Serra Cassano [12, 13].

THEOREM. *Let E be a set with finite perimeter in H_n. Then for any $x \in \mathcal{F}E$ the rescaled sets $\delta_r(x^{-1}E)$ converge as $r \to \infty$ to a vertical subgroup of H_n. As a consequence, the reduced boundary $\mathcal{F}E$ is intrinsically rectifiable.*

The proof requires, amon other things, the understanding of classical results (implicit function theorem, Whitney extension theorem) in the Carnot group setting.

The result has been recently extended in [14] to any step 2 Carnot group.

10. Open problems

1. Extend the rectifiability result to general Carnot groups and Carnot spaces.

In general Carnot groups there are remarkable examples showing that this extension is a non-trivial task. Franchi, Serapioni and Serra Cassano give in [14] an example in the Engel group (step 3, with grading (2,1,1)) arising from the 4-dimensional Lie algebra with commutator relations

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4.$$

This group has metric dimension 2+2+3 and the 6-dimensional cone (in exponential coordinates)

$$\frac{1}{6} x_2 (x_1^2 + x_2^2) - \frac{1}{2} x_1 x_3 + x_4 = 0$$

is «flat» with respect to the intrinsic geometry (*i.e.* it has a constant normal), is an entire minimal surface but it is not a subgroup. In lower dimensional groups, as in H_1, one can still build entire minimal surfaces which are not subgroups but their normal is not constant [19, 6].

2. The regularity theory for minimal surfaces in the Heisenberg group

3. Develop a general theory of currents which embodies on the one hand the extrinsic metric theory of De Giorgi (modelled on the Lipschitz embedding of Euclidean spaces) and on the other hand the intrinsic theories of BV functions and sets of finite perimeter, now well estabilished in Carnot groups and sub-Riemannian metric spaces.

References

[1] L. Ambrosio, *Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces*. Adv. Math., 59, 2001, 51-67.

[2] L. Ambrosio, *Fine properties of sets of finite perimeter in doubling metric measure spaces*. Set Valued Analysis, to appear.

[3] L. Ambrosio - B. Kirchheim, *Rectifiable sets in metric and Banach spaces*. Math. Ann., 318, 2000, 527-555.

[4] L. Ambrosio - B. Kirchheim, *Currents in metric spaces*. Acta Math., 185, 2000, 1-80.

[5] A. Baldi, *Weighted BV functions.* Houston J. of Math., 27(3), 2001, 683-705.

[6] I. Birindelli - E. Lanconelli, *A negative answer to a one-dimensional symmetry problem in the Heisenberg group*. Calc. Var., to appear.

[7] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces.* Geometric and Functional Analysis, 9, 1999, 428-517.

[8] E. De Giorgi, *Nuovi teoremi relativi alle misure* $(r-1)$*-dimensionali in uno spazio a r dimensioni.* Ricerche Mat., 4, 1955, 95-113.

[9] E. De Giorgi, *Problema di Plateau generale e funzionali geodetici.* Atti Sem. Mat. Fis. Univ. Modena, 43, 1995, 285-292.

[10] H. Federer - W.H. Fleming, *Normal and integral currents.* Ann. of Math., 72, 1960, 458-520.

[11] H. Federer, *A note on the Gauss-Green theorem.* Proc. Amer. Math. Soc., 9, 1958, 447-451.

[12] B. Franchi - R. Serapioni - F. Serra Cassano, *Sets of finite perimeter in the Heisenberg group.* C. R. Acad. Sci. Paris, 329, 1999, 183-188.

[13] B. Franchi - R. Serapioni - F. Serra Cassano, *Rectifiability and Perimeter in the Heisenberg group*. Math. Ann., to appear.

[14] B. Franchi - R. Serapioni - F. Serra Cassano, *On the structure of finite perimeter sets in step 2 Carnot groups*. Preprint.

[15] N. Garofalo - D.M. Nhieu, *Isoperimetric and Sobolev Inequalities for Carnot-Carathéodory Spaces and the Existence of Minimal Surfaces*. Comm. Pure Appl. Math., 49, 1996, 1081-1144.

[16] E. Giusti, *Minimal surfaces and functions of bounded variation*. Birkhäuser, 1984.

[17] V. Magnani, *Differentiability and area formula on stratified Lie groups.* Houston J. of Math., 27(2), 2001, 297-323.

[18] M. Miranda Jr., *Functions of bounded variation on «good» metric measure spaces.* Preprint Scuola Normale Superiore 2000, to appear in J. Math. Pures Appliquees.

[19] S.D. Pauls, *Minimal surfaces in the Heisenberg group*. Forthcoming.

[20] S.K. Vodop'yanov, *$\mathcal{P}$-differentiability on Carnot groups in different topologies and related topics.* Proc. on Analysis and Geometry, Sobolev Institute Press, Novosibirsk 2000, 603-670.

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