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Arina A. Arkhipova

Quasireverse Hölder inequalities and a priori estimates for strongly nonlinear systems

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Analisi matematica. — *Quasireverse Hölder inequalities and a priori estimates for strongly nonlinear systems.* Nota (*) di ARINA A. ARKHIPOVA, presentata dal Socio O.A. Ladyzhenskaya.

ABSTRACT. — It is proved that a function can be estimated in the norm with a higher degree of summability if it satisfies some integral relations similar to the reverse Hölder inequalities (quasireverse Hölder inequalities). As an example, we apply this result to derive an a priori estimate of the Hölder norm for a solution of strongly nonlinear elliptic system.

KEY WORDS: Boundary value problem; Reverse Hölder inequalities; Elliptic system.

RIASSUNTO. — *Disuguaglianze di Hölder quasi-inverse e stime a priori per sistemi fortemente non lineari.* Si prova che una funzione può essere stimata nella norma con un grado più alto di sommabilità se soddisfa alcune relazioni integrali simili alle disuguaglianze di Hölder inverse (disuguaglianze di Hölder quasi-inverse). Come esempio applichiamo questo risultato per desumere una stima a priori di una norma di Hölder per una soluzione di un sistema ellittico fortemente non lineare.

Nowadays, a variety of modifications of the Gehring lemma [1] are known. The theorems of M. Giaquinta-G. Modica [2] and E. W. Stredulinsky [3] were generalized in different directions (see [4-6] and references therein). Some integral relations similar to the reverse Hölder inequalities were obtained by the author of the present paper when studying solvability and regularity problems for strongly nonlinear elliptic systems. We call such relations «quasireverse Hölder inequalities» (QRHI).

In the present paper, we prove that a function can be estimated in the norm with a higher degree of summability if it satisfies the QRHI (§ 1, Theorem 1).

As an example, we apply Theorem 1 to obtain an a priori estimate of the Hölder norm for a solution of a strongly nonlinear elliptic system (§ 2, Theorem 2). In particular, to estimate the L_s-norm, $s > 2$, of the gradient of the solution, we assume, that the BMO-norm rather than L_∞-norm of the solution is small. It turned out that the QRHI are also very helpful in proving the solvability of boundary-value problems for different classes of strongly nonlinear elliptic systems. Results of such type will be published in other papers of the author.

1. A THEOREM ON QUASIREVERSE HÖLDER INEQUALITIES

The following notation is used:
$x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ n \geq 2,$
$Q_r(x^0) = \{x \in \mathbb{R}^n \mid |x_i - x_i^0| < r,\ i = 1, \dots, n\},$
$B_r(x^0) = \{x \in \mathbb{R}^n \mid |x - x^0| < r\},$
$S_r(x^0) = \partial B_r(x^0),$

(*) Nella seduta del 14 febbraio 2003.

$|A| = \mathrm{meas}_n A$ is the Lebesgue measure of a set $A \subset \mathbb{R}^n$,

$\fint_A f(x)\,dx = \frac{1}{|A|}\int_A f(x)\,dx$,

$\|v\|_{m,A} = \|v\|_{L_m(A)}$, $m \in [1, \infty)$.

Theorem 1. *Let numbers $q, l > 1$, $\omega > 0$ and $m > ql$ be fixed. Let for $g \in L_m(Q_{\widehat{R}}(\widehat{x}))$ the following inequalities hold*:

$$\fint_{B_{r/2}} g^q\,dx \leqslant c_0\left[\left(\fint_{B_r} g\,dx\right)^q + b(l)\,\omega\left(\fint_{B_r} g^{ql}\,dx\right)^{\frac{1}{l}}\right], \qquad \forall B_r \subset Q_{\widehat{R}}(\widehat{x}), \tag{1.1}$$

where c_0 and b are positive constants, and b may depend on l.

There exists a number $p_0 > q$ depending on q and c_0 only such that for a fixed $p \in (q, \min(p_0, m))$

$$\left(\fint_{Q_{r/2}} g^p\,dx\right)^{\frac{1}{p}} \leqslant c_*\left(\fint_{Q_r} g^q\,dx\right)^{\frac{1}{q}}, \qquad \forall Q_r \subset Q_{\widehat{R}}(\widehat{x}), \tag{1.2}$$

provided that $l \in \left(1, \frac{p}{q}\right)$ and $\omega < \frac{\kappa_0}{b}$ in (1.1) (κ_0 is a number dependent on p, q, l, and c_0).

In particular,

$$\|g\|_{p, Q_r(\widehat{x})} \leqslant c_{**}\|g\|_{q, Q_{\widehat{R}}(\widehat{x})}, \qquad r < \widehat{R}. \tag{1.3}$$

The constants c_* and c_{**} depend on q, c_0, p, l, $b(l)$ and ω, and the constant c_{**} depends also on $(\widehat{R} - r)^{-1}$.

Remark 1.1. Inequalities (1.1) with $\omega = 0$ are reverse Hölder inequalities for the function g, and one can prove a higher integrability of g in the known way. Here $\omega \neq 0$ and we assume a higher integrability of g. Theorem 1 yields only an L_p estimate of g in terms of the L_q norm, $p > q$.

Remark 1.2. Theorem 1 holds if we add some other terms on the right-hand side in inequalities (1.1). For example, we can add the expression

$$\theta \int_{B_r(x^0)} g^q\,dx, \qquad \theta \ll 1.$$

Other terms known from different modifications are also allowed in (1.1) (see [4-6]).

The proof of Theorem 1 is given in the following lemmas.

First we prove a modification of a well-known result on the Stiltjes integral (see [7, Chapter V, Lemma 1.2]) in appropriate form.

LEMMA 1. *Let numbers l, q, $a > 1$ and $m > ql$ be fixed. Consider a non-increasing function $h: [1, \infty) \to [0, \infty)$ such that*

$$\lim_{t\to+\infty} h(t) = 0, \qquad \lim_{t\to+\infty} \int_t^{+\infty} \xi^{m-1}\, dh(\xi) = 0, \tag{1.4}$$

$$-\int_t^{+\infty} \tau^{q-1}\, dh(\tau) \leqslant a\left[t^{q-1}h(t) + \frac{\theta b}{t^{q(l-1)}}\left(-\int_t^{+\infty} \tau^{ql-1}\, dh(\tau)\right)\right], \qquad t \geqslant 1, \tag{1.5}$$

$b = b(l) = \mathrm{const} > 0$. *For a number* $p \in \left(q, \min\left(m, \frac{qa-1}{a-1}\right)\right)$ *the following inequality holds*:

$$-\int_1^{+\infty} t^{p-1}\, dh(t) \leqslant b_0\left(-\int_1^{+\infty} t^{q-1}\, dh(t)\right), \tag{1.6}$$

provided that $l \in (1, p/q)$ and $\theta < \kappa_*/b$ in (1.5) (κ_* *is a number dependent on q, p, l, and a*), *the constant b_0 depends on q, p, l, a, $b(l)$ and θ.*

PROOF. Fix $p \in (q, m)$ and denote

$$I^s(t) = -\int_t^{+\infty} t^{s-1}\, dh(t), \qquad s > 1.$$

From the monotonicity of h it follows that

$$\tau^{s-1}h(\tau) \leqslant -\int_\tau^{+\infty} \lambda^{s-1}\, dh(\lambda) = I^s(\tau), \qquad \tau \geqslant 1, \qquad s > 1.$$

The second relation in (1.4) and the last inequality yield

$$\lim_{\tau\to+\infty} \tau^{s-1}h(\tau) = 0, \qquad s \leqslant m. \tag{1.7}$$

Moreover,

$$\tau^{p-q}I^q(\tau) \leqslant I^p(\tau) \to 0, \qquad \tau^{p-ql}I^{ql} \to 0 \text{ as } \tau \to +\infty. \tag{1.8}$$

The following relations are valid:

$$I^p(1) = -\int_1^{+\infty} \tau^{p-q}\, dI^q(\tau) = (p-q)\int_1^{+\infty} \tau^{p-q-1}I^q(\tau)\, d\tau - \tau^{p-q}I^q(\tau)\Big|_1^{+\infty} \overset{(1.5),(1.8)}{\leqslant}$$

$$\overset{(1.5),(1.8)}{\leqslant} a(p-q)\int_1^{+\infty} \tau^{p-q-1}\left[\tau^{q-1}h(\tau) + \frac{\theta b}{\tau^{q(l-1)}}\left(-\int_\tau^{+\infty} \xi^{ql-1}\, dh(\xi)\right)\right] d\tau + I^q(1) =$$

$$= a(p-q)\int_1^{+\infty} \tau^{p-2}h(\tau)\, d\tau + a(p-q)\,\theta b\int_1^{+\infty} \tau^{p-ql-1}\left(-\int_\tau^{+\infty} \xi^{ql-1}\, dh(\xi)\right) d\tau + I^q(1) =$$

$$= a(p-q)\int_1^{+\infty}\left(\frac{\tau^{p-1}}{p-1}\right)' h(\tau)\,d\tau + a(p-q)\,\theta b\int_1^{+\infty}\left(\frac{\tau^{p-ql}}{p-ql}\right)' I^{ql}(\tau)\,d\tau + I^q(1) =$$

$$= -\frac{a(p-q)}{p-1}\int_1^{+\infty}\tau^{p-1}dh(\tau) + \frac{a(p-q)}{p-1}\tau^{p-1}h(\tau)\Big|_1^{+\infty} - \frac{a(p-q)\,\theta b}{p-ql}\int_1^{+\infty}\tau^{p-ql}\tau^{ql-1}dh(\tau) +$$

$$+\frac{a(p-q)\,\theta b}{p-ql}\tau^{p-ql}I^{ql}(\tau)\Big|_1^{+\infty} + I^q(1) \overset{(1.7),(1.8)}{=} \frac{a(p-q)}{p-1}I^p(1) - \frac{a(p-q)}{p-1}h(1) +$$

$$+\frac{ab\theta(p-q)}{p-ql}I^p(1) - \frac{ab\theta(p-q)}{p-ql}I^{ql}(1) + I^q(1).$$

Now we put $t=1$ in (1.5) and obtain the inequality

$$ah(1) \geqslant I^q(1) - ab\theta I^{ql}(1).$$

Then,

$$I^p(1)\left[1 - \frac{a(p-q)}{p-1} - \frac{ab\theta(p-q)}{p-ql}\right] + I^{ql}(1)\,ab\theta\frac{(p-q)(ql-1)}{(p-1)(p-ql)} \leqslant I^q(1)\frac{q-1}{p-1},$$

and we have the estimate

$$I^p(1)\left[1 - \frac{a(p-q)}{p-1} - \frac{ab\theta(p-q)}{p-ql}\right] \leqslant I^q(1)\frac{q-1}{p-1}. \tag{1.9}$$

Now we fix $p < m$ so that the following inequality is satisfied:

$$1 - \frac{a(p-q)}{p-1} > 0.$$

It means that $p < \frac{qa-1}{a-1}$, and henceforth we consider

$$p \in \left(q, \min\left(\frac{qa-1}{a-1}, m\right)\right).$$

Next we fix $l \in \left(1, \frac{p}{q}\right)$ and choose θ in such a way as to obtain a positive expression in the square brackets of (1.9):

$$\theta < \frac{\kappa_0}{b(l)}, \qquad \kappa_0 = \left(1 - \frac{a(p-q)}{p-1}\right)\frac{(p-ql)}{a(p-q)}.$$

With fixed parameters p, l and θ we have an estimate

$$I^p(1) \leqslant b_0 I^q(1),$$

where $b_0 = \frac{q-1}{p-1}\left(1 - \frac{a(p-q)}{p-1} - \frac{a\theta b(p-q)}{p-ql}\right)^{-1}$. □

Remark 1.3. If $\theta = 0$, then the second condition in (1.4) can be removed and we claim by the classical result that estimate (1.6) is valid for $p < \frac{qa-1}{a-1}$.

LEMMA 2. *Let numbers* $q, l > 1$ *and* $\omega > 0$ *be fixed. Let* $g \in L_m(Q)$, *where* $Q = Q_{3/2}(0) \subset \mathbb{R}^n$, $m > ql$, *satisfy the inequalities*

$$\fint_{B_{r/2}} g^q \leqslant c_0 \left[\left(\fint_{B_r} g\,dx \right)^q + \beta\omega \left(\fint_{B_r} g^{ql}\,dx \right)^{\frac{1}{l}} \right], \qquad \forall B_r \subset Q, \tag{1.10}$$

with positive constants c_0 *and* $\beta = \beta(l)$.

There exists a number $p_0 = p_0(q, c_0) > q$ *such that for* $p \in (q, \min(p_0, m))$

$$\|g\|^p_{p, Q_r(0)} \leqslant \frac{c_1}{\left(\frac{3}{2} - r\right)^{\frac{np}{q}}} \left\{ \|g\|^p_{q, Q} + \kappa_0^{\frac{p}{q} - 1} \|g\|^{p-q}_{p, Q} \|g\|^q_{q, Q} \right\}, \qquad r \in \left[\frac{1}{2}, \frac{3}{2} \right), \tag{1.11}$$

provided that $l \in \left(1, \frac{p}{q}\right)$ *and* $\omega \leqslant \frac{\kappa_0}{\beta}$, ($\kappa_0$ *is a number dependent on* q, p, l, *and* c_0), *the constant* c_1 *depends on the same parameters as* κ_0 *does.*

REMARK 1.4. Inequalities (1.10) with $\omega = 0$ are known to ensure a higher integrability of $g \in L_q(Q)$ and estimate (1.11) with $\kappa_0 = 0$. If $\omega \neq 0$, then (1.11) does not imply an L_p-estimate of g but helps us to derive estimate (1.3) in the sequel.

PROOF OF LEMMA 2. To save space here we make use of the notation accepted in [7, Chapter V, Proposition 1.1]. Moreover, we explain in detail only new points in the proof of this proposition.

First, following [7] we put

$$\mathbb{C}_0 = \{x \in \mathbb{R}^n \mid |x_i| < 1/2,\ i = 1, \dots, b\},$$
$$\mathbb{C}_k = \{x \in Q \mid 1/2^k < \operatorname{dist}(x, \partial Q) \leqslant 1/2^{k-1}\}, \qquad k \in N.$$

Obviously, $Q = \bigcup_{k \geqslant 0} \mathbb{C}_k$.

We define

$$G(x) = \frac{g(x)}{M}, \qquad M = \|g\|_{q, Q} + (\beta\omega)^{1/q} \|g\|_{ql, Q}, \tag{1.12}$$

and write (1.10) in the form

$$\fint_{B_{r/2}} G^q\,dx \leqslant c_0 \left[\left(\fint_{B_r} G\,dx \right)^q + \beta\omega \left(\fint_{B_r} G^{ql}\,dx \right)^{\frac{1}{l}} \right]. \tag{1.13}$$

We also put

$$\alpha_k = (\sigma^n 2^{nk})^{1/q}, \qquad \sigma = \text{const} > 6\sqrt{n}, \quad \text{and}$$

$$G_0(x) = \sum_{k \geqslant 0} \frac{G(x)}{\alpha_k} \chi_{\mathbb{C}_k}(x),$$

where $\chi_A(x)$ is the characteristic function of the set A.

We apply the well-known Calderon-Zygmund lemma [7, Chapter V, Lemma 1.3] in appropriate form and claim that there exists a sequence of n-cubes $Q^j_{(k)} \subset \mathbb{C}_k$, $j \geqslant 1$,

$k \geqslant 0$, with disjoint interiors such that

$$|Q^j_{(k)}| \leqslant \left(\frac{3}{\sigma}\right)^n 2^{-nk} < \frac{1}{2}\operatorname{dist}(Q^j_{(k)}, \partial Q), \tag{1.14}$$

$$(\alpha_k s)^q < \fint_{Q^j_{(k)}} G^q(x)\,dx \leqslant \sigma^n(\alpha_k s)^q, \quad s \geqslant 1, \tag{1.15}$$

$$G(x) \leqslant \alpha_k s \text{ a.e. on } \mathbb{C}_k \setminus \bigcup_j Q^j_{(k)}. \tag{1.16}$$

Then

$$s^q < \fint_{Q^j_{(k)}} G_0^q\,dx \leqslant \sigma^n s^q, \tag{1.17}$$

$$G_0(x) \leqslant s \text{ a.e. on } \mathbb{C}_k \setminus \bigcup_j Q^j_{(k)}. \tag{1.18}$$

From (1.18) it follows that $|E(G_0, s) \setminus \bigcup_{j,k} Q^j_{(k)}| = 0$, where

$$E(h, \tau) = \{x \in Q \,|\, h(x) > \tau\}.$$

Hence,

$$\int_{E(G_0, s)} G_0^q(x)\,dx \leqslant \sum_{j,k} \int_{Q^j_{(k)}} G_0^q(x)\,dx \leqslant \sigma^n s^q \sum_{j,k} |Q^j_{(k)}|. \tag{1.19}$$

As a consequence of (1.14), we have the inequalities

$$\operatorname{diam} Q^j_{(k)} \leqslant \frac{3\sqrt{n}}{\sigma} 2^{-k} < \frac{1}{2}\operatorname{dist}(Q^j_{(k)}, \partial Q).$$

Then

$$Q^j_{(k)} \subset B_R(x), \quad \forall x \in Q^j_k, \qquad R = \operatorname{diam} Q^j_{(k)} < \frac{1}{2}\operatorname{dist}(x, \partial Q) \equiv \frac{1}{2} d(x).$$

We derive from (1.15) that

$$(\alpha_k s)^q < \fint_{Q^j_{(k)}} G^q(x)\,dx \leqslant \alpha(n) \fint_{B_R(x)} G^q\,dx, \qquad \alpha(n) = |B_1|\, n^{n/2}.$$

Inequality (1.13) allows us to estimate the right-hand side of the last inequality, and we have

$$(\alpha_k s)^q < \alpha(n)\, c_0 \left[\left(\fint_{B_{2R}(x)} G\,dy\right)^q + \beta\omega\left(\fint_{B_{2R}(x)} G^{ql}\,dy\right)^{\frac{1}{l}}\right], \qquad 2R < d(x). \tag{1.20}$$

Note that from the defenition of G (see (1.12)) it follows that $\|G\|_{q,Q} \leqslant 1$ and $\beta\omega\|G\|^q_{ql,Q} \leqslant 1$.

If we put $s = \lambda t$, where $t \geqslant 1$ and $\lambda > 1$ is a constant to be chosen, then (1.20) implies that

$$|B_{2R}| \leqslant \frac{c_2}{2^{nk}\lambda}, \qquad c_2 = c_2(c_0, \sigma, n). \tag{1.21}$$

Let

$$\lambda \geqslant \frac{c_2 2^n}{|B_1|}, \tag{1.22}$$

then from (1.21) we have $r = 2R < \frac{1}{2^{k+1}}$, and the ball $B_r(x)$ can intersect at most $\mathbb{C}_{k-1}$, $\mathbb{C}_k$ and $\mathbb{C}_{k+1}$.

Now from (1.20) with $r = 2R$ we obtain the inequality

$$(\lambda t)^q \leqslant c_3 \left[\left(\fint_{B_r(x)} G_0 \, dy \right)^q + \beta\omega \left(\fint_{B_r(x)} G_0^{ql} \, dy \right)^{\frac{1}{l}} \right], \qquad r < d(x). \tag{1.23}$$

Hence,

$$\lambda t |B_r| \leqslant c_4 \left[\int_{B_r(x)} G_0 \, dy + (\beta\omega)^{\frac{1}{q}} \left(\int_{B_r(x)} G_0^{ql} \, dy \right)^{\frac{1}{ql}} |B_r|^{1 - \frac{1}{ql}} \right]. \tag{1.24}$$

We apply the Cauchy inequality to estimate the term J_r with ω in (1.24):

$$J_r = c_4 (|B_r| t)^{\frac{ql-1}{ql}} \left(\frac{(\beta\omega)^l}{t^{ql-1}} \int_{B_r} G_0^{ql} dx \right)^{\frac{1}{ql}} \leqslant c_4 |B_r| t + c_4 \frac{(\beta\omega)^l}{t^{ql-1}} \int_{B_r} G_0^{ql} dx. \tag{1.25}$$

Henceforth the c_i may depend on q, c_0, and n. We put

$$E_r(t) = \{ y \in B_r(x) \mid G_0(y) > t \}$$

and derive from (1.24) and (1.25) the inequality

$$\lambda t |B_r| \leqslant c_4 (2 + (\beta\omega)^l) t |B_r| + c_4 \int_{E_r(t)} G_0 \, dx + c_4 \frac{(\beta\omega)^l}{t^{ql-1}} \int_{E_r(t)} G_0^{ql} \, dx.$$

Let us assume that in this theorem $\beta(l)\omega < \kappa_0$, where $\kappa_0 \leqslant 1$, and the parameter λ satisfies (in addition to (1.22)) the inequality

$$\lambda - 3 c_4 \geqslant \frac{\lambda}{2}. \tag{1.26}$$

Then

$$|B_r| \leqslant \frac{c_5}{t} \int_{E_r(t)} G_0 \, dx + \frac{c_5 (\beta\omega)^l}{t^{ql}} \int_{E_r(t)} G_0^{ql} \, dx \, t \geqslant 1. \tag{1.27}$$

The family of such balls $B_r(x)$ covers the set $T = \bigcup_{k,j} Q^j_{(k)}$. We can assert that there exists a numerable subfamily of disjoint B_i such that

$$\sum_{j,k} |Q^j_{(k)}| \leqslant c(n) \sum_i |B_i|$$

(see [7, Chapter V, Lemma 1.1]).

Now, from (1.19) and (1.27) it follows that

$$\int_{E(G_0,s)} G_0^q\,dx \leqslant c_6\left(t^{q-1}\int_{E(G_0,t)} G_0\,dx + \frac{(\beta\omega)^l}{t^{ql-q}}\int_{E(G_0,t)} G_0^{ql}\,dx\right), \quad s=\lambda t. \tag{1.28}$$

Hence,

$$\int_{E(G_0,t)} G_0^q\,dx \leqslant c_7\left(t^{q-1}\int_{E(G_0,t)} G_0\,dx + \frac{(\beta\omega)^l}{t^{q(l-1)}}\int_{E(G_0,t)} G_0^{ql}\,dx\right), \quad t\geqslant 1. \tag{1.29}$$

We put $h(t) = \fint_{E(G_0,t)} G_0(x)\,dx$ and note that

$$\int_{E(G_0,t)} G_0^m\,dx = -\int_t^{+\infty} \tau^{m-1}\,dh(\tau), \qquad m>1.$$

Now inequality (1.29) takes the form

$$-\int_t^{+\infty}\tau^{q-1}\,dh(\tau) \leqslant c_8\left\{t^{q-1}h(t) + \frac{(\beta\omega)^l}{t^{q(l-1)}}\left(-\int_t^{+\infty}\tau^{ql-1}\,dh(\tau)\right)\right\}, \quad t\geqslant 1. \tag{1.30}$$

The function h satisfies the assumptions of Lemma 1 with $a=c_8>1$, $b=\beta^l$, and $\theta=\omega^l$. By Lemma 1, we fix $l\in\left(1,\frac{p}{q}\right)$ for $p\in\left(q,\min\left(m,\frac{qc_8}{c_8-1}\right)\right)$ and $\omega<\frac{\kappa_*^{1/l}}{\beta(l)}$, $\kappa_*=\kappa_*(q,p,l,c_8)$.

In Theorem 1, we fix $\kappa_0=\min\{1,\kappa_*^{1/l}\}$. By Lemma 1, we have

$$-\int_1^{+\infty} t^{p-1}\;dh(t) \leqslant b_0\left(-\int_1^{+\infty} t^{q-1}\,dh(t)\right), \tag{1.31}$$

whence

$$\int_{E(G_0,1)} G_0^p\,dx \leqslant b_0\int_{E(G_0,1)} G_0^q\,dx, \tag{1.32}$$

$b_0=b_0(q,p,l,c_8,\beta,\omega)$.

From (1.32) it follows that

$$\int_Q G_0^p\,dx \leqslant (b_0+1)\int_Q G_0^q\,dx \leqslant (b_0+1)\int_Q G^q\,dx. \tag{1.33}$$

Note that $\operatorname{dist}(Q_r(0),\partial Q)=\frac{3}{2}-r$ for $r\in\left[\frac{1}{2},\frac{3}{2}\right)$, and there exists a number $k_0\in N$ such that $\frac{3}{2}-r\approx\frac{1}{2^{k_0}}$ and $Q_r(0)\subseteq\mathbb{C}_0\cup\ldots\cup\mathbb{C}_{k_0}$. From the definition of α_k, it follows that $\alpha_{k_0}^q=\sigma^n 2^{nk_0}$, *i.e.*, $\frac{3}{2}-r\approx\frac{\sigma}{\alpha_{k_0}^{q/n}}$. Hence,

$$\int_Q G_0^p\,dx \geqslant \int_{\mathbb{C}_0\cup\ldots\cup\mathbb{C}_{k_0}} G_0^p\,dx \geqslant \frac{c_9}{\alpha_{k_0}^p}\int_{Q_r(0)} G^p\,dx \geqslant c_{10}\left(\frac{3}{2}-r\right)^{\frac{np}{q}}\int_{Q_r(0)} G^p\,dx.$$

Now from (1.33) we deduce the inequality

$$\int\limits_{Q_r(0)} G^p\,dx \leqslant \frac{c_{11}}{\left(\frac{3}{2}-r\right)^{\frac{np}{q}}} \int\limits_{Q} G^q\,dx.$$

By (1.12), it guarantees the validity of estimate (1.11). □

LEMMA 3. *Let numbers* l, $q>1$, c_0, ω, *and* $\beta=\beta(l)>0$ *be fixed. Let* $g\in L_m(Q_{R_0}(x^0))$, $m>ql$, *satisfy inequalities* (1.10) *in the balls* $B_r\subset Q_{R_0}(x^0)$. *If* p, l, β, ω *are fixed by Lemma* 2, *then the following inequalities hold:*

$$(1.34)\qquad \fint\limits_{Q_\varrho(x^0)} g^p\,dx \leqslant \frac{c_{12}}{(R-\varrho)^{\frac{np}{q}}}\left\{\left(\int\limits_{Q_R(x^0)} g^q\,dx\right)^{\frac{p}{q}} + \left(\fint\limits_{Q_R(x^0)} g^p\,dx\right)^{1-\frac{q}{p}} \int\limits_{Q_R(x^0)} g^q\,dx\cdot R^{n(\frac{p}{q}-1)}\right\},$$

$\varrho\in\left[\frac{R}{3}, R\right)$, $R\leqslant R_0$. *The constant* c_{12} *depends on* c_0 (*from* (1.10)), p, q, l, β, *and* ω.

PROOF. Inequalities (1.34) are a consequence of estimate (1.11). Indeed, fix $R\leqslant R_0$ and, changing the variables, put $y(x)=\dfrac{3(x-x^0)}{2R}$. As a result, $y(Q_R(x^0))=Q_{\frac{3}{2}}(0)$ and $y(Q_{\frac{R}{3}}(x^0))=Q_{\frac{1}{2}}(0)$. From (1.10) for $g(x)$ in the balls $B_r\subset Q_R(x^0)$, we find that $z(y)=$ $=g(x(y))$ satisfies the inequalities

$$(1.35)\qquad \fint\limits_{B_\varrho} z^p\,dy \leqslant c_0\left[\left(\fint\limits_{B_\varrho} z\,dy\right)^q + \beta\omega\left(\fint\limits_{B_\varrho} z^{ql}\,dy\right)^{\frac{1}{l}}\right], \qquad \forall B_\varrho\subset Q_{\frac{3}{2}}(0).$$

By Lemma 2, there exists a number $p_0=p_0(q, c_0, n)>q$ such that for a fixed $l\in\left(1, \frac{p}{q}\right)$, where $p\in(q, \min(p_0, m))$, and $\beta\omega<\kappa_0$ $(\kappa_0=\kappa_0(p, l, q, c_0)\leqslant 1)$ in (1.35), estimate (1.11) is valid for $z(y)$ in $Q=Q_{\frac{3}{2}}(0)$:

$$(1.36)\qquad \int\limits_{Q_R(0)} z^p\,dy \leqslant \frac{c_1}{\left(\frac{3}{2}-r\right)^{\frac{np}{q}}}\left\{\left(\int\limits_{Q} z^q\,dy\right)^{\frac{p}{q}} + \kappa_0^{\frac{p}{q}-1}\left(\int\limits_{Q} z^p\,dy\right)^{\frac{p-q}{p}} \int\limits_{Q} z^q\,dy\right\},$$

which yields (1.34). □

LEMMA 4. *If a function* g *satisfies inequalities* (1.34) *in* $Q_\varrho(x^0)$ *and* $Q_R(x^0)$ *for any* $R\leqslant R_0$ *and* $\varrho\in\left[\frac{R}{3}, R\right)$, *then*

$$(1.37)\qquad \left(\fint\limits_{Q_{R/2}(x^0)} g^p\,dx\right)^{\frac{1}{p}} \leqslant c_{13}\left(\fint\limits_{Q_R(x^0)} g^q\right)^{\frac{1}{q}}, \qquad \forall R\leqslant R_0,$$

$c_{13}=c_{13}(p, q, c_{12}, n)$.

PROOF. For a fixed $\widetilde{R} \leqslant R_0$ consider the sequence $\varrho_k = \widetilde{R} - \frac{\widetilde{R}}{2^{k+1}}$, $k = 0, 1, \dots,$ $\left(\varrho_0 = \frac{\widetilde{R}}{2}, \varrho_k \nearrow \widetilde{R}\right)$.

We put

$$H = c_{12}\left(\int_{Q_{\widetilde{R}}(x^0)} g^q\,dx\right)^{\frac{p}{q}}, \qquad B = c_{12}\int_{Q_{\widetilde{R}}(x^0)} g^q\,dx\cdot \widetilde{R}^{n(\frac{p}{q}-1)}, \tag{1.38}$$

and from (1.34) with $\varrho = \varrho_k$ and $R = \varrho_{k+1}$ we derive the inequality

$$\fint_{Q_{\varrho_k}(x^0)} g^p\,dx \leqslant \frac{1}{(\varrho_{k+1}-\varrho_k)^{\frac{np}{q}}}\left(H + B\left(\fint_{Q_{\varrho_{k+1}}(x^0)} g^p\,dx\right)^{1-\frac{q}{p}}\right). \tag{1.39}$$

Now put

$$\psi(\varrho_k) = \fint_{Q_{\varrho_k}(x^0)} g^p\,dx, \qquad m = \frac{np}{q}, \qquad \gamma = 1 - \frac{q}{p} \in (0, 1).$$

Inequalities (1.39) take the form

$$\psi(\varrho_k) \leqslant \frac{1}{(\varrho_{k+1}-\varrho_k)^m}(H + B\psi^\gamma(\varrho_{k+1})), \qquad k \geqslant 0. \tag{1.40}$$

One of the two situations may occur.

a) There exists a number k_0 such that $B\psi^\gamma(\varrho_{k_0+1}) \leqslant H$. By (1.38), we obtain

$$\int_{Q_{\widetilde{R}}(x^0)} g^q\,dx\cdot\widetilde{R}^{n(\frac{p}{q}-1)}\left(\int_{Q_{\varrho_{k_0+1}}(x^0)} g^p\,dx\right)^\gamma \leqslant \left(\int_{Q_{\widetilde{R}}(x^0)} g^q\,dx\right)^{\frac{p}{q}}.$$

Hence,

$$\left(\fint_{Q_{\varrho_{k_0+1}}(x^0)} g^p\,dx\right)^{\frac{p-q}{p}} \leqslant \frac{1}{\widetilde{R}^{n(\frac{p}{q}-1)}}\left(\int_{Q_{\widetilde{R}}(x^0)} g^q\,dx\right)^{\frac{p}{q}-1}$$

and

$$\left(\fint_{Q_{\widetilde{R}/2}(x^0)} g^p\,dx\right)^{\frac{1}{p}} \leqslant c_{14}(p, q)\left(\fint_{Q_{\widetilde{R}}(x^0)} g^q\,dx\right)^{\frac{1}{q}}. \tag{1.41}$$

b) For any $k \geqslant 0$, the following inequalities hold:

$$H \leqslant B\psi^\gamma(\varrho_{k+1}).$$

Then, by (1.40),

$$\psi(\varrho_k) \leqslant \frac{2B\psi^\gamma(\varrho_{k+1})}{(\varrho_{k+1}-\varrho_k)^m}. \tag{1.42}$$

After iterations, we have the relations

$$\psi\left(\frac{\tilde{R}}{2}\right)=\psi(\varrho_0)\leqslant\ldots\leqslant\frac{(2B)^{\frac{1}{1-\gamma}}2^{m\sum\limits_{j=0}^{\infty}(2+j)\gamma^j}}{\left(\frac{\tilde{R}}{2}\right)^{m(1+\gamma+\ldots+\gamma^k)}}(\psi(\varrho_k))^{\gamma^k}. \tag{1.43}$$

Note that $\sum\limits_{j=0}^{\infty}(2+j)\,\gamma^j<+\infty$, and $\psi(\varrho_k)\to\psi(\tilde{R})$, $\gamma^k\to 0$ as $k\to+\infty$. Passing to the limit in (1.43), we obtain

$$\psi\left(\frac{\tilde{R}}{2}\right)\leqslant\frac{c_{15}B^{\frac{1}{1-\gamma}}}{\tilde{R}^{\frac{m}{1-\gamma}}}=c_{16}\left(\;\fint_{Q_{\tilde{R}}(x^0)} g^q\,dx\right)^{\frac{p}{q}}, \tag{1.44}$$

$c_{16}=c_{16}(p,q,c_{12},n)$.

Inequalities (1.41) and (1.44) with any $\tilde{R}\leqslant R_0$ imply estimate (1.37). □

Proof of Theorem 1. The assumptions of the theorem ensure the validity of the assumptions of Lemma 3. This means that inequalities (1.34) are valid in $Q_\varrho(x^0)$ and $Q_R(x^0)$ for the mentioned ϱ and R. By Lemma 4, inequalities (1.37) are valid and they coincide with inequalities (1.2). Theorem 1 is proved.

2. A priori estimates for solutions of strongly nonlinear elliptic systems

Let Ω be a bounded domain in $\mathbb{R}^n$, $n\geqslant 2$ and let $u:\Omega\to\mathbb{R}^N$, $N>1$, $u=(u^1,\ldots,u^N)$, be a solution of the system

$$-(A_{kl}^{\alpha\beta}(x,u)u_{x_\beta}^l)_{x_\alpha}+b^k(x,u,u_x)=0,\quad k\leqslant N,\quad x\in\Omega. \tag{2.1}$$

We assume that the matrix $A=\{A_{kl}^{\alpha\beta}\}_{k,l\leqslant N}^{\alpha,\beta\leqslant n}$ is defined on the set $\mathfrak{M}=\overline{\Omega}\times\mathbb{R}^N$ and satisfies the conditions

a)

$$(A(x,u)\,\xi,\xi)\equiv\sum_{\substack{\alpha\beta\leqslant n\\ k,l\leqslant N}}A_{kl}^{\alpha\beta}(x,u)\,\xi_\alpha^k\xi_\beta^l\geqslant\nu|\xi|^2,\qquad\forall\xi\in\mathbb{R}^{nN}, \tag{2.2}$$

$$\sup_{\mathfrak{M}}\|A(x,u)\|\leqslant\mu. \tag{2.3}$$

b) The $A_{kl}^{\alpha\beta}$ are uniformly continuous functions on $\mathfrak{M}$. More precisely, there exists a continuous, bounded and nonincreasing function $\omega(s,t)$ such that

$$\|A(x,u)-A(y,v)\|\leqslant\omega(|x-y|^2,|u-v|^2),\quad x,y\in\overline{\Omega},\quad u,v\in\mathbb{R}^N, \tag{2.4}$$

$\omega(s,t)\to 0$ as $s,t\to 0$, $\omega(s,t)$ is a function convex in t.

c) $b=\{b^k\}^{k\leqslant N}$ is the Caratheodory function on $\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}$ and

$$|b(x,u,z)|\leqslant a|z|^2+\mu,\qquad(x,u)\in\mathfrak{M},\qquad z\in\mathbb{R}^{nN}. \tag{2.5}$$

The positive constants ν, μ and a in *a*) and *c*) are arbitrarily fixed.

In general, the solvability of boundary-value problems for systems (2.1) under assumptions (2.2), (2.3) and (2.5) has yet to be proved. In the two-dimentional case,

J. Frehse obtained the existence of a smooth solution of the Dirichlet problem for a class of nonlinear elliptic systems with quadratic nonlinearity (2.5) in the gradient [10].

The smoothness of bounded weak solutions of (2.1), $n \geq 2$, under conditions *a*), *b*), *c*) was also studied [8, 11]. In the local setting the following result was proved.

If $u \in W_2^1(\Omega) \cap L_\infty(\Omega)$ is a solution of (2.1) and

$$a \operatorname*{osc}_{B_{R_0}} u < \nu \tag{2.6}$$

in a ball $B_{R_0} \subset\subset \Omega$, then u is a Hölder continuous function in $B_{\frac{R_0}{2}} \backslash \sigma$ and $H_{n-2-\varepsilon}(\sigma) = 0$ for some $\varepsilon > 0$ ($H_s(\sigma)$ is the Hausdorff measure of dimension s of a set σ). To obtain this result in the context of the direct method of investigation, we need the «smallness condition» (2.6) to prove only a higher integrability of the gradient u_x and to estimate locally an L_p norm of u_x, $p > 2$.

Here we apply Theorem 1 to derive an a priori estimate of the L_p norm, $p > 2$, of a solution of (2.1) under the smallness condition on the seminorm $[u]_{2, n; B_{R_0}}$ in the space $BMO(B_{R_0})$ instead of condition (2.6) on the smallness of $\|u\|_{\infty, B_{R_0}}$. Moreover, we derive an a priori estimate of the Hölder norm of a solution u smooth in a ball $B_{R_0} \subset\subset \Omega$ if $\|u_x\|_{L^{2, n-2}(B_{R_0})}$ is small enough (no control of $\|u\|_{\infty, B_{R_0}}$ is implied).

THEOREM 2. *Let assumptions a*), *b*), *c*) *hold and let u be a smooth solution of* (2.1) *in a ball* $B_{R_0} \subset\subset \Omega$. *There exists a number* $\theta = \theta(\nu, \mu) > 0$ *such that if*

$$[u]_{2, n; B_{R_0}} < \theta \min\left\{\frac{\nu}{a}, 1\right\} \tag{2.7}$$

then

$$\|u_x\|_{s, B_{R_1}} \leq c_*(\|u_x\|_{2, B_{R_0}} + 1) \tag{2.8}$$

for some $s = s(\nu, \mu) > 2$ *and* $R_1 = R_1(R_0) < R_0$, $c_* = c_*(\nu, \mu, \theta, R_1)$.

Moreover, there exists a number $\theta_1 > 0$ *such that the inequality*

$$\|u_x\|_{2, n-2; B_{R_0}} < \theta_1 \min\left\{\frac{\nu}{a}, 1\right\} \tag{2.9}$$

provides the estimate

$$\|u\|_{C^\alpha(\overline{B_{R_2}})} \leq c_{**}(\|u\|_{W_2^1(B_{R_0})} + 1) \tag{2.10}$$

for any $\alpha \in (0, 1)$ *with some* $R_2 < R_0$. *The parameter* θ_1 *depends on* ν, μ *and the function* ω *from* (2.4), *the constant* c_{**} *depends also on* R_0, θ_1, *and* α.

First we recall that

$$\|v\|_{2, n-2; B_{R_0}} = \left(\sup_{\substack{\varrho \leq R_0 \\ y \in B_{R_0}}} \frac{1}{\varrho^{n-2}} \int_{B_\varrho(y) \cap B_{R_0}} |v|^2 \, dx \right)^{\frac{1}{2}}$$

is a norm in the Morrey space $L^{2, n-2}(B_{R_0})$. A norm in the Campanato space

$\mathcal{L}^{m,n}(B_{R_0})$ is given by

$$\|v\|_{\mathcal{L}^{m,n}(B_{R_0})} = \|v\|_{m,B_{R_0}} + [v]_{m,n;B_{R_0}}$$

with seminorm

$$[v]_{m,n;B_{R_0}} = \left(\sup_{\substack{\varrho \leqslant R_0\\ y\in B_{R_0}}} \frac{1}{\varrho^n} \int_{B_\varrho(y)\cap B_{R_0}} |v - v_{\varrho,y}|^m\,dx\right)^{\frac{1}{m}}, \qquad v_{\varrho,y} = \fint_{B_\varrho(y)\cap B_{R_0}} v\,dx.$$

Note that the spaces $\mathcal{L}^{m,n}(\cdot)$ are isomorphic for different $m \geqslant 1$, and, in particular,

$$[v]_{m,n;B_{R_0}} \leqslant c(m,n)[v]_{2,n;B_{R_0}}, \tag{2.11}$$

(for the definitions and properties of the spaces see [9, Chapter IV] and [12, Chapter I]).

Remark 2.1. By the known estimate

$$[u]_{2,n;B_{R_0}} \leqslant c(n)\|u_x\|_{2,n-2;B_{R_0}}, \tag{2.12}$$

and (2.9) with some θ_1, inequality (2.7) follows with $\theta = c(n)\theta_1$.

Proof of Theorem 2. Let condition (2.7) hold. We fix a ball $B_r(y^0) \subset B_{R_0}$ and write the integral identity with a test-function $h \in C_0^1(\overline{B_r(y^0)})$:

$$\int_{B_r(y^0)} [A_{kl}^{\alpha\beta}(x,u)u^l_{x_\beta} h^k_{x_\alpha} + b^k(x,u,u_x)h^k]\,dx = 0. \tag{2.13}$$

From (2.13), where $h = (u - u_{r,y^0})\,\xi^2$, $u_{r,y^0} = \fint_{B_r(y^0)} u(x)\,dx$, and ξ is a cut-off function for $B_r(y^0)$, $\xi = 1$ in $B_{\frac{r}{2}}(y^0)$, we derive the inequality

$$\int_{B_{\frac{r}{2}}(y^0)} |u_x|^2\,dx \leqslant \frac{4\mu}{\nu r^2}\int_{B_r(y^0)} |u-u_{r,y^0}|^2\,dx + \frac{2a}{\nu}\int_{B_r(y^0)} |u_x|^2\,|u-u_{r,y^0}|\,dx + \frac{2\mu}{\nu}\,|B_r|. \tag{2.14}$$

By the embedding theorem $W^1_{\frac{2n}{n+2}}(B_r) \hookrightarrow L_2(B_r)$, the estimate

$$\int_{B_r(y^0)} |u - u_{r,y^0}|^2\,dx \leqslant c(n)\left(\int_{B_r(y^0)} |u_x|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}}$$

is valid, and from (2.14) we obtain

$$\int_{B_{\frac{r}{2}}(y^0)} |u_x|^2\,dx \leqslant \frac{c(\nu,\mu)}{r^2}\left(\int_{B_r(y^0)} |u_x|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}} + \frac{2a}{\nu}J_r + \frac{2\mu}{\nu}\,|B_r|\,. \tag{2.15}$$

Now we estimate the integral $J_r = \int_{B_r(y^0)} |u_x|^2\,|u - u_{r,y^0}|\,dx$ in (2.15) by the Hölder inequality with an arbitrary $l>1$:

$$J_r \leqslant \left(\fint_{B_r(y^0)} |u_x|^{2l}\,dx \right)^{\frac{1}{l}} \left(\fint_{B_r(y^0)} |u-u_{r,y^0}|^{\frac{l}{l-1}} \right)^{\frac{l-1}{l}} |B_r| \leqslant |B_r|\,[u]_{\frac{l}{l-1},n;B_{R_0}} \cdot$$

$$\cdot \left(\fint_{B_r(y^0)} |u_x|^{2l}\,dx \right)^{\frac{1}{l}} \overset{(2.11)}{\leqslant} c(l,n)\,|B_r|\,[u]_{2,n;B_{R_0}} \left(\fint_{B_r(y^0)} |u_x|^{2l}\,dx \right)^{\frac{1}{l}} \overset{(2.7)}{\leqslant}$$

$$\overset{(2.7)}{\leqslant} c(l,n)\,|B_r|\,\theta \min\left\{ \frac{\nu}{a}, 1 \right\} \left(\int_{B_r(y^0)} |u_x|^{2l}\,dx \right)^{\frac{1}{l}}.$$

Then from (2.15) it follows that

$$(2.16)\quad \fint_{B_{\frac{r}{2}}(y^0)} |u_x|^2\,dx \leqslant c_1 \left[\left(\fint_{B_r(y^0)} |u_x|^{\frac{2n}{n+2}}\,dx \right)^{\frac{n+2}{n}} + 1 + c_2(l)\,\theta \left(\fint_{B_r(y^0)} |u_x|^{2l}\,dx \right)^{\frac{1}{l}} \right],$$

$c_1 = c_1(\nu,\mu)$. For $g(x) = |u_x(x)|^{\frac{2n}{n+2}} + 1$ and $q = \frac{n+2}{n}$, inequality (2.16) implies the inequality

$$(2.17)\quad \fint_{B_{\frac{r}{2}}} g^q\,dx \leqslant c_3(\nu,\mu) \left[\left(\fint_{B_r} g\,dx \right)^q + c_4(l)\,\theta \left(\fint_{B_r} g^{ql}\,dx \right)^{\frac{1}{l}} \right], \quad \forall B_r \subset B_{R_0}.$$

By Theorem 1, there exists a number $p_0 = p_0(c_3, q) > q$ such that if $l \in \left(1, \frac{p}{q}\right)$ for $p \in (q, p_0)$, and

$$(2.18)\quad c_4(l)\,\theta < \kappa_0,$$

for some $\kappa_0 = \kappa_0(p,q,l) > 0$ in (2.17), then

$$(2.19)\quad \fint_{Q_{\frac{R}{2}}} g^p\,dx \leqslant c_5 \fint_{Q_R} g^q\,dx, \quad \forall Q_R \subset B_{R_0}$$

with a constant c_5 dependent on p, q, l, ν, μ and θ. Now we fix $p = \frac{p_0+q}{2}$ and $l = \frac{p}{2q} + \frac{1}{2}$. Then $l = l(\nu,\mu,n) \in \left(1, \frac{p}{q}\right)$ and the parameter $s = \frac{2p}{q} > 2$ depends only on ν, μ and n.

Estimate (2.19) yields

$$(2.20)\quad \left(\fint_{B_{\frac{r}{2}}(y_0)} |u_x|^s\,dx \right)^{\frac{1}{s}} \leqslant c_6 \left(\fint_{B_{r\sqrt{n}}(y_0)} (|u_x|^2+1)\,dx \right)^{\frac{1}{2}}, \quad \forall B_{r\sqrt{n}}(y_0) \subset B_{R_0}.$$

The constant c_6 in (2.20) and all other constants c_i, $i \geqslant 6$, may depend on ν, μ, n, and θ. The dependence of c_i on other parameters will be shown additionally. In particular, estimate (2.20) implies

$$\|u_x\|_{s,\,B_{R_1}} \leqslant c_7(R_1)(\|u_x\|_{2,\,B_{R_0}} + 1), \qquad s = \frac{2p}{q} > 2, \qquad R_1 = \frac{R_0}{2\sqrt{n}}. \tag{2.21}$$

Henceforth, we assume that condition (2.9) holds with $\theta_1 = \frac{\theta}{c(n)}$, where θ was fixed by (2.18) (see Remark 2.1).

It is obvious that estimates (2.20) and (2.21) are valid. We apply them to estimate the Hölder norm of the solution u in the ball B_{R_2}, $R_2 = \frac{R_1}{2\sqrt{n}} = \frac{R_0}{4n}$.

For a fixed $y^0 \in B_{R_2}$ and $R < R_2$ we consider the linear problem

$$\begin{aligned} &A_{kl}^{\alpha\beta}(y^0, u^0)\, v^l_{x_\beta x_\alpha} = 0 \quad \text{in } B_R(y^0), \qquad u^0 = u_{R,\,y^0}, \\ &v|_{S_R(y^0)} = u. \end{aligned} \tag{2.22}$$

We use a well-known Campanato estimate of the solution v of problem (2.22):

$$\int\limits_{B_\varrho(y^0)} |v_x|^2\, dx \leqslant c_{12}\left(\frac{\varrho}{R}\right)^n \int\limits_{B_R(y^0)} |v_x|^2\, dx, \qquad \forall \varrho \leqslant R. \tag{2.23}$$

The function $w = u - v$ is a solution of the Dirichlet problem

$$\begin{aligned} &A_{kl}^{\alpha\beta}(y^0, u^0)\, w^l_{x_\beta x_\alpha} = (F^k_\alpha(x))_{x_\alpha}, \qquad x \in B_R(y^0), \\ &w|_{S_R(y^0)} = 0, \end{aligned} \tag{2.24}$$

where $F^k_\alpha = A_{kl}^{\alpha\beta}(y^0, u^0)\, u^l_{x_\beta} \in L^{2,\,n-2}(B_{R_0})$.

From regularity theory in Morrey and Campanato spaces [9, 12] it follows that the solution $w \in W^1_2(B_R)$ of (2.24) has derivatives $w_{x_\alpha} \in L^{2,\,n-2}(B_R(y^0))$, $\alpha = 1, \ldots, n$, and,

$$\|w_x\|_{2,\,B_R(y^0)} \leqslant c_8 \|u_x\|_{2,\,B_R(y^0)}, \tag{2.25}$$

$$\|w_x\|_{2,\,n-2;\,B_R(y^0)} \leqslant c_9\{\|u_x\|_{2,\,n-2;\,B_R(y^0)} + R^{-\frac{(n-2)}{2}} \|w_x\|_{2,\,B_R(y^0)}\}. \tag{2.26}$$

Estimates (2.25) and (2.26) imply

$$\|w_x\|_{2,\,n-2;\,B_R(y^0)} \leqslant c_{10}\|u_x\|_{2,\,n-2;\,B_{R_0}} \leqslant c_{11} \min\left\{\frac{\nu}{a}, 1\right\} \theta_1. \tag{2.27}$$

From the integral identities for u and v in $B_R(y^0)$, it follows the relation

$$\begin{aligned} \int\limits_{B_R(y^0)} [&A_{kl}^{\alpha\beta}(y^0, u^0)\, w^l_{x_\beta} w^k_{x_\alpha} + (A_{kl}^{\alpha\beta}(x, u) - A_{kl}^{\alpha\beta}(y^0, u^0))\, u^l_{x_\beta} w^k_{x_\alpha} + \\ &+ b^k(x, u, u_x)\, w^k]\, dx = 0. \end{aligned} \tag{2.28}$$

With the help of assumptions (2.2)-(2.5), from (2.28) we deduce the inequality

$$\int_{B_R(y^0)} |w_x|^2\,dx \leqslant c(\nu,\mu) \int_{B_R(y^0)} \omega^2(|x-y^0|^2,\ |u-u^0|^2)\,|u_x|^2\,dx + \tag{2.29}$$
$$+\frac{2a}{\nu}\int_{B_R(y^0)} |u_x|^2\,|w|\,dx + \frac{2M}{\nu}\int_{B_R(y^0)} |w|\,dx,$$

where $\omega(\cdot,\cdot)$ is a function from (2.4).

The right-hand side of (2.29) can be estimated by (2.20):

$$\int_{B_R(y^0)} |w_x|^2\,dx \leqslant c_{13}\left(\fint_{B_R(y_0)} |u_x|^s\,dx\right)^{\frac{2}{s}}\left(\fint_{B_R(y_0)} \omega^{\frac{2s}{s-2}}(R^2,\ |u-u^0|^2)\,dx\right)^{\frac{s-2}{s}} |B_R| +$$
$$+\frac{2a}{\nu}\left(\fint_{B_R(y_0)} |u_x|^s\,dx\right)^{\frac{2}{s}}\left(\fint_{B_R(y_0)} |w|^{\frac{s}{s-2}}\,dx\right)^{\frac{s-2}{s}} |B_R| + c_{13}R^{n+2} \overset{(2.20)}{\leqslant} \tag{2.30}$$
$$\overset{(2.20)}{\leqslant} c_{14}\fint_{B_{2\sqrt{n}R}} (1+|u_x|^2)\,dx\left[\omega_0\left(R^2,\ \fint_{B_R(y_0)} |u-u^0|^2\,dx\right) + \frac{2a}{\nu}\cdot\right.$$
$$\left.\cdot\left(\fint_{B_R(y_0)} |w|^{\frac{s}{s-2}}\,dx\right)^{\frac{s-2}{s}} + R^2\right].$$

Here $\omega_0(t,\tau)$ is a continuous function nonincreasing in t,τ, and $\omega_0(t,\tau)\to 0$ as $t,\tau\to 0$.

Now we estimate the integral $P_R = \left(\fint_{B_R(y_0)} |w|^{\frac{s}{s-2}}\,dx\right)^{\frac{s-2}{s}}$. We have

$$P_R \leqslant \left(\fint_{B_R(y_0)} |w - w_{R,y^0}|^{\frac{s}{s-2}}\,dx\right)^{\frac{s-2}{s}} + |w_{R,y^0}| \leqslant [w]_{\frac{s}{s-2},n;\,B_R(y^0)} +$$
$$+\|w\|_{2,B_R(y^0)}\cdot R^{\frac{n}{2}} \overset{(2.11)}{\leqslant} c(s,n)[w]_{2,n;\,B_R(y^0)} + c(n)R^{\frac{n-2}{2}}\|w_x\|_{2,B_R(y^0)} \overset{(2.12)}{\leqslant}$$
$$\overset{(2.12)}{\leqslant} c_{15}(s,n)\|w_x\|_{2,n-2;\,B_R(y^0)} \overset{(2.27)}{\leqslant} c_{16}\min\left\{\frac{\nu}{a},\ 1\right\}\theta_1.$$

Hence,

$$\fint_{B_R(y_0)} |w_x|^2\,dx \leqslant \tag{2.31}$$
$$\leqslant c_{17}\left[\omega_0\left(R^2,\ c_*(n)\,R^{-(n-2)}\int_{B_r(y^0)} |u_x|^2\,dx\right) + \theta_1 + R^2\right]\int_{B_{2\sqrt{n}R}(y^0)} (1+|u_x|^2)\,dx.$$

If we rename the number $2\sqrt{n}R$ by R and put $\Phi(r,y^0) = \int_{B_r(y^0)} (1+|u_x|^2)\,dx$, then from (2.23) and (2.31) it is easy to derive the inequality

$$\Phi(\varrho, y^0) \leq c_{18}\left[\left(\frac{\varrho}{R}\right)^n + \omega_0(R^2, c_*(n), \theta_1^2) + \theta_1 + R^2\right]\Phi(R, y^0), \quad \forall \varrho \leq R \leq R_1. \tag{2.32}$$

By a well-known algebraic lemma (see, for example, [7, Chapter 3, Lemma 2.1]), there exist a number $\varepsilon_0 = \varepsilon_0(c_{18}, n) > 0$ such that if

$$\omega_0(R^2, c_*(n)\,\theta_1^2) + \theta_1 + R^2 < \varepsilon_0, \tag{2.33}$$

then from (2.32) it follows that

$$\Phi(\varrho, y^0) \leq c_{19}\left(\frac{\varrho}{R}\right)^{n-\delta}\Phi(R, y^0), \quad \forall \varrho \leq R \leq R_1, \tag{2.34}$$

for any $\delta \in (0, n)$, $c_{19} = c_{19}(c_{18}, \delta) > 0$.

Now we fix r_0 and $\tau_0 > 0$ in such a way as to have the inequality

$$\omega_0(r_0^2, \tau_0^2) < \frac{\varepsilon_0}{3}. \tag{2.35}$$

Let θ_1 satisfy the restriction

$$\theta_1 \leq \min\left\{\frac{\theta}{c(n)}, \frac{\varepsilon_0}{3}, \frac{\tau_0}{\sqrt{c_*(n)}}\right\} \tag{2.36}$$

and

$$r_0 \leq \min\left\{R_1, \sqrt{\frac{\varepsilon_0}{3}}\right\}. \tag{2.37}$$

Then inequality (2.33) holds for $R \leq r_0$ and, consequently, estimate (2.34) is valid for $\varrho \leq R \leq r_0$.

Estimate (2.34) holds for any point $y^0 \in B_{R_2}$, and, therefore,

$$\|u_x\|_{2,n-\delta;B_{R_2}} \leq c_{20}(\delta, r_0)(\|u_x\|_{2,B_{R_0}} + 1), \quad \forall \delta > 0. \tag{2.38}$$

This implies

$$[u]_{2,n+2-\delta;B_{R_2}} \leq c_{21}(\delta, r_0)(\|u_x\|_{2,B_{R_0}} + 1), \quad \forall \delta > 0. \tag{2.39}$$

By the isomorphism of the spaces $\mathcal{L}^{2,n+2-\delta}(B_{R_2})$ and $C^{1-\frac{\delta}{2}}(\overline{B_{R_2}})$,

$$\|u\|_{C^{1-\frac{\delta}{2}}(\overline{B_{R_2}})} \leq c_{22}(\delta, r_0)(\|u\|_{W_2^1(B_{R_0})} + 1), \quad \forall \delta > 0. \tag{2.40}$$

Finally, we note that (2.36) describes the dependence of θ_1 on ν, μ, n and the function ω from (2.4). The number r_0 depends on the same parameters and R_0 (see (2.37)). Hence, estimate (2.10) is obtained and Theorem 2 is proved. □

References

[1] F.W. Gehring, *The L^p-integrability of the partial derivatives of a quasi conformal mapping.* Acta Math., 130, 1973, 265-277.

[2] M. Giaquinta - G. Modica, *Regularity results for some classes of higher order non-linear elliptic systems.* J. für Reine u. Angew. Math., 311/312, 1979, 145-169.

[3] E.W. STREDULINSKY, *Higher integrability from reverce Hölder inequalities.* Indiana Univ. Math. J., 29, 3, 1980, 408-417.

[4] A.A. ARKHIPOVA, *Reverse Hölder inequalities with boundary integrals and L^p-estimates for solution of nonlinear elliptic and parabolic boundary-value problems.* Advances in Math. Sci. Translations Ser., 2, 164, 1995, 15-42.

[5] A.A. ARKHIPOVA - O.A. LADYZHENSKAYA, *On a modification of Gehring lemma.* Zapiski Nauchn, Semin. POMI, St-Petersburg, 259, 1999, 7-18.

[6] A.A. ARKHIPOVA, *On the regularity of the solution of the Neumann problem for quasilinear parabolic systems.* Russian Acad. Sci. Izv. Math., 45, 1995, 231-253.

[7] M. GIAQUINTA, *Multiple integrals in the Calculus of Variations and Nonlinear Elliptic Systems.* Annals of Math., Studies 105, Princeton Univ. Press, Princeton 1983.

[8] M. GIAQUINTA - E. GIUSTI, *Non linear elliptic systems with Quadratic growth.* Manuscripta Math., 24, 1978, 323-349.

[9] A. KUFNER - O. JOHN - S. FUČIK, *Functional Spases.* Academia, Prague 1977.

[10] J. FREHSE, *On two-dimensional quasi-linear elliptic systems.* Manuscripta Math., 28, 1979, 21-50.

[11] C. HAMBURGER, *A new partial regularity proof for solutions of nonlinear elliptic systems.* Manuscripta Math., 95, n. 1, 1998, 11-31.

[12] G.M. TROIANIELLO, *Elliptic differential equations and obstacle problems.* Plenum, New York-London 1987.

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St. Petersburg State University
Math. & Mech. Faculty 198504
University Street, 28, Petergof
SAINT-PETERSBURG (Russia)
arina@aa1101.spb.edu