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The fourth tautological group of $\overline{\mathfrak{M}}_{g,n}$ and relations with the cohomology

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Geometria algebrica. — *The fourth tautological group of $\overline{\mathcal{M}}_{g,n}$ and relations with the cohomology.* Nota di MARZIA POLITO, presentata (*) dal Socio C. De Concini.

ABSTRACT. — We give a complete description of the fourth tautological group of the moduli space of pointed stable curves, $\overline{\mathcal{M}}_{g,n}$, and prove that for $g \geq 8$ it coincides with the cohomology group with rational coefficients. We further give a conjectural upper bound depending on the genus for the degree of new tautological relations.

KEY WORDS: Algebraic geometry; Algebraic curves; Moduli spaces; Cohomology.

RIASSUNTO. — *Il quarto gruppo tautologico di $\overline{\mathcal{M}}_{g,n}$ e relazioni con la coomologia.* Si dà una descrizione completa del quarto gruppo tautologico dello spazio di moduli delle curve puntate stabili, $\overline{\mathcal{M}}_{g,n}$, e si dimostra che per $g \geq 8$ tale gruppo coincide con il gruppo di coomologia a coefficienti razionali. Si formula inoltre una congettura sulla dimensione massima del grado delle nuove relazioni tautologiche, in funzione del genere.

1. INTRODUCTION

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of n -pointed complex stable algebraic curves of genus g .

The existence of some degree 4 relations among tautological classes has been proved with various methods by E. Getzler, C. Faber, R. Pandharipande and P. Belorousski, while other relations are obtained as a consequence of the well known ones in degree 2.

We actually prove that no other relations can arise, and that for genus $g \geq 8$, the cohomology group $H^4(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ coincides with its tautological subgroup. The main results of this paper are formally stated in Theorems 10 and 19. It turns out that new relations appear only in genus up to 5, whereas for higher genus all possible relations arise only as a consequence of degree 2 ones. The proof of this fact allows us to suggest in Conjecture 18 an upper bound depending on the genus for higher degree new tautological relations.

As for the methods, E. Arbarello and M. Cornalba proposed in [1] new methods for computing the cohomology groups with rational coefficients of $\overline{\mathcal{M}}_{g,n}$; their strategy is to establish a strict relation between the cohomology of the moduli space and the one of the irreducible components of the boundary, which in turn can be expressed in terms of moduli spaces of curves with lower genus or with lower number of marked points. With similar arguments, we establish inductive procedures on genus and/or number of markings to derive constraints among coefficients in possible relations.

(*) Nella seduta del 13 dicembre 2002.

We will therefore be able to give the explicit expression of a new relation in $H^4(\overline{\mathcal{M}}_{3,2})$, whose existence was proved by Faber as a consequence of the existence of a tautological relation on the open part $\mathcal{M}_{3,2}$. Furthermore, we will exclude the existence of any relation other than the known ones.

A description of $H^4(\overline{\mathcal{M}}_g, \mathbb{Q})$, for $g \geq 12$, has been given by D. Edidin in [8], and once the tautological group is known, we can adapt his argument to prove that for $g \geq 8$, it coincides with the cohomology. For this, we make use of the results by Harer [16], Ivanov [17] and Loojenga [19] on the homology of the mapping class group.

This paper is extracted from my *Tesi di Perfezionamento* at the Scuola Normale Superiore, Pisa. In the present exposition, many of the calculations will be omitted. The interested reader can find them all in the thesis [22], available upon request from the author.

2. STABLE GRAPHS AND TAUTOLOGICAL CLASSES

To every stable curve C of genus g , with P as a set of markings, one can associate a labelled graph Γ in the following way:

1. draw a vertex v for every irreducible component $C(v)$ of the normalization \tilde{C} of C , and label it with the genus $g(v)$ of that component,
2. draw an edge between two vertices v_1, v_2 (possibly a loop if $v_1 = v_2$) whenever the normalization map $\nu : \tilde{C} \rightarrow C$ identifies two points lying respectively in $C(v_1)$ and $C(v_2)$,
3. draw a half-edge with vertex v whenever there is a marking in $\nu(C(v))$, and label it with the marking's name. We denote by $P(v)$ the set of these markings.

We call **marked half-edges** the half-edges constructed in 3. The total set of half-edges is the union of the set of marked half-edges with the set consisting of the halves of the edges constructed in 2.

Let $r(v)$ be the valence of a vertex, namely the number of half-edges with vertex v . The stability condition translates to: $2g(v) + r(v) \geq 3$, for every vertex v . The genus of a curve corresponding to the graph Γ is $g(\Gamma) = \chi(\Gamma) + \sum_v g(v)$. Observe that the construction of the graph is only based on the topological type of the curve.

DEFINITION 1. *A **P-marked stable graph of genus g** (briefly a (g, P) graph), is a connected graph with $n = |P|$ marked half-edges, with the following additional data:*

- 1) each vertex v is labelled with an integer $g(v)$,
- 2) the valence $r(v)$ of any vertex satisfies the stability condition $2g(v) + r(v) \geq 3$,
- 3) there is a bijection between marked half-edges and elements in P ,
- 4) $g = \chi(\Gamma) + \sum_v g(v)$.

The codimension of a graph is defined as the number of its edges.

Given a P -marked stable graph of genus g and codimension d , with set of vertices

V , one can associate to it a closed stratum of codimension d in $\overline{\mathfrak{M}}_{g,P}$. For every vertex $v \in V$, we let $S(v)$, denote the set of unmarked half-edges with vertex v .

Let $\overline{\mathfrak{M}}_\Gamma := \prod_{v \in V} \mathfrak{M}_{g(v), P(v) \cup S(v)}$.
 The map

$$\xi_\Gamma: \overline{\mathfrak{M}}_\Gamma \rightarrow \overline{\mathfrak{M}}_{g,P}$$

is called a boundary map, and has the closed stratum $\Delta_\Gamma = \xi_\Gamma(\overline{\mathfrak{M}}_\Gamma)$ as image.

The notation $\overline{\mathfrak{M}}_\Gamma$ will be used also when Γ is disconnected: if $\Gamma = \Gamma_1 \sqcup \Gamma_2$, then $\overline{\mathfrak{M}}_\Gamma = \overline{\mathfrak{M}}_{\Gamma_1} \times \overline{\mathfrak{M}}_{\Gamma_2}$.

Let Γ be a (g, P) -graph.

DEFINITION 2. *The graph G is a Γ -graph if it is the disjoint union of a collection of $(g(v), P(v) \cup S(v))$ -graphs.*

Look at a Γ -graph G . Set $G = \sqcup_{v \in V} G_v$. We can define the map

$$\overline{\mathfrak{M}}_G = \prod \overline{\mathfrak{M}}_{G_v} \xrightarrow{\xi_G} \overline{\mathfrak{M}}_\Gamma$$

as $\xi_G = \{\xi_{G_v}\}_{v \in V}$.

Let π be the forgetful map:

$$\begin{aligned} \pi: \overline{\mathfrak{M}}_{g,n+1} &\rightarrow \overline{\mathfrak{M}}_{g,n} \\ [C, p_1, \dots, p_n, p_{n+1}] &\rightarrow [C, p_1, \dots, p_n]. \end{aligned}$$

We will also refer to the map π as the universal curve, or the projection map.

Let $\sigma_1, \dots, \sigma_n$ be the n canonical sections of the forgetful map, and let D_i be the image of σ_i . Finally, let ω_π be the relative dualizing sheaf of π .

We recall the definition of the basic cohomology classes in $\overline{\mathfrak{M}}_{g,P}$ (see [2]):

DEFINITION 3.

$$\begin{aligned} \psi_i &= \sigma_i^*(c_1(\omega_\pi)), \quad i = 1, \dots, n \\ \kappa_a &= \pi_*((c_1(\omega_\pi(\sum D_j)))^{a+1}), \quad a = 0, \dots, 3g-3+n \end{aligned}$$

The class ψ_i can be interpreted as the first Chern class of the orbifold bundle whose fiber over the point $[C, p_1, \dots, p_n]$ is the cotangent bundle to the curve C evaluated at the point p_i .

DEFINITION 4. *A **Mumford class** in $H^*(\overline{\mathfrak{M}}_{g,P}, \mathbb{Q})$ is a polynomial in the classes ψ_i, κ_a . The **Mumford ring** is*

$$\mathbb{Q}[\psi_1, \dots, \psi_n, \kappa_1, \dots, \kappa_{3g-3+n}].$$

The Mumford ring on a product or a disjoint union of moduli spaces is the tensor product or the direct sum of the Mumford rings.

It is worth noticing that the following formula (see [2, Formula 1.7]) holds:

$$\kappa_a = \pi_* (\psi_{n+1}^{a+1}).$$

DEFINITION 5. A **Mumford class** in $H^*(\mathfrak{M}_{g,p}, \mathbb{Q})$ is the pull-back under the inclusion

$$\mathfrak{M}_{g,p} \rightarrow \overline{\mathfrak{M}}_{g,p}$$

of a polynomial in the classes ψ_i, κ_a .

DEFINITION 6. A **tautological class** is the push-forward of a Mumford class via a boundary map. The k -th **tautological group** $T_{g,p}^k$ is the subspace of $H^k(\overline{\mathfrak{M}}_{g,p}, \mathbb{Q})$ generated by these classes.

In figures 1 and 2 we draw all the graphs of codimension 1 and codimension 2 which we need in our study of $T_{g,p}^4$. In each figure we will also write the name of the corresponding graph. Every time half-edges are drawn, one should imagine them labelled with the correspondent markings.

If p is a Mumford class, we use the following notation:

$$p | \delta_\Gamma := \frac{\xi_{\Gamma^*}(p)}{|Aut \Gamma|}.$$

We will often write δ_{irr}, ξ_{irr} instead of $\delta_{\Gamma_{irr}}, \xi_{\Gamma_{irr}}$, and $\delta_{a,A}, \xi_{a,A}$ instead of $\delta_{\Gamma_{a,A}}, \xi_{\Gamma_{a,A}}$.

Degree 4 autological classes are:

1. **Pure boundary classes:** let Γ be a graph of codimension 2, then we define:

$$\delta_\Gamma := \frac{\xi_{\Gamma^*}(1)}{|Aut \Gamma|}.$$

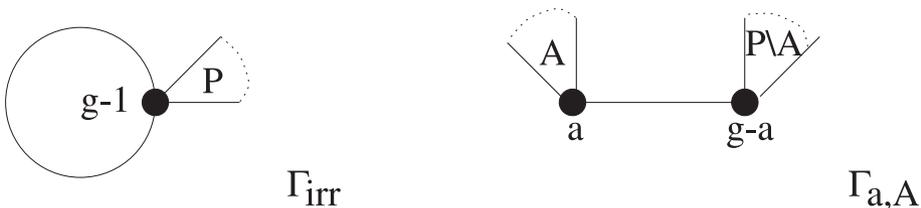


Fig. 1. – Graphs of codimension 1.

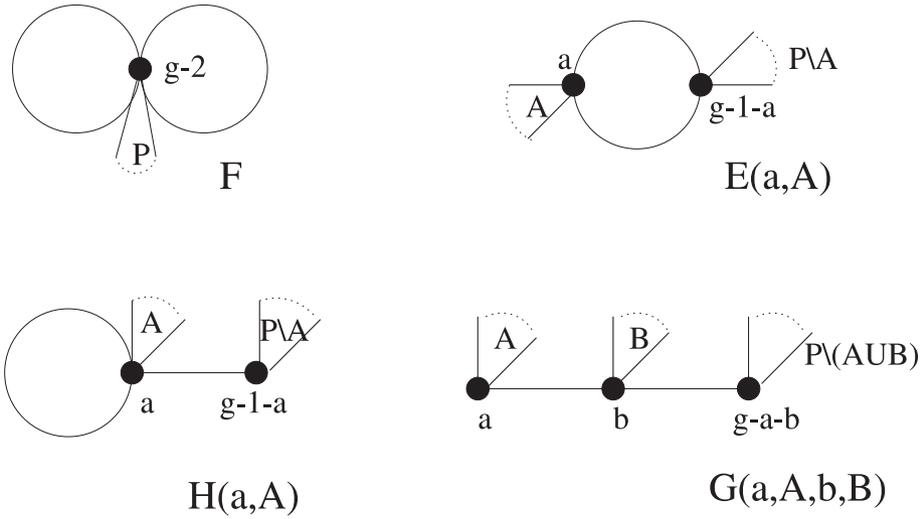


Fig. 2. – Graphs of codimension 2.

2. Mixed boundary classes: if $\text{codim } \Gamma = 1$, and p is a Mumford class of degree 2 in $\overline{\mathfrak{M}}_\Gamma$, then

$$p | \delta_\Gamma := \frac{\xi_{\Gamma^*}(p)}{|Aut \Gamma|}.$$

We will often use the following simplified notation:

- $\psi_i \delta_{a,A} = (\psi_i \otimes 1) | \delta_{a,A} = \frac{1}{|Aut \Gamma_{a,A}|} \xi_{a,A^*}(\psi_i \otimes 1),$
- $\psi | \delta_{a,A} = (\psi_s \otimes 1) | \delta_{a,A} = \frac{1}{|Aut \Gamma_{a,A}|} \xi_{a,A^*}(\psi_s \otimes 1),$
- $\delta_{a,A} | \psi = (1 \otimes \psi_t) | \delta_{a,A} = \frac{1}{|Aut \Gamma_{a,A}|} \xi_{a,A^*}(1 \otimes \psi_t) = \psi | \delta_{g-a,A^c},$
- $\kappa | \delta_{a,A} = (\kappa_1 \otimes 1) | \delta_{a,A} = \frac{1}{|Aut \Gamma_{a,A}|} \xi_{a,A^*}(\kappa_1 \otimes 1),$
- $\delta_{a,A} | \kappa = (1 \otimes \kappa_1) | \delta_{a,A} = \frac{1}{|Aut \Gamma_{a,A}|} \xi_{a,A^*}(1 \otimes \kappa_1) = \kappa | \delta_{g-a,A^c},$
- $\psi_i \delta_{irr} = (\psi_i) | \delta_{irr} = \frac{1}{|Aut \Gamma_{irr}|} \xi_{irr^*}(\psi_i),$
- $\psi | \delta_{irr} = (\psi_q + \psi_r) | \delta_{irr} = \frac{1}{|Aut \Gamma_{irr}|} \xi_{irr^*}(\psi_q + \psi_r),$
- $\kappa_1 \delta_{irr} = \kappa_1 | \delta_{irr} = \frac{1}{|Aut \Gamma_{irr}|} \xi_{irr^*}(\kappa_1).$

3. Mumford classes: these are simply monomials in Mumford classes (considered as push-forward via the map corresponding to the trivial graph).

In the mixed boundary classes we intentionally used ambiguous notation. Some of the classes $(\psi_i \delta_{a,A}, \psi_i \delta_{irr}, \kappa_1 \delta_{irr})$ turn out to be written as a product of a codimension 1 boundary class with a Mumford class. In the proof of the next Proposition we will show that the above notation is unambiguous.

PROPOSITION 7. *The image of the map:*

$$H^2(\overline{\mathfrak{M}}_{g,p}) \times H^2(\overline{\mathfrak{M}}_{g,p}) \rightarrow H^4(\overline{\mathfrak{M}}_{g,p}) \quad (\alpha, \beta) \rightarrow \alpha \cdot \beta$$

lies in $T_{g,p}^4$.

PROOF. Recall that $H^2(\overline{\mathfrak{M}}_{g,p}) = T_{g,p}^2$. Two irreducible codimension 1 boundary classes either coincide or intersect transversally. In the latter case, it is trivial to check that their intersection is a linear combination of tautological pure boundary classes. The product of two Mumford classes is clearly a Mumford class.

Finally, using the push-pull formula, one is able to express the product of a Mumford class and a boundary class, and the square of a boundary class, as linear combination of tautological classes:

$$\begin{aligned} \psi_i \cdot \delta_{a,A} &= \psi_i | \delta_{a,A} & \psi_i \cdot \delta_{irr} &= \psi_i | \delta_{irr} \\ \kappa_1 \cdot \delta_{a,A} &= \kappa_1 | \delta_{a,A} + \delta_{a,A} | \kappa_1 & \kappa_1 \cdot \delta_{irr} &= \kappa_1 | \delta_{irr} \end{aligned}$$

$$\delta_{a,A}^2 = -\psi | \delta_{a,A} - \delta_{a,A} | \psi + \begin{cases} \frac{2}{|Aut \Gamma_{a,A}|} \delta_{G(g-a, \emptyset, 2a-g, P)} & \text{if } A = P \\ \frac{2}{|Aut \Gamma_{a,A}|} \delta_{G(a, \emptyset, g-2a, P)} & \text{if } A = \emptyset \end{cases}$$

$$\delta_{irr}^2 = -\frac{1}{2} \xi_{irr^*}(\psi_q + \psi_r) + 2\delta_F + 2 \sum \delta_{E(a,A)} - \psi | \delta_{irr} + 2\delta_F + 2 \sum \delta_{E(a,A)}.$$

We compute explicitly one sample case. Since

$$\xi_{irr^*}(\delta_{irr}) = \delta_{irr} + \sum \delta_{a, A \cup \{q\}} - \psi_q - \psi_r,$$

then

$$(2.13) \quad 2\delta_{irr}^2 = \xi_{irr^*} \xi_{irr^*}(\delta_{irr}) = \frac{1}{2} \xi_{irr^*} \tilde{\xi}_{irr^*}(1) + \sum \xi_{irr^*} \tilde{\xi}_{a, A \cup \{q\}^*}(1) - \xi_{irr^*}(\psi_q + \psi_r),$$

where the symbol $\tilde{\xi}$ is used for boundary maps of $\overline{\mathfrak{M}}_{g-1, P \cup \{q, r\}}$. In fact, from now on, when composing two boundary maps, we will append the second one with the twiddle.

We easily compute: $\frac{1}{2} \xi_{irr^*} \tilde{\xi}_{irr^*}(1) = \frac{1}{2} \xi_{F^*}(1) = 4\delta_F$, and then observe that $\xi_{irr} \tilde{\xi}_{a, A \cup \{q\}} = \xi_{E(a,A)}$ and that the corresponding graph has automorphism order 2, unless $P = \emptyset$, $a = g/2$, when the order is 4. Moreover, $\xi_{irr^*} \tilde{\xi}_{a, A \cup \{q\}^*}(1) = \xi_{irr^*} \tilde{\xi}_{g-a, A^c \cup \{q\}^*}(1) = |Aut \Gamma_{E(a,A)}| \delta_{E(a,A)}$. Whenever $|Aut \Gamma_{E(a,A)}| = 4$, then by symmetry only one of the summands above does appear, hence we can write

$$\delta_{irr}^2 = -\frac{1}{2} \xi_{irr^*}(\psi_q + \psi_r) + 2\delta_F + 2 \sum \delta_{E(a,A)} - \psi | \delta_{irr} + 2\delta_F + 2 \sum \delta_{E(a,A)}. \quad \square$$

3. ESSENTIAL TAUTOLOGICAL CLASSES

It is well known that, for genus up to 2, there are some relations between degree 2 tautological classes; thus, certain tautological classes could be expressed as linear combination of other ones; they are: κ_1 and $\psi_i, i \in P$ for genera $g = 0, 1, \kappa_1$ for genus $g = 2$.

Moreover, there are Keel's relations among boundary classes in genus 0.

All these relations reproduce themselves in every genus. The reason is quite clear: every time there is a relation among tautological classes in the second cohomology group of a codimension 1 boundary component, we can push it forward to $H^4(\overline{\mathfrak{M}}_{g,p})$.

In this section we will choose a set of degree 4 tautological classes which generate $T_{g,p}^4$, by eliminating the above relations. We will call these classes the **essential** tautological classes. The set of essential tautological classes will be denoted by $\mathcal{B}_{g,p}^4$ and it is obtained from the set of all tautological classes by removing the **unessential** classes which we are presently going to list.

The unessential tautological classes are:

$$\begin{aligned}
 \psi | \delta_{0,A} &= \xi_{0,A^*}(\psi_s \otimes 1) & \psi_i \delta_{0,A} &= \xi_{0,A^*}(\psi_i \otimes 1) & \kappa | \delta_{0,A} &= \xi_{0,A^*}(\kappa_1 \otimes 1) \\
 & & & & & \text{for any } g, \\
 \psi | \delta_{1,A} &= \xi_{1,A^*} \left(\frac{\psi_s \otimes 1}{|Aut \Gamma_{1,A}|} \right) & \psi_i \delta_{1,A} &= \xi_{1,A^*}(\psi_i \otimes 1) & \kappa | \delta_{1,A} &= \xi_{1,A^*} \left(\frac{\kappa_1 \otimes 1}{|Aut \Gamma_{1,A}|} \right) \\
 & & & & & \text{for any } g, \\
 \kappa | \delta_{2,A} &= \xi_{2,A^*} \left(\frac{\kappa_1 \otimes 1}{|Aut \Gamma_{2,A}|} \right) & & & & \text{for any } g, \\
 \psi | \delta_{irr} &= \xi_{irr^*} \left(\frac{\psi_q + \psi_r}{2} \right) & & & & \text{for } g = 1, 2, \\
 \psi_i \delta_{irr} &= \xi_{irr^*} \left(\frac{\psi_i}{2} \right) & & & & \text{for } g = 1, 2, \\
 \kappa | \delta_{irr} &= \xi_{irr^*} \left(\frac{1\kappa_1}{2} \right) & & & & \text{for } g = 1, 2, 3, \\
 \psi_i^2, \psi_i \psi_j & & \kappa_1^2, \kappa_1 \psi_i & & & \text{for } g = 0, 1, \\
 & & \kappa_1^2, \kappa_1 \psi_i & & & \text{for } g = 2.
 \end{aligned}$$

Moreover, some classes $\delta_{G(0,A,0,B)}$ are unessential (see below); in fact, in genus 0 there are Keel's relations [18] among boundary classes: we can push them forward by means of the maps

$$H^2(\overline{\mathfrak{M}}_{0,A \cup \{s\}}) \xrightarrow{\phi_{0,A^*}} H^4(\overline{\mathfrak{M}}_{g,p})$$

to obtain the following relations:

$$\sum_{\substack{x, y \in B, \\ z, w \in C, \\ B \cup C = A}} \delta_{G(0, B, 0, C)} + \delta_{G(0, C, 0, B)} = \sum_{\substack{x, z \in B, \\ y, w \in C, \\ B \cup C = A}} \delta_{G(0, B, 0, C)} + \delta_{G(0, C, 0, B)},$$

$$\sum_{\substack{x, y \in B, \\ z \in C, \\ B \cup C = A}} \delta_{G(0, B, 0, C)} = \sum_{\substack{x, z \in B, \\ y \in C, \\ B \cup C = A}} \delta_{G(0, B, 0, C)}.$$

We now describe a subset of essential classes of this type; if we fix an ordering in P , this induces an ordering of every subset A ; a basis for $H^2(\overline{\mathcal{M}}_{0, A \cup \{s\}})$ consists of classes $\delta_{0, \{s\} \cup C}$, with $B = A \setminus C$, $|B| \geq 3$, or $|B| = 2$ and $b < c \forall b \in B, \forall c \in C$. This implies that we are going to consider only classes $\delta_{G(0, B, 0, C)}$, with $|B| \geq 3$, or $|B| = 2$ and $b < c \forall b \in B, \forall c \in C$.

4. PULL-BACK FORMULAS

In this section we show how to pull back tautological classes to the codimension 1 boundary components and to the universal curve. Let A be a stable (g, P) -graph of codimension 1, as defined in the introduction, and let Γ be a stable connected (g, P) -graph of codimension ≤ 2 . We fix our attention on a class of the form $p|\delta_\Gamma = \frac{1}{|\text{Aut } \Gamma|} \xi_{\Gamma^*}(p)$. We want to describe the boundary components of $\overline{\mathcal{M}}_A$ on which the pull-back $\xi_A^*(p|\delta_\Gamma)$ is supported.

Given any stable A -graph G , let $j_{s,t}(G)$ be the graph obtained by gluing the half edges s and t , and let $f_{s,t}(G)$ be the graph obtained from $j_{s,t}(G)$ by collapsing the new edge. Via the operation $j_{s,t}$ we are either creating a node on an irreducible component, or joining two irreducible components at a point. In either case we are creating a node. Via the operation $f_{s,t}$ we are smoothing the new node.

We claim that the boundary components we are looking for correspond to A -graphs G such that $j_{s,t}(G) = \Gamma$ or $f_{s,t}(G) = \Gamma$. It is very simple to produce graphs G of this sort.

Either $\Delta_\Gamma \subseteq \Delta_A$, or Δ_Γ and Δ_A intersect transversally. If Δ_Γ and Δ_A intersect transversally there must be at least a vertex v of Γ and a simple Feynman move based at v making Γ a degeneration of A . Cutting into a half the edge produced by the Feynman move, and calling the two new half edges s and t , creates a stable A -graph G having the property that $f_{s,t}(G) = \Gamma$.

Suppose, on the other hand, that Δ_Γ is contained in Δ_A . This simply means that there is at least one edge of Γ cutting which produces two half edges s and t and a stable A -graph G with the property that $j_{s,t}(G) = \Gamma$.

Furthermore we can say that $\Delta_\Gamma \subseteq \Delta_A$ if and only if there exist a graph G such that $j_{s,t}(G) = \Gamma$.

In conclusion, whatever the position of Δ_Γ is with respect to Δ_A , we can build a diagram:

$$\begin{array}{ccc} \overline{\mathfrak{M}}_G & \xrightarrow{\zeta_G} & \mathfrak{M}_A \\ \downarrow \eta_G & & \downarrow \xi_A \\ \overline{\mathfrak{M}}_\Gamma & \xrightarrow{\xi_\Gamma} & \overline{\mathfrak{M}}_{g,P} \end{array}$$

for any graph G such that $j_{s,t}(G) = \Gamma$ or $f_{s,t}(G) = \Gamma$. The maps ξ_A and ξ_Γ are boundary maps, the map ζ_G has been defined in Section 2, and the map η_G consists in joining the two half-edges s and t of the graph G .

Observe that some of these maps could be the identity: e.g. if $\Gamma = A = \Gamma_{irr}$, then the trivial A -graph G satisfies $j_{s,t}(G) = \Gamma$, and the map ζ_G is the identity.

PROPOSITION 8. *Let Γ be any stable graph, of codimension ≤ 2 . Let A be any graph of codimension 1. Then the following formula holds:*

$$\frac{\xi_A^*(\xi_\Gamma^*(p))}{Aut \Gamma} = \sum_{f_{s,t}(G)=\Gamma} \frac{\zeta_G^*(\eta_G^*(p))}{Aut G} + \sum_{j_{s,t}(G)=\Gamma} \frac{\zeta_G^*(\eta_G^*(p))}{Aut G} \cdot c_1(N_{\xi_A}),$$

where we denote by N_{ξ_A} the normal bundle to the map ξ_A .

As usual, we will adopt the simplified notation:

$$\xi_A^*(p|\delta_\Gamma) = \sum_{f_{s,t}(G)=\Gamma} (\eta_G^*(p))|\delta_G + \sum_{j_{s,t}(G)=\Gamma} (\eta_G^*(p))|\delta_G \cdot c_1(N_{\xi_A}).$$

PROOF. As we already explained, the two cycles Δ_Γ and Δ_A do not intersect transversally in $\overline{\mathfrak{M}}_{g,P}$ if and only if there exist a graph G such that $j_{s,t}(G) = \Gamma$. In this case, we consider a tubular neighborhood T of the divisor with normal crossing $\Delta_A \subset \overline{\mathfrak{M}}_{g,P}$.

Consider the diagram:

$$\begin{array}{ccc} \overline{\mathfrak{M}}_A & \xrightarrow{g_A} & \overline{\mathfrak{M}}_A / Aut A \\ & \searrow \xi_A & \downarrow f_A \\ & & \overline{\mathfrak{M}}_{g,P} \end{array}$$

and the normal bundle N_{f_A} to the map f_A . Also observe that $g_A^* N_{f_A} = N_{\xi_A}$.

Introduce a metric in N_{f_A} , construct a tubular neighborhood \tilde{T} of its zero section, and extend f_A in the obvious way to a C^∞ map

$$\tilde{f}_A: \tilde{T} \rightarrow T.$$

Take then a sufficiently generic C^∞ section s of N_{f_A} lying in \tilde{T} . The composition $\tilde{f}_A \circ s \circ g_A$ yields a C^∞ map

$$s_A: \overline{\mathfrak{M}}_A \rightarrow \overline{\mathfrak{M}}_{g,P}$$

homotopic to ξ_A .

As Poincaré duality holds for smooth compact orbifolds, we may pull back cycles from $\overline{\mathfrak{M}}_{g,P}$ to $\overline{\mathfrak{M}}_A$. If Δ is any irreducible boundary component, then because of our

generic choice of the sections, we have, by transverse intersection,

$$(1) \quad s_A^*([\Delta]) = \sum_i [\Delta_i]$$

where the sum ranges over the irreducible components Δ_i of the preimage of Δ in $\overline{\mathfrak{M}}_A$.

The first step is to describe the irreducible components Δ_i . We claim that they are of two types, which can combinatorially described as follows. The first one is simply a cycle $\Delta_G \subset \overline{\mathfrak{M}}_A$ for each graph G such that $f_{s,t}(G) = \Gamma$. If Δ_A and Δ_Γ intersect transversally, these are the only components Δ_i appearing in the above expression. If not, the remaining Δ_i 's are all of the form

$$\frac{\xi_{G^*} \xi_G^*(c_1(N_{\xi_A}))}{Aut G},$$

where G is a graph such that $j_{s,t}(G) = \Gamma$.

Once this is established, we get the Proposition for the case $p=1$, that is:

$$\xi_A^*(\delta_\Gamma) = \sum_{f_{s,t}(G) = \Gamma} \delta_G + \sum_{j_{s,t}(G) = \Gamma} \delta_G \cdot c_1(N_{\xi_A}).$$

Instead of proving our assertion about the Δ_i 's in general, we shall restrict ourselves to some typical examples. The first example is $\Gamma = A = \Gamma_{b,B}$, with $B \neq \emptyset$, $B^c \neq \emptyset$. There is only one Δ_i , which is the zero locus of a section of the normal bundle to the map ξ_A . One may notice that Δ_i corresponds to the trivial A -graph G , drawn on the right, and that one has that

$$\xi_{b,B}^*(\delta_{b,B}) = (\eta_G^*(1)) | \delta_G \cdot c_1(N_{\xi_A}) = c_1(N_{\xi_A}).$$

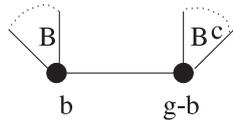


Fig. 3. - The graph $A = \Gamma = \Gamma_{b,B}$.

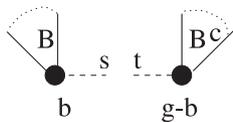


Fig. 4. - The graph G .



Fig. 5. - The graphs G_1 and G_2 .

This is the standard situation of excess intersection, and there is no surprise in finding this term in the general formula of Proposition 8 we are discussing.

The opposite situation occurs for example in the formula for

$$\xi_{irr}^*(\delta_{b,B}) = \xi_{irr}^*(1 | \delta_{b,B})$$

where we further assume that $b \geq 1, g - b \geq 1$. There are two components Δ_i , corresponding to the A -graphs G_1 and G_2 having the property that $f_{s,t}(G_i) = \Gamma_{b,B}$. In this case

$$\xi_{irr}^*(\delta_{b,B}) = \delta_{G_1} + \delta_{G_2}.$$

This is the standard situation of transverse intersection.

What is somewhat unexpected in the formula we are discussing, is the mixture between terms related to excess intersection and terms related to transverse intersection. To illustrate this phenomenon, let us consider the case

$$\xi_{irr}^*(\delta_F).$$

The formula in the statement tells us that

$$\xi_{irr}^*(\delta_F) = -(\psi_q + \psi_r) \delta_{irr} + \delta_F + \sum \delta_{E(a, A \cup \{q\})} + \sum (\delta_{H(a, A \cup \{q\})} + \delta_{H(a, A \cup \{r\})}),$$

where the two sums range over all the possible graphs of the corresponding type.

The first term is clear: it comes from excess intersection, and corresponds to the only graph G such that $j_{s,t}(G) = F$, *i.e.* the graph with one vertex of genus $g - 2$, one loop, and half-edges with labels in $P \cup \{s, t\}$.

As a sample case, let us explain the presence of the term δ_F . The presence of the other terms can be justified by similar arguments. Draw a picture of Δ_{irr} in a neighborhood of a generic point of the cycle Δ' corresponding to the locus of irreducible curves with at least three nodes (fig. 6). We cut it with a codimension three generic subspace, in order to draw the picture. The cycle Δ' is drawn as a triple point of Δ_{irr} , which is locally the union of three planes, intersecting each other in the three lines belonging to Δ_F .

Now we «move» a little bit Δ_{irr} (fig. 7), we call it $\tilde{\Delta}_{irr}$, and draw it with a dotted

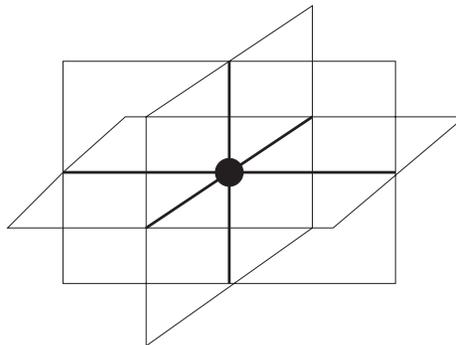


Fig. 6. – A neighborhood of the three-nodes locus.

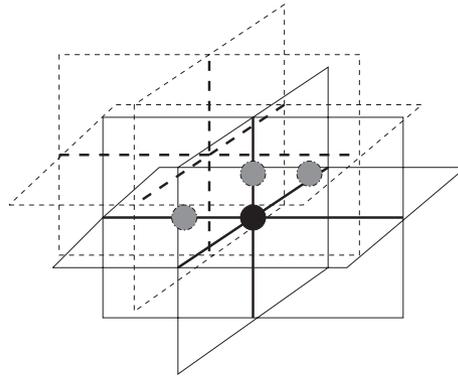


Fig. 7. – A modified neighborhood of the three-nodes locus.

line. There are three points of transverse intersection between $\tilde{\Delta}_{irr}$ and Δ_F . This shows that $s_A^*(\delta_F)$ contains, with multiplicity 1, the codimension 2 cocycle in $\overline{\mathcal{M}}_{g-1, P \cup \{s, t\}}$ corresponding to the locus of irreducible two-noded curves, which by abuse of notation is again denoted by Δ_F .

The formula in the statement, in the case $p = 1$,

$$\xi_A^*(\delta_\Gamma) = \sum_{f_{s,t}(G)=\Gamma} \delta_G + \sum_{j_{s,t}(G)=\Gamma} \delta_G \cdot c_1(N_{\xi_A})$$

is now completely justified.

To prove the general formula we make the following preliminary remark; we seek a formula for the pull-back under a ξ_A map of one of the following classes:

- pure boundary classes, hence orbifold Poincaré duals of cycles;
- ψ -mixed classes, hence orbifold Chern classes of bundles supported on cycles;
- κ -mixed classes. These are linear combinations of the above two types. In fact, we recall Mumford theorem

$$\kappa_1 = 12\lambda_1 + \sum \psi_i - \sum \delta_G,$$

where the second sum ranges over the set of stable graphs of codimension 1, and λ_1 is the first Chern class of the Hodge bundle; this implies that κ_1 is a linear combination of Poincaré duals of cycles and of Chern classes of bundles;

- pure Mumford classes, hence polynomials in classes of the above types.

In order to pull-back a tautological class, we first decompose it into a linear combination of Mumford classes supported on cycles, and then pull back each summand separately.

We therefore seek a formula for

$$\xi_A^* \left(\frac{\xi_{\Gamma^*}(c_1(F))}{Aut \Gamma} \right)$$

where F is a line bundle on $\overline{\mathcal{M}}_\Gamma$.

Suppose first that Δ_Γ and Δ_A intersect transversally. Take a sufficiently generic C^∞ section σ_F of the line bundle F . For every graph G such that $f_{s,t}(G) = \Gamma$, we denote by F_G the bundle $\eta_G^*(F)$, and by σ_{F_G} its section $\eta_G^*(\sigma_F)$.

By Poincaré duality, we can pull back cycles. We claim that

$$\xi_A^* \left(\frac{\xi_{\Gamma^*}([\{\sigma_F = 0\}])}{\text{Aut } \Gamma} \right) = \sum_{f_{s,t}(G) = \Gamma} \frac{\xi_G^*([\{\sigma_{F_G} = 0\}])}{\text{Aut } G}.$$

Let Δ be a cycle in in $\overline{\mathfrak{M}}_{g,p}$ such that

$$[\Delta] = \frac{\xi_{\Gamma^*}([\{\sigma_F = 0\}])}{\text{Aut } \Gamma};$$

we can pick

$$\Delta = \{x \in \overline{\mathfrak{M}}_{g,p} \mid x = \xi_\Gamma(y), \sigma_F(y) = 0\}$$

with orbifold multiplicity 1. Because of transverse intersection of Δ_Γ and Δ_A , Formula 1 applies in this case too. Δ is a cycle contained in Δ_Γ . We therefore seek the irreducible components Δ_i inside the irreducible components of the preimage of Δ_Γ in $\overline{\mathfrak{M}}_A$, that is, inside the Δ_G 's, where $f_{s,t}(G) = \Gamma$. One can easily check that

$$\begin{aligned} \Delta_G \cap \xi_A^{-1}(\Delta) &= \{z \in \overline{\mathfrak{M}}_A \cap \Delta_G \mid \xi_A(z) = \xi_\Gamma(y) \text{ for some } y \text{ such that } \sigma_F(y) = 0\} \\ &= \{z \in \overline{\mathfrak{M}}_A \mid z = \xi_G(w) \text{ for some } w, \xi_A(z) = \xi_\Gamma(y) \\ &\hspace{15em} \text{for some } y \text{ such that } \sigma_F(y) = 0\} \\ &= \{z \in \overline{\mathfrak{M}}_A \mid z = \xi_G(w) \text{ for some } w \text{ such that } \sigma_{F_G}(w) = 0\}, \end{aligned}$$

again with orbifold multiplicity 1.

Suppose, on the other hand, that $\Delta_\Gamma \subseteq \Delta_A$. We need formulas for degree 4 classes, hence the only new and significant situation occurs when $\Delta_\Gamma = \Delta_A$, and $\Gamma = A$ is a graph of codimension 2.

From the construction of the map s_A , we see that the diagram

$$\begin{array}{ccc} \overline{\mathfrak{M}}_G & \xrightarrow{\xi_G} & \overline{\mathfrak{M}}_A \\ \downarrow \eta_G & & \downarrow s_A \\ \overline{\mathfrak{M}}_\Gamma & \xrightarrow{\xi_\Gamma} & \overline{\mathfrak{M}}_{g,p} \end{array}$$

commutes only up to homotopy. To explain the presence of the transverse intersection terms in the pull-back formula,

$$\sum_{f_{s,t}(G) = \Gamma} \frac{\xi_{G^*}(\eta_G^*(c_1(F)))}{\text{Aut } G},$$

we observe that the induced diagram in cohomology commutes, hence, if one chooses suitable sections σ_{F_G} 's of the bundles $\eta_G^*(F)$, one can proceed as in the transverse intersection case. We now pass to justify the self-intersection term. In our specific situation this term is

$$\eta_G^*(c_1(F)) \circ c_1(N_{\xi_A}),$$

in fact, since $\Gamma = A$, the only A -graph G such that $f_{s,t}(G) = \Gamma$ is the trivial A -graph

and the map ξ_G is the identity. The corresponding component in the preimage of Δ_Γ under the map s_A is the Poincaré dual to $c_1(N_{\xi_A})$. Take a section of such bundle, call it τ . The component we are looking for is the Poincaré dual of

$$\{x \in \overline{\mathcal{M}}_A \mid \sigma_{F_G}(x) = 0, \tau(x) = 0\},$$

that is, the first Chern class of the bundle

$$\eta_G^*(F) \oplus N_{\xi_A},$$

as we claimed. \square

4.1. Formulas for π^* .

Let

$$\pi_A: \overline{\mathcal{M}}_{g, P \cup A} \rightarrow \overline{\mathcal{M}}_{g, P}$$

be the map forgetting the A markings. We first recall pull-back formulas for degree 2 classes (see [1, 2]).

$$\begin{aligned} \pi_A^*(\delta_{c, C}) &= \sum_{BCA} \delta_{c, C \cup B} & \pi_A^*(\psi_i) &= \psi_i - \sum_{BCA} \delta_{0, B \cup \{i\}}, \\ \pi_A^*(\delta_{irr}) &= \delta_{irr} & \pi_A^*(\kappa_1) &= \kappa_1 - \sum_{i \in A} \psi_i + \sum_{BCA} \delta_{0, B}. \end{aligned}$$

The pull-back formulas for Mumford classes are recursively deduced from Formula (1.10) in [2] and Lemma (1.2) in [1]; if $\pi: \overline{\mathcal{M}}_{0, n} \rightarrow \overline{\mathcal{M}}_{0, n-1}$ is the forgetful map, then

$$(2) \quad \psi_i = \pi^*(\psi_i) + \delta_{0, \{i, n\}},$$

and

$$(3) \quad \kappa_i = \pi^*(\kappa_i) + \psi_n^i.$$

Let us now come to degree 4 classes.

Mumford classes are pulled back via formulas 2 and 3:

$$\begin{aligned} \pi_A^*(\psi_i^2) &= \psi_i^2 - \sum_{BCA} \delta_{0, B \cup \{i\}} \mid \psi + \text{type } G \text{ classes,} \\ \pi_A^*(\psi_i \psi_j) &= \psi_i \psi_j - \psi_j \sum_{BCA} \delta_{0, B \cup \{i\}} - \psi_i \sum_{BCA} \delta_{0, B \cup \{j\}} + \text{type } G \text{ classes,} \\ \pi_A^*(\kappa_1 \psi_i) &= \kappa_1 \psi_i - \psi_i \sum_{j \in A} \psi_j - \sum_{BCA} \delta_{0, B \cup \{i\}} \mid \kappa + \sum_{BCA, j \in A \setminus B} \psi_j \delta_{0, B \cup \{i\}} + \sum_{BCA} \psi_i \delta_{0, B} + \\ &\quad + \text{type } G \text{ classes,} \\ \pi_A^*(\kappa_1^2) &= \kappa_1^2 - 2 \sum_{i \in A} \kappa_1 \psi_i + \sum_{i \in A} \psi_i^2 + 2 \sum_{i, j \in A, i \neq j} \psi_i \psi_j + \\ &\quad + 2 \sum_{BCA} \delta_{0, B} \mid \kappa - 2 \sum_{BCA, i \in A \setminus B} \psi_i \delta_{0, B} - \sum_{BCA} \delta_{0, B} \mid \psi + \text{type } G \text{ classes,} \\ \pi_A^*(\kappa_2) &= \kappa_2 - \sum_{i \in A} \psi_i^2 + \sum_{BCA} \delta_{0, B} \mid \psi + \text{type } G \text{ classes;} \end{aligned}$$

this last formula is computed by induction on $|A|$.

With arguments similar to the ones used in Proposition 8, one can easily prove the following:

PROPOSITION 9. *The following formulas hold:*

$$\begin{aligned} \pi_A^*(p|\delta_{irr}) &= (\tilde{\pi}_A^*(p))|\delta_{irr}, & \pi_A^*(\delta_{E(c,C)}) &= \sum_{BCA} \delta_{E(c,C \cup B)}, \\ \pi_A^*(p|\delta_{c,C}) &= \sum_{BCA} (\tilde{\pi}_B^*(p))|\delta_{c,C \cup B}, & \pi_A^*(\delta_{H(c,C)}) &= \sum_{BCA} \delta_{H(c,C \cup B)}, \\ \pi_A^*(\delta_F) &= \delta_F, & \pi_A^*(\delta_{G(c,C,d,D)}) &= \sum_{(B \cup B')CA} (\delta_{G(c,C \cup B,d,D \cup B')}) \end{aligned}$$

where

$$\begin{aligned} \tilde{\pi}_A: \overline{\mathfrak{M}}_{g-1,P \cup A \cup \{q,r\}} &\rightarrow \overline{\mathfrak{M}}_{g-1,P \cup \{q,r\}}, \\ \tilde{\pi}_B: \overline{\mathfrak{M}}_{c,C \cup B \cup \{s\}} \times \overline{\mathfrak{M}}_{g-c,(P \setminus C) \cup (A \setminus B) \cup \{t\}} &\rightarrow \overline{\mathfrak{M}}_{c,C \cup \{s\}} \times \overline{\mathfrak{M}}_{g-c,(P \setminus C) \cup \{t\}}. \quad \square \end{aligned}$$

5. RELATIONS IN DEGREE 4

New relations arising in degree 4 appear in $\overline{\mathfrak{M}}_{g,n}$ for $g \leq 5$ and for suitable n , and can be pulled back with formulas in 4.1. They have been computed with different techniques by E. Getzler, R. Pandharipande, P. Belorousski, and C. Faber. Most of them can be found in the literature, and we will give below the precise reference. The existence of some of them follows from [13], as a consequence of the existence of tautological relations on $\mathfrak{M}_{g,n}$, while their explicit expression on $\overline{\mathfrak{M}}_{g,n}$ has been recently computed by C. Faber and privately communicated to the author [12]. The only exception is the new relation in $\overline{\mathfrak{M}}_{3,2}$, whose coefficients will be determined in Section by the «pull-back to the boundary» techniques.

5.1. Genus 0.

The only new result is that

$$\kappa_2 = 0 \text{ in } H^4(\overline{\mathfrak{M}}_{0,4})$$

for dimension reasons.

5.2. Genus 1.

As above,

$$\kappa_2 = 0 \text{ in } H^4(\overline{\mathfrak{M}}_{1,1}).$$

Moreover, as observed by Faber in [11],

$$\delta_{irr}^2 = 0.$$

There are other relations: the first one originates in $H^4(\overline{\mathfrak{M}}_{1,2})$:

$$\delta_{E(0,\{i\})} - \delta_{H(0,\emptyset)} = 0,$$

as the push-forward of Keel relation with the map $\xi_{irr}: \overline{\mathfrak{M}}_{0,4} \rightarrow \overline{\mathfrak{M}}_{1,2}$. The second one originates in $H^4(\overline{\mathfrak{M}}_{1,4})$:

$$\begin{aligned} 0 &= 12 \sum_i \delta_{G(0,\{1,i\},1,\emptyset)} - 12 \sum_i \delta_{G(1,\{i\},0,\{*\})} - 2 \sum_{i,j} \delta_{G(1,\emptyset,0,\{i,j\})} + \\ &\quad + 6 \sum_i \delta_{G(1,\emptyset,0,\{i\})} - 2 \sum_i \delta_{E(\{1,i\})} + \sum_i \delta_{H(\{i\})} + \delta_{H(\emptyset)}. \end{aligned}$$

This was discovered by Getzler [14], while Pandharipande [21] then proved it is algebraic.

5.3. Genus 2.

Following Mumford [20],

$$60\kappa_2 = \delta_F + 6\delta_{H(0, \emptyset)}$$

in $H^4(\overline{\mathcal{M}}_{2,0})$. Faber proves that in $H^4(\overline{\mathcal{M}}_{2,1})$

$$\psi_i^2 = \frac{1}{120}\delta_F + \frac{1}{5}\delta_{E(1, \emptyset)} + \frac{13}{120}\delta_{H(0, i)} - \frac{1}{120}\delta_{H(0, \emptyset)} + \frac{7}{5}\delta_{G(1, \emptyset, 0, i)}.$$

Getzler proves in [15] that, in $H^4(\overline{\mathcal{M}}_{2,2})$,

$$\begin{aligned} \psi_i \psi_j = 3\psi | \delta_{2, \emptyset} + \frac{1}{72}\delta_F + \frac{7}{15}\delta_{E(1, \emptyset)} + \frac{1}{15}(\delta_{E(1, i)} + \delta_{E(1, j)}) + \\ + \frac{23}{120}\delta_{H(0, ij)} + \frac{1}{24}(\delta_{H(0, i)} + \delta_{H(0, j)}) - \frac{1}{40}\delta_{H(0, \emptyset)} - \frac{1}{15}\delta_{H(1, \emptyset)} + \\ + \frac{13}{5}\delta_{G(1, \emptyset, 0, ij)} + \frac{4}{5}(\delta_{G(1, i, 0, j)} + \delta_{G(1, j, 0, i)}) - \frac{4}{5}\delta_{G(0, ij, 1, \emptyset)}. \end{aligned}$$

A new algebraic relation was discovered by Belorousski and Pandharipande [6] in $H^4(\overline{\mathcal{M}}_{2,3})$:

$$\begin{aligned} 0 = 12\psi | \delta_{2, \emptyset} - 6 \sum_{i=1}^3 \psi | \delta_{2, i} + 6 \sum_{i=1}^3 \psi_i \delta_{2, i} + \frac{6}{5}\delta_{E(1, \emptyset)} - \frac{6}{5} \sum_{i=1}^3 \delta_{E(1, i)} + \frac{2}{5} \sum_{i=1}^3 \delta_{E(0, i)} + \\ + \frac{1}{10}\delta_{H(0, 123)} - \frac{3}{10} \sum_{i=1}^3 \delta_{H(0, jk)} + \frac{3}{10} \sum_{i=1}^3 \delta_{H(0, i)} - \frac{1}{10}\delta_{H(0, \emptyset)} - \frac{3}{5}\delta_{H(1, \emptyset)} - \frac{1}{5} \sum_{i=1}^3 \delta_{H(1, i)} - \\ - 12\delta_{G(2, \emptyset, 0, *)} + \frac{12}{5}\delta_{G(1, \emptyset, 0, 123)} - \frac{12}{5} \sum_{i=1}^3 \delta_{G(1, i, 0, jk)} + \frac{24}{5} \sum_{i=1}^3 \delta_{G(1, \emptyset, 0, i)} - \\ - \frac{36}{5} \sum_{i=1}^3 \delta_{G(1, *, 0, i)} - \frac{36}{5} \sum_{i=1}^3 \delta_{G(1, \emptyset, 1, \emptyset)} + \frac{18}{5} \sum_{i=1}^3 \delta_{G(1, i, 1, \emptyset)} - \frac{12}{5} \sum_{i=1}^3 \delta_{G(1, \emptyset, 1, i)}. \end{aligned}$$

Here, and from now on, every time we write the symbol $*$ instead of a marking's name, we mean that any marking which does not appear elsewhere in the notation could replace the $*$.

5.4. Genus 3.

In $H^4(\overline{\mathcal{M}}_{3,0})$ [12, 9]:

$$0 = \kappa_1^2 - \frac{5}{7}\psi | \delta_{irr} - \frac{89}{7}\psi | \delta_{2, \emptyset} - \frac{2}{35}\delta_F - \frac{94}{35}\delta_{E(1, \emptyset)} + \frac{103}{84}\delta_{H(0, \emptyset)} - \frac{2}{7}\delta_{H(1, \emptyset)} - \frac{22}{35}\delta_{G(1, \emptyset, 1, \emptyset)},$$

$$0 = \kappa_2 - \frac{5}{42}\psi | \delta_{irr} - \frac{41}{21}\psi | \delta_{2, \emptyset} + \frac{1}{630}\delta_F - \frac{11}{35}\delta_{E(1, \emptyset)} + \frac{41}{252}\delta_{H(0, \emptyset)} + \frac{2}{105}\delta_{H(1, \emptyset)} + \frac{8}{35}\delta_{G(1, \emptyset, 1, \emptyset)},$$

whereas in $H^4(\overline{\mathcal{M}}_{3,1})$ a new relation involving $\kappa_1 \psi_i$ appears, and the three of them

could be written as follows [12]:

$$\begin{aligned}
 0 = & \kappa_1 \psi_i - 5\psi_i^2 - \frac{1}{7} \psi_i \delta_{irr} - \frac{1}{42} \psi | \delta_{irr} - \frac{5}{7} \psi_i \delta_{2,i} - \frac{16}{21} \psi | \delta_{2,i} - \frac{40}{21} \psi \delta_{2,\emptyset} - \frac{1}{630} \delta_F + \\
 & + \frac{13}{21} \delta_{E(0,i)} - \frac{9}{35} \delta_{E(1,i)} + \frac{61}{252} \delta_{H(0,i)} + - \frac{2}{105} \delta_{H(1,i)} + \frac{4}{105} \delta_{H(1,\emptyset)} + \frac{4}{63} \delta_{H(0,\emptyset)} + \\
 & + \frac{16}{35} \delta_{G(1,i,1,\emptyset)} + \frac{61}{21} \delta_{G(1,\emptyset,0,i)} - \frac{8}{35} \delta_{G(1,\emptyset,1,i)},
 \end{aligned}$$

$$\begin{aligned}
 0 = & \kappa_1^2 - 9\psi_i^2 - \frac{2}{7} \psi_i \delta_{irr} - \frac{16}{21} \psi | \delta_{irr} - \frac{10}{7} \psi_i \delta_{2,i} - \frac{299}{21} \psi | \delta_{2,i} - \frac{347}{21} \psi \delta_{2,\emptyset} - \frac{19}{315} \delta_F + \\
 & + \frac{83}{3} \delta_{E(0,i)} - \frac{16}{5} \delta_{E(1,i)} + \frac{431}{252} \delta_{H(0,i)} - \frac{34}{105} \delta_{H(1,i)} - \frac{22}{105} \delta_{H(1,\emptyset)} + \frac{341}{252} \delta_{H(0,\emptyset)} + \\
 & + \frac{2}{7} \delta_{G(1,i,1,\emptyset)} + \frac{389}{21} \delta_{G(1,\emptyset,0,i)} - \frac{38}{35} \delta_{G(1,\emptyset,1,i)},
 \end{aligned}$$

$$\begin{aligned}
 0 = & \kappa_2 - \psi_i^2 - \frac{5}{42} \psi | \delta_{irr} - \frac{41}{21} \psi | \delta_{2,i} - \frac{347}{21} \psi \delta_{2,\emptyset} + \frac{1}{630} \delta_F + \\
 & + \frac{5}{21} \delta_{E(0,i)} - \frac{11}{35} \delta_{E(1,i)} + \frac{41}{252} \delta_{H(0,i)} + \frac{2}{105} \delta_{H(1,i)} + \frac{2}{105} \delta_{H(1,\emptyset)} + \frac{41}{252} \delta_{H(0,\emptyset)} + \\
 & + \frac{8}{35} \delta_{G(1,i,1,\emptyset)} + \frac{41}{21} \delta_{G(1,\emptyset,0,i)} + \frac{8}{35} \delta_{G(1,\emptyset,1,i)}.
 \end{aligned}$$

Finally, in $H^4(\overline{\mathfrak{M}}_{3,2})$, we have:

$$\begin{aligned}
 0 = & \psi_a^2 + \psi_b^2 - \frac{6}{5} \psi_a \psi_b - \kappa | \delta_{3,\emptyset} + 5\psi | \delta_{3,\emptyset} - \frac{40}{21} \psi | \delta_{2,\emptyset} + \frac{5}{3} (\psi | \delta_{2,a} + \psi | \delta_{2,b}) - \\
 & - \frac{6}{7} (\psi_a \delta_{2,a} + \psi_b \delta_{2,b}) - \frac{16}{21} \psi | \delta_{2,ab} + \frac{12}{35} (\psi_a \delta_{2,ab} + \psi_b \delta_{2,ab}) - \frac{1}{42} \psi | \delta_{irr} + \\
 & + \frac{1}{35} (\psi_a \delta_{irr} + \psi_b \delta_{irr}) - \frac{1}{630} \delta_F + \frac{13}{21} \delta_{E(2,\emptyset)} - \frac{4}{15} (\delta_{E(2,a)} + \delta_{E(2,b)}) - \\
 & - \frac{9}{35} \delta_{E(1,\emptyset)} - \frac{34}{105} \delta_{E(1,a)} + \frac{1}{7} \delta_{H(2,\emptyset)} - \frac{2}{105} \delta_{H(1,ab)} + \frac{4}{105} \delta_{H(1,\emptyset)} + \\
 & + \frac{1}{105} (\delta_{H(1,a)} + \delta_{H(1,b)}) + \frac{4}{63} \delta_{H(0,\emptyset)} + \frac{10}{63} \delta_{H(0,ab)} - \frac{5}{36} (\delta_{H(0,a)} + \delta_{H(0,b)}) + \\
 & + \frac{40}{21} \delta_{G(2,\emptyset,0,ab)} - \delta_{G(2,\emptyset,1,\emptyset)} + \frac{16}{35} \delta_{G(1,ab,1,\emptyset)} - \frac{8}{35} \delta_{G(1,\emptyset,1,ab)} - \\
 & - \frac{5}{3} (\delta_{G(2,b,0,a)} + \delta_{G(2,a,0,b)}) - \frac{40}{21} (\delta_{G(2,\emptyset,0,a)} + \delta_{G(2,\emptyset,0,b)}).
 \end{aligned}$$

5.5. Genus 4.

In $H^4(\overline{\mathfrak{M}}_{4,\emptyset})$ [12, 10]:

$$\begin{aligned}
 0 = & \frac{45}{2} \kappa_1^2 - 240 \kappa_2 - 7 \kappa_1 \delta_{irr} + \frac{35}{2} \psi | \delta_{irr} - 39 \kappa | \delta_{3,\emptyset} + \frac{315}{2} \psi | \delta_{3,\emptyset} + \frac{45}{2} \psi | \delta_{2,\emptyset} + \\
 & + \delta_F + 13 \delta_{E(2,\emptyset)} - \frac{105}{8} \delta_{H(0,\emptyset)} + 2 \delta_{H(1,\emptyset)} + 5 \delta_{H(2,\emptyset)} + 24 \delta_{G(1,\emptyset,1,\emptyset)} + 21 \delta_{G(1,\emptyset,2,\emptyset)},
 \end{aligned}$$

and since another relation appears in $H^4(\overline{\mathcal{M}}_{4,1})$ [12], we get there the following two relations:

$$\begin{aligned} 0 = & 5\kappa_1^2 - 30\kappa_2 - 40\kappa_1\psi_i + 245\psi_i^2 - \kappa_1\delta_{irr} + 7\psi_i\delta_{irr} - 2\kappa|\delta_{3,i} + 44\psi_i\delta_{3,i} - \\ & - 35\psi|\delta_{3,i} - 32\kappa|\delta_{3,0} + 175\psi|\delta_{3,0} - 30\psi_i\delta_{2,i} + 95\psi|\delta_{2,i} - 85\psi|\delta_{2,0} - \\ & - 36\delta_{E(0,i)} + 24\delta_{E(1,i)} - 12\delta_{E(2,i)} + \frac{35}{12}\delta_{H(0,0)} + \delta_{H(1,0)} + 5\delta_{H(2,0)} - \\ & - \frac{175}{12}\delta_{H(0,i)} + \delta_{H(1,i)} - \delta_{H(2,i)} - 18\delta_{G(1,i,1,0)} + 28\delta_{G(1,i,2,0)} + \\ & + 12\delta_{G(2,i,1,0)} - 175\delta_{G(3,0,0,i)} - 10\delta_{G(2,0,0,i)} + 12\delta_{G(1,0,1,i)} - 4\delta_{G(1,0,2,i)}, \end{aligned}$$

$$\begin{aligned} 0 = & \frac{25}{2}\kappa_1^2 - 180\kappa_2 + 35\kappa_1\psi_i - \frac{455}{2}\psi_i^2 - 5\kappa_1\delta_{irr} + \frac{35}{2}\psi|\delta_{irr} - 7\psi_i\delta_{irr} - \\ & - 35\kappa|\delta_{3,i} - 49\psi_i\delta_{3,i} + \frac{455}{2}\psi|\delta_{3,i} + 25\kappa|\delta_{3,0} - \frac{385}{2}\psi|\delta_{3,0} + \\ & + 60\psi_i\delta_{2,i} - \frac{335}{2}\psi|\delta_{2,i} + \frac{385}{2}\psi|\delta_{2,0} + \delta_F + 37\delta_{E(0,i)} - 35\delta_{E(1,i)} + \\ & + 37\delta_{E(2,i)} - \frac{455}{24}\delta_{H(0,0)} - 5\delta_{H(2,0)} + \frac{385}{24}\delta_{H(0,i)} + 7\delta_{H(2,i)} + \\ & + 60\delta_{G(1,i,1,0)} - 35\delta_{G(1,i,2,0)} + \frac{385}{2}\delta_{G(3,0,0,i)} - 25\delta_{G(2,0,0,i)} + 49\delta_{G(1,0,2,i)}. \end{aligned}$$

5.6. Genus 5.

Finally, in $H^4(\overline{\mathcal{M}}_{5,0})$ [12]:

$$\begin{aligned} 0 = & \frac{25}{2}\kappa_1^2 - 180\kappa_2 - 5\kappa_1\delta_{irr} + \frac{35}{2}\psi|\delta_{irr} - 35\kappa|\delta_{4,0} + \frac{455}{2}\psi|\delta_{4,0} + 25\kappa|\delta_{3,0} - \\ & - \frac{385}{2}\psi|\delta_{3,0} - \frac{385}{2}\delta_{3,0}|\psi + \delta_F + 37\delta_{E(1,0)} - 35\delta_{E(2,0)} - \frac{455}{24}\delta_{E(0,0)} - \\ & - 5\delta_{H(2,0)} + 7\delta_{H(3,0)} - 35\delta_{G(1,0,2,0)} + 49\delta_{G(1,0,3,0)} + 25\delta_{G(2,0,1,0)}. \end{aligned}$$

6. DEGREE 4 RELATIONS IN THE TAUTOLOGICAL GROUP

THEOREM 10. *For $g \geq 6$, $\mathcal{B}_{g,p}^4$ is a basis for $T_{g,p}^4$. For $2 \leq g \leq 5$, the relations among elements of $\mathcal{B}_{g,p}^4$ are the ones listed in Section 5.*

We will prove this Theorem by induction on g . We start with a sketchy exposition of an argument which covers the cases $g \geq 6$, once the previous ones are established. Unfortunately, this argument fails to extend to the low genus cases. We will therefore give a second, less direct argument. The initial cases require more involved computations, because of the presence of many relations among tautological classes. We will work out two sample cases in Lemmas 14 and 16, and recover the coefficients of the new relation in $\overline{\mathcal{M}}_{3,2}$ in Proposition 15.

PROPOSITION 11. *Suppose that Theorem 10 holds for $g = 5$. Then it holds for every genus $g \geq 6$.*

PROOF. For the first proof we make an induction on g . Consider the boundary maps:

$$\xi_{a,A}: \overline{\mathcal{M}}_{a,A \cup \{s\}} \times \overline{\mathcal{M}}_{g-a,A \cup \{t\}} \rightarrow \overline{\mathcal{M}}_{g,P},$$

on varying (a, A) in such a way that $a \geq 3, g - a \geq 3$. Consider the composition of the induced pull-back map with the projection on $H^2 \otimes H^2$:

$$g_{a,A}: H^4(\overline{\mathcal{M}}_{g,P}) \rightarrow H^2(\overline{\mathcal{M}}_{a,A \cup \{s\}}) \otimes H^2(\overline{\mathcal{M}}_{g-a,A \cup \{t\}}).$$

We need a few remarks:

- Under the above hypotheses on genera, there are no relation among tautological classes in $H^2(\overline{\mathcal{M}}_{a,A \cup \{s\}}) \otimes H^2(\overline{\mathcal{M}}_{g-a,A \cup \{t\}})$.

- Every class of the standard basis in $H^2(\overline{\mathcal{M}}_{a,A \cup \{s\}}) \otimes H^2(\overline{\mathcal{M}}_{g-a,A \cup \{t\}})$ (by the standard basis we mean the one described in [1]), appears, with the suitable sign, as a summand in the pull-back of at most one tautological class of $H^4(\overline{\mathcal{M}}_{g,P})$, with the exception of $-\psi_s \otimes \psi_t$, which is a summand both of $\xi_{a,A}^*(\psi | \delta_{a,A})$ and $\xi_{a,A}^*(\delta_{a,A} | \psi)$. This is a combinatorial remark which follows from the description of pull-backs of Section 4. In particular, one should look at the description of the operations on graphs denoted by $f_{s,t}$ and $j_{s,t}$.

- Almost every essential tautological class α in $\mathcal{B}_{g,P}^4$ satisfies $g_{a,A}(\alpha) \neq 0$ for at least one (a, A) satisfying the hypotheses. This is also a combinatorial remark, and it is based on the relative position of boundary cycles in $\overline{\mathcal{M}}_{g,P}$. The exceptions are:

$$\begin{aligned} &\kappa_2, \psi_x \text{ for every } x \in P, \delta_{E(b,B)}, \\ &\delta_{G(c,C,d,D)}, \text{ if } c + d \leq 2. \end{aligned}$$

Suppose there is a relation among essential tautological classes in $H^4(\overline{\mathcal{M}}_{g,P})$. Applying all the maps $g_{a,A}$, one obtains that many coefficients have to vanish. The relation should then be:

$$c\kappa_2 + \sum_{x \in P} c_x \psi_x^2 + \sum c_{b,B} \delta_{E(b,B)} + \sum_{c+d \leq 2} c_{c,C,d,D} \delta_{G(c,C,d,D)} = 0.$$

We pull it back with the map

$$\xi^*: H^4(\overline{\mathcal{M}}_{g,P}) \rightarrow H^4(\overline{\mathcal{M}}_{g-1,P \cup \{q,r\}}),$$

and get

$$\begin{aligned} c\kappa_2 + \sum_{x \in P} c_x \psi_x^2 + \sum c_{b,B} (\delta_{E(b,B)} + \delta_{E(b-1,B \cup \{q,r\})} + \dots) + \\ + \sum_{c+d \leq 2} c_{c,C,d,D} (\delta_{G(c-1,C \cup \{q,r\},d,D)} + \delta_{G(c,C,d-1,D \cup \{q,r\})} + \delta_{G(c,C,d,D)}) = 0. \end{aligned}$$

By induction hypothesis, the coefficients $c, c_x, c_{b,B}$ all have to vanish. Every type G class appears at most once as a summand in the image of at type G class. If we call «critical» the classes corresponding to graphs $G(0, A, 0, B)$, *i.e.* the possibly unessential ones, we observe that every non-critical class has at least one non-critical summand

in its pull-back. On the other hand, if we extend the ordering of P to an ordering for $P \cup \{q, r\}$ imposing $\{q, r\}$ to be the last two elements, then a basis of critical classes maps to a set of linearly independent critical classes. Thus, the coefficients $c_{c, C, d, D}$ vanish. \square

The main tool used in the second proof is the map:

$$\xi^*: H^4(\overline{\mathcal{M}}_{g, P}) \rightarrow H^4(\overline{\mathcal{M}}_{g-1, P \cup \{q, r\}}).$$

The combinatorics of tautological classes and pull-back formulas becomes rather intricate, but nevertheless it suggests a partition of $\mathcal{B}_{g, P}^4$, corresponding to any given partition of P , which, inductively, turns out to give a direct sum decomposition of the tautological group.

DEFINITION 12.

1. Pure boundary classes of type E and F

are essential pure boundary classes corresponding to graphs F and $E(a, A)$.

They generate the subspace W_{EF} of $T_{g, P}^4$.

2. Pure boundary classes of type H and G

are essential pure boundary classes corresponding to graphs $H(a, A)$ and $G(a, A, b, B)$.

They generate the subspace W_{GH} of $T_{g, P}^4$.

3. Ψ -mixed classes

are essential mixed boundary classes $\psi | \delta_{irr}$ and $\psi | \delta_{a, A}$, generating W_{Ψ} .

4. Ψ_I -mixed classes

are essential mixed boundary classes $\psi_i \delta_{irr}$ and $\psi_i \delta_{a, A}$, with $i \in I \cap A$, generating W_{Ψ_I} .

5. K -mixed classes

are essential mixed boundary classes $\kappa_1 \delta_{irr}$ and $\kappa | \delta_{a, A}$, generating W_K .

6. Mumford K classes

are essential classes $\begin{cases} \kappa_1^2, \kappa_2, & \text{for } g \geq 6 \\ \kappa_1^2, & \text{for } g = 5 \text{ and } g = 4, P = \emptyset, \text{ and generate } K \\ \emptyset, & \text{for } g = 4, P \neq \emptyset, \text{ and } g \leq 3. \end{cases}$

7. Mumford Ψ_I classes

are essential classes $\begin{cases} \kappa_1 \psi_i, \psi_i^2, \psi_i \psi_j, & \text{for } g \geq 4 \\ \psi_i^2, \psi_i \psi_j, & \text{for } g = 3, \text{ with } i, j \in I, \text{ and generate } \Psi_I \\ \emptyset, & \text{for } g \leq 2. \end{cases}$

8. Mumford Ψ_{IJ} classes

are essential classes $\begin{cases} \psi_i \psi_j, & \text{for } g \geq 3 \\ \emptyset, & \text{for } g \leq 2 \end{cases}, \text{ with } i \in I, j \in J, \text{ and generate } \Psi_{IJ}.$

PROPOSITION 13. *Suppose that Theorem 10 holds for $g = 6$. Then it holds for every genus $g \geq 6$.*

PROOF. Let $O = \{q, r\}$, so that $P \cup \{q, r\} = P \cup O$. Following formulas of Section 4, we describe how the above subspaces of $T_{g,P}^4$ behave with respect to the map

$$\xi^*: H^4(\overline{\mathcal{M}}_{g,P}) \rightarrow H^4(\overline{\mathcal{M}}_{g-1,P \cup \{q,r\}}).$$

We write down the behavior for genus $g \geq 4$. When no confusion will arise, we will denote by the same letter the subspaces of the same type in $H^4(\overline{\mathcal{M}}_{g,P})$ and $H^4(\overline{\mathcal{M}}_{g-1,P \cup \{q,r\}})$.

$$\begin{aligned} K &\rightarrow K, \text{ for } g \geq 7 \\ \Psi_P &\rightarrow \Psi_P, \text{ for } g \geq 5 \\ W_K &\rightarrow W_K + W_\psi + W_{\psi P} + W_{EF} + W_{GH} + W_{\psi O} + \Psi_O \\ W_\psi &\rightarrow W_\psi + W_{GH} + \Psi_O \\ W_{\psi P} &\rightarrow W_{\psi P} + W_{GH} + \Psi_{OP} \\ W_{EF} &\rightarrow W_{EF} + W_{GH} + W_{\psi O} \\ W_{GH} &\rightarrow W_{GH} + W_{\psi O}. \end{aligned}$$

We prove the Proposition by induction on g . Suppose that

$$T_{g-1,P \cup \{q,r\}}^4 = W_{EF} \oplus W_{GH} \oplus W_\psi \oplus W_{\psi P} \oplus W_{\psi O} \oplus W_K \oplus \Psi_P \oplus \Psi_O \oplus \Psi_{OP}$$

and that every summand is freely generated by essential tautological classes. We write down in block form the matrix of the map

$$\xi^*: H^4(\overline{\mathcal{M}}_{g,P}) \rightarrow H^4(\overline{\mathcal{M}}_{g-1,P \cup \{q,r\}}).$$

	K	Ψ_P	W_K	W_ψ	$W_{\psi P}$	W_{EF}	W_{GH}	$W_{\psi O}$	Ψ_O	Ψ_{OP}
K	A	0	0	0	0	0	0	0	0	0
Ψ_P	0	B	0	0	0	0	0	0	0	0
W_K	0	0	C	0	0
W_ψ	0	0	0	D	0	0	...	0	...	0
$W_{\psi P}$	0	0	0	0	E	0	...	0	0	...
W_{EF}	0	0	0	0	0	F	0	0
W_{GH}	0	0	0	0	0	0	G	...	0	0

We claim that the elements of $\mathcal{B}_{g,P}^4$ form a basis for $T_{g,P}^4$. Because of the form of the above matrix, it is sufficient to check that every subset generating each subspace consists of independent classes. For this, we look at blocks A, \dots, G , and check that each of them has maximal rank, equal to the number of rows. It is easy to see that A

and B are both the identity matrix, whereas from

$$\kappa_1 \delta_{irr} \rightarrow \kappa_1 \delta_{irr} + \dots, \text{ if } g \geq 5$$

$$\kappa | \delta_{a,A} \rightarrow \begin{cases} \kappa | \delta_{a,A} + \kappa | \delta_{a-1,A \cup \{q,r\}}, & \text{if } g - a \geq 1, \ a \geq 4 \\ \kappa | \delta_{a-1,A \cup \{q,r\}}, & \text{if } g = a \geq 4 \\ \kappa | \delta_{a,A}, & \text{if } g - a \geq 1, \ a \leq 3 \\ 0, & \text{if } g = a \leq 3 \end{cases}$$

we observe that C has maximal rank for $g \geq 5$.

Similarly, D and E have maximal rank for $g \geq 3$, whereas F has maximal rank for $g \geq 2$.

As for the block G , from

$$\delta_{H(a,A)} \rightarrow \begin{cases} \delta_{H(a,A)} + \delta_{H(a-1,A \cup \{q,r\})} + \dots, & \text{if } g - 1 - a \geq 1, \ a \geq 2 \\ \delta_{H(a,A)} + \frac{5}{6} \delta_{H(a-1,A \cup \{q,r\})} + \dots, & \text{if } g - 1 - a \geq 1, \ a = 1 \\ \delta_{H(a-1,A \cup \{q,r\})} + \dots, & \text{if } g = a + 1 \geq 3 \\ \frac{5}{6} \delta_{H(a-1,A \cup \{q,r\})} + \dots, & \text{if } g = a + 1 = 2 \\ \delta_{H(a,A)} + \dots, & \text{if } g - 1 - a \geq 1, \ a = 0 \\ 0, & \text{if } g = a + 1 = 1 \end{cases}$$

$$\delta_{G(a,A,b,B)} \rightarrow \delta_{G(a,A,b,B)} + \delta_{G(a-1,A \cup \{q,r\},b,B)} + \delta_{G(a,A,b-1,B \cup \{q,r\})};$$

we observe that type H classes are independent, and independent from type G ones. For the type G class, the argument used in the proof of Proposition 6 works in this case as well. One can write the block G in a triangular form, and see that it has maximal rank for $g \geq 3$. \square

LEMMA 14. *Suppose that Theorem 10 holds for $g = 5$. Then it holds for genus $g = 6$.*

PROOF. The same proof of Proposition 13 can be repeated to prove that $\mathcal{B}_{6,p}^4 \setminus \{\kappa_2\}$ is a set of linearly independent classes. Thus, if a relation does exist, it should be of the form

$$\kappa_2 + \dots = 0;$$

since $\xi^*(\kappa_2 + \dots) = \kappa_2 + \dots = 0$, then the relation should be a pull-back of the relation in $H^4(\overline{\mathcal{M}}_{5,0})$ (see Section 5):

$$\kappa_2 - \frac{1}{180} \delta_F + \frac{37}{180} \delta_{E(1)} + \dots = 0,$$

and hence it should be of the form

$$\kappa_2 - \frac{1}{180} \delta_F + \frac{37}{180} (\delta_{E(1,q)} + \delta_{E(1,r)}) + \dots = 0;$$

but one can easily observe that classes δ_F and $\delta_{E(1,q)} + \delta_{E(1,r)}$ do only appear

in the pull-back $\xi^*(\delta_F) = \delta_F + (\delta_{E(1,q)} + \delta_{E(1,r)}) + \dots$, hence cannot have different coefficients. This leads to a contradiction.

PROPOSITION 15. *There is a unique new relation in $\overline{\mathcal{M}}_{3,2}$, and it is the one described in Section 5.*

PROOF. We know from [9] and [12] the relations arising in $H^4(\overline{\mathcal{M}}_{3,0})$ and $H^4(\overline{\mathcal{M}}_{3,1})$, and further we know that a new relation does exist in $H^4(\overline{\mathcal{M}}_{3,2})$, involving pure Mumford classes $\psi_a, \psi_b, \psi_a \psi_b$. We need to prove that the relation has exactly the form described in Section 5, and that no other relation appears. We also recall that the group $H^4(\overline{\mathcal{M}}_{2,2})$ has been computed in [15].

The relations in $H^4(\overline{\mathcal{M}}_{3,0})$ and $H^4(\overline{\mathcal{M}}_{3,1})$ can be all used to write classes $\kappa_1^2, \kappa_2, \kappa_1 \psi_i$ in terms of other boundary classes, when $|P| \geq 2$.

Therefore, a possible new relation in $H^4(\overline{\mathcal{M}}_{3,2})$, can be written as follows:

$$(4) \quad \sum_{\Gamma} c_{\Gamma} \delta_{\Gamma} + \sum_p c_{p(irr)} p | \delta_{irr} + \sum_{p(a,A)} c_{p(a,A)} p | \delta_{a,A} + \sum c_i \psi_i^2 + c_{ab} \psi_a \psi_b = 0.$$

The first constraints on coefficients in (4) are derived by writing down explicitly the non-vanishing pull-backs of tautological classes under the map

$$\overline{\mathcal{M}}_{3,s} \rightarrow \overline{\mathcal{M}}_{3,ab},$$

which glues a fixed rational tail marked by $P \cup t$ by identifying t and s , and observing that the pull-back in $H^4(\overline{\mathcal{M}}_{3,s})$ of (4) must be a multiple of Faber’s relation involving $\kappa_1 \psi_s$ (see Section 5). They are:

$$\begin{aligned} c_F &= -\frac{1}{630}k & c_{H(2,0)} &= \frac{1}{7}k & c_{G(1,P,1,0)} &= \frac{16}{35}k & c_{\psi(irr)} &= -\frac{1}{42}k \\ c_{E(1,P)} &= -\frac{9}{35}k & c_{H(1,0)} &= \frac{4}{105}k & c_{G(1,0,1,P)} &= -\frac{8}{35}k & c_{\psi(3,0)} &= 5k \\ c_{E(0,P)} &= \frac{13}{21}k & c_{H(1,P)} &= -\frac{2}{105}k & c_{G(1,0,2,0)} &= \frac{5}{7}k & c_{\psi(2,0)} &= -\frac{40}{21}k \\ & & c_{H(0,0)} &= \frac{4}{63}k & c_{G(2,0,0,P)} &= \frac{40}{21}k & c_{\psi(2,P)} &= -\frac{16}{21}k \\ & & c_{H(0,P)} &= \frac{10}{63}k & c_{G(2,0,1,0)} &= -k & c_{\kappa(3,0)} &= -k. \end{aligned}$$

To determine the coefficient of some classes of type H and G we also need to use the map

$$H^4(\overline{\mathcal{M}}_{3,P}) \rightarrow H^2(\overline{\mathcal{M}}_{2,s}) \otimes H^2(\overline{\mathcal{M}}_{1,P \cup t}).$$

We then know by [12] and [13] that a new relation does actually exist, and therefore we fix the value of the constant k to be 1.

We consider the following maps:

$$\begin{aligned} H^4(\overline{\mathcal{M}}_{3,ab}) &\rightarrow H^2(\overline{\mathcal{M}}_{2,s}) \otimes H^2(\overline{\mathcal{M}}_{1,abt}) \\ H^4(\overline{\mathcal{M}}_{3,ab}) &\rightarrow H^2(\overline{\mathcal{M}}_{2,as}) \otimes H^2(\overline{\mathcal{M}}_{1,bt}) \\ H^4(\overline{\mathcal{M}}_{3,ab}) &\rightarrow H^4(\overline{\mathcal{M}}_{2,as}); \end{aligned}$$

the constraints on the coefficient derived by pulling back (4) force all of them to be the ones indicated in Section 5. \square

LEMMA 16. *Theorem 10 holds for genus $g = 2$.*

PROOF. The cases $n = 0, 1$ are well known (see [20]); the cases $n = 2, 3$ are entirely described in [15, 6]. Recall that a new relation appears in $H^4(\overline{\mathcal{M}}_{2,3})$ (see Section 5).

For every set $\{i, j, k\} \subset P$, only the relation pulled back from $\overline{\mathcal{M}}_{2,\{i,j,k\}}$ contains the summand:

$$\psi_i \delta_{2, P \setminus \{j, k\}} + \psi_j \delta_{2, P \setminus \{i, k\}} + \psi_k \delta_{2, P \setminus \{i, j\}};$$

we fix an ordering on P , and use the relation in $H^4(\overline{\mathcal{M}}_{2,\{i,j,k\}})$ to express $\psi_i \delta_{2, P \setminus \{j, k\}}$, for $i < j, i < k$, as linear combination of other classes.

Let $\mathcal{C}_{2,P}^4$ be the set obtained from the set of essential classes $\mathcal{B}_{2,P}^4$ after having eliminated the relations arising in degree 4, that is, after having removed all pure Mumford classes, and the classes $\psi_i \delta_{2, P \setminus \{j, k\}}$, for $i < j, i < k$. Observe that the definition of $\mathcal{C}_{2,P}^4$ depends on the choice of an ordering on P .

If $n = 4$, there is no new relation among essential tautological classes; we postpone the proof of this fact. If $n \geq 5$, let $F_{2,P}^4$ be the free vector space generated by classes in $\mathcal{C}_{2,P}^4$. One can define every pull-back map on $F_{2,P}^4$, following formulas in Section 4. Our claim is that the map

$$f = \{f_{ij}^*\}: F_{2,P}^4 \rightarrow \bigoplus_{\{i,j\} \subset P} F_{2, P \setminus \{i,j\} \cup \{s\}}^4$$

is injective for $|P| \geq 5$. This implies, by induction, that no new relation among tautological classes can appear for $n \geq 5$: any new one should map to zero with f .

We use a decomposition of $F_{2,P}^4$ similar to the one described at the beginning of this section.

- W_F is generated by δ_F ,
- W_E is generated by classes $\delta_{E(1,A)}$,
- $W_{H(0)}$ is generated by classes $\delta_{H(0,A)}$,
- $W_{H(1)}$ is generated by classes $\delta_{H(1,A)}$,
- $W_{G(2,0)}$ is generated by classes $\delta_{G(2,A,0,B)}$,
- $W_{G(0,2)}$ is generated by classes $\delta_{G(0,A,2,B)}$,
- $W_{G(1,1)}$ is generated by classes $\delta_{G(1,A,1,B)}$,
- $W_{G(1,0)}$ is generated by classes $\delta_{G(1,A,0,B)}$,
- W_ψ is generated by classes $\psi | \delta_{2,A}$,
- W_{ψ_I} is generated by classes $\psi_i \delta_{2,A}$, with $i \in I$.

In the space $\bigoplus_{\{i,j\} \subset P} F_{2, P \setminus \{i,j\} \cup \{s\}}^4$, we denote by $W_X = \bigoplus_{ij} W_X^{ij}$ the direct sum of subspaces $W_X^{ij} \subset F_{2, P \setminus \{i,j\} \cup \{s\}}^4$. The matrix of the map f can be written in triangular block

form (we omit all zeroes):

	W_F	W_E	$W_{H(0)}$	$W_{G(1,0)}$	$W_{G(0,2)} \oplus W_{\psi_S}$	$W_{G(2,0)}$	$W_{H(1)}$	$W_{G(1,1)}$	W_ψ	W_{ψ_P}
W_F	A									
W_E		B								
$W_{H(0)}$			C							
$W_{G(1,0)}$				D						
$W_{G(0,2)}$					E					
$W_{G(2,0)}$						F				
$W_{H(1)}$				G			
$W_{G(1,1)}$						H		
W_ψ			I	
W_{ψ_P}	L

We just need to check that the blocks on the diagonal have maximal rank. This is completely trivial for the blocks A, B, C, D, E . We check block G , and observe that blocks H and I present a very similar combinatorics. G is of the form

$$(G^{12} \quad G^{13} \quad \dots \quad G^{ij} \quad \dots),$$

where G^{ij} is a block of the matrix of the map f_{ij}^* . We can write G^{ij} as

	$\delta_{H(1, B \cup \{s\})}$	$\delta_{H(1, B)}$
$\delta_{H(1, A), \{i, j\} \subset A}$	Id	0
$\delta_{H(1, A), \{i, j\} \not\subset A^c}$	0	Id
$\delta_{H(1, P \setminus \{i, j\})}$	0	...
$\delta_{H(1, A), \{i, j\} \cap A = 1}$	0	0

We consider the matrix G' obtained removing the second column of blocks from each G^{ij} , except for the columns corresponding to $\delta_{H(1, \emptyset)}, \delta_{H(1, x)}$. Finally, we can extract such a triangular matrix

	$\delta_{H(1, B), B \leq 1}$	$\delta_{H(1, B \cup \{s\})}$
$\delta_{H(1, A), A \leq 1}$	Id	0
$\delta_{H(1, A), A \geq 2}$...	Id

Observe that we just need the weaker assumption $|P| \geq 4$.

As for the block L , observe that any essential class maps to essential classes, except for

$$\psi_i \delta_{2, P \setminus \{j, k\}} \xrightarrow{f_{jk}^*} -\psi_i \psi_s = - \sum_{|C^c| \geq 3} \psi_i \delta_{2, C} - \sum_{x < i} \psi_i \delta_{2, P \setminus \{x, s\}} + \sum_{x > i} \psi_x \delta_{2, P \setminus \{i, s\}};$$

but this doesn't prevent us from extracting a non-degenerate matrix

	$\psi_i \delta_{2,B}, B \leq 1$	$\psi_i \delta_{2,B \cup \{s\}}$
$\psi_i \delta_{2,A}, A \leq 1$	Id	0
$\psi_i \delta_{2,A}, A \geq 2$	\dots	Id

With the same argument, one can write a sub-block of F of the form

	$\delta_{G(2,A,0,D \cup \{s\})}, \delta_{G(2,A,0,D)}$	$\delta_{G(2,C \cup \{s\},0,D)}$
$\delta_{G(2,A,0,B)}, A \leq 1$	K	0
$\delta_{G(2,A,0,B)}, A \geq 2$	\dots	Id

The set $P \setminus \{i, j\} \cup \{s\}$ inherits an ordering from P , assuming s to be the last point; therefore the second column of blocks gives no problem. As for the matrix K , write it in sub-blocks K_A , where K_A involves classes $\delta_{G(2,A,0,B)}$. These classes are all obtained pushing forward from $H^2(\overline{\mathcal{M}}_{0,A^C \cup \{z\}})$, and so are the relations among them in $F_{2,p}^4$. The combinatorics of the map corresponding to the block K_A is then exactly the same of the map

$$H^2(\overline{\mathcal{M}}_{0,A^C \cup \{z\}}) \rightarrow \bigoplus_{\{i,j\} \subset A^C} H^2(\overline{\mathcal{M}}_{0,A^C \setminus \{i,j\} \cup \{z,s\}})$$

which will be proved in Lemma 17 to be injective for $|A^C| \geq 4$. Therefore each K_A , and consequently K , has maximal rank.

As for the case $n = 4$, we first prove by using the pull-back map

$$H^4(\overline{\mathcal{M}}_{2,\{i,j,k,l\}}) \rightarrow H^2(\overline{\mathcal{M}}_{2,\{i,s\}}) \otimes H^2(\overline{\mathcal{M}}_{0,\{j,k,l,t\}});$$

that in a possible new relation, the coefficients of ψ -mixed classes and of classes of type G vanish.

We now restrict the map f to the free vector space generated by the classes with non vanishing coefficient in a possible new relation in $T_{2,4}^4$. By the same arguments used for the general case, the new map f is injective, and the proof of our Lemma is complete. \square

LEMMA 17. For $|P| \geq 5$, the map

$$H^2(\overline{\mathcal{M}}_{0,p}) \rightarrow \bigoplus_{\{x,y\} \subset P \setminus \{b\}} H^2(\overline{\mathcal{M}}_{0,P \setminus \{x,y\} \cup \{s\}})$$

is injective.

PROOF. The case $|P| = 5$ is trivial.

We can consider

$$\phi_A^*: H^2(\overline{\mathcal{M}}_{0,p}) \rightarrow H^2(\overline{\mathcal{M}}_{0,A \cup \{s\}} \times \overline{\mathcal{M}}_{0,A^C \cup \{t\}})$$

as the sum of the two maps

$$f_A^* : H^2(\overline{\mathcal{M}}_{0,P}) \rightarrow H^2(\overline{\mathcal{M}}_{0,A \cup \{s\}}),$$

$$f_A^{*c} : H^2(\overline{\mathcal{M}}_{0,P}) \rightarrow H^2(\overline{\mathcal{M}}_{0,A^c \cup \{t\}}),$$

where the two maps are the pull-back of the map that glues any fixed rational tail to the extra marked point. For any such A , there exist $\{x, y\} \subset P$ such that $A \subset P \setminus \{x, y\}$. For a suitable choice of the rational tail to glue, we can write a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,P \setminus \{x,y\} \cup \{b\}} & \xrightarrow{f_{P \setminus \{x,y\}}} & \overline{\mathcal{M}}_{0,P} \\ \uparrow & \nearrow f_A & \\ \overline{\mathcal{M}}_{0,A \cup \{s\}} & & \end{array}$$

so that from the induced diagram on H^2 we read: $\ker f_{P \setminus \{x,y\}}^* \subset \ker f_A^*$. Therefore, by Proposition 2.8 in [1],

$$(\cap_{\{x,y\} \subset P} \ker f_{P \setminus \{x,y\}}^*) \subset (\cap_{A \subset P} \ker f_A^*) = 0.$$

The statement is proved by induction on $|P|$. Suppose that $x \in (\cap_{\{x,y\} \subset P \setminus \{b\}} \ker f_{P \setminus \{x,y\}}^*)$, but there exist $k \in P \setminus \{b\}$, such that $y \in f_{P \setminus \{b,k\}}^*(x) \neq 0$. By the commutativity of

$$\begin{array}{ccc} H^2(\overline{\mathcal{M}}_{0,P}) & \xrightarrow{f_{P \setminus \{x,y\}}^*} & H^2(\overline{\mathcal{M}}_{0,P \setminus \{x,y\} \cup \{t\}}) \\ \downarrow f_{P \setminus \{b,k\}}^* & & \downarrow \\ H^2(\overline{\mathcal{M}}_{0,P \setminus \{b,k\} \cup \{u\}}) & \longrightarrow & H^2(\overline{\mathcal{M}}_{0,P \setminus \{b,k,x,y\} \cup \{u,v\}}) \end{array}$$

we see that $y \in (\cap_{\{x,y\} \subset P \setminus \{b,k\}} \ker f_{P \setminus \{x,y\}}^*)$, hence by induction hypothesis, $y = 0$, and we are done. \square

PROOF OF THEOREM 10. The induction on the genus starts with Lemma 16; then one can perform the next few steps by arguments similar to the one used in Lemma 14, and get the result for genus up to 6. The procedure is then completed with Proposition 13. \square

7. A CONJECTURE ON HIGHER DEGREE TAUTOLOGICAL RELATIONS

At this point it is natural to formulate a conjecture which is suggested by the proof of Proposition 11.

This conjecture agrees with Harer’s and Ivanov’s stability theorems (see [16, 17]), and with Faber’s results and conjectures concerning the tautological ring of the open part $\mathcal{M}_{g,n}$ (see [13]).

CONJECTURE 18. *There are no relations between essential tautological classes in $H^{2k}(\overline{\mathcal{M}}_{g,p})$ whenever $g \geq 3k$.*

We justify our conjecture. We first need to extend some definitions. A tautological class of degree $2k$ is a push-forward of a degree $2l$ Mumford class from a codimension $k - l$ boundary component; a degree $2k$ class is unessential if it can be eliminated by means of a relation among tautological classes arising in degree $< 2k$.

Then we need to build new pull-back formulas, but we can give conjectural ones starting from the ones we proved for degree 4. In particular, we claim that they preserve the tautological group.

Under the above hypotheses, there are plenty of boundary components $\prod \overline{\mathcal{M}}_{\Gamma_i}$ in $\overline{\mathcal{M}}_{g,P}$ such that

$$H^{2k}(\prod \overline{\mathcal{M}}_{\Gamma_i}) = \bigoplus_{\sum i=2k} (\otimes H^i(\overline{\mathcal{M}}_{g_i, P_i}))$$

contains at least one summand $\otimes H^{2j}(\overline{\mathcal{M}}_{g_j, P_j})$ with $g_j \geq 3j$, and $g_j < g$.

Write a generic linear combination of tautological classes in $H^{2k}(\overline{\mathcal{M}}_{g,P})$, and suppose it is equal to 0; by pulling back these relation to the above components we can prove that many coefficients do vanish: in fact, inductively, there are no relations among essential classes in these summands of the cohomology. We also conjecture that the pull-back maps in higher degree still satisfy the property that each class is generically a summand in the pull-back of at most one class.

It is then hard to believe that a new relation holds among the few classes whose coefficient has not yet been showed to be zero.

8. GENERATORS OF THE COHOMOLOGY GROUP

THEOREM 19. $H^4(\overline{\mathcal{M}}_{g,P}, \mathbb{Q})$ is generated by tautological classes for all $g \geq 8$.

PROOF. We are following Edidin's scheme of Proof [8].

In the proof of this Proposition we plan to give an upper bound for the dimension of the cohomology group, and then to use the knowledge of the tautological group and of the homology of the mapping class group to prove that, this bound is achieved.

Let $n = 3g - 3 + |P|$ be the complex dimension of $\overline{\mathcal{M}}_{g,P}$. We write a part of the exact homology sequence of the pair $(\overline{\mathcal{M}}_{g,P}, \overline{\mathcal{M}}_{g,P} \setminus \mathcal{M}_{g,P})$:

$$\dots \rightarrow H_{2n-4}(\overline{\mathcal{M}}_{g,P} \setminus \mathcal{M}_{g,P}) \xrightarrow{j_*} H_{2n-4}(\overline{\mathcal{M}}_{g,P}) \rightarrow H_{2n-4}(\overline{\mathcal{M}}_{g,P}, \overline{\mathcal{M}}_{g,P} \setminus \mathcal{M}_{g,P}) \rightarrow \dots$$

hence, using Poincaré duality for smooth orbifolds:

$$\dim H^4(\overline{\mathcal{M}}_{g,P}) = \dim H_{2n-4}(\overline{\mathcal{M}}_{g,P}) \leq \dim j_* H_{2n-4}(\overline{\mathcal{M}}_{g,P} \setminus \mathcal{M}_{g,P}) + \dim H^4(\mathcal{M}_{g,P})$$

We refer to the description of the stratified structure of $\overline{\mathcal{M}}_{g,P}$ which has been explained in Section 2. For any stable graph Γ , we further denote by Δ_Γ^0 the open stratum $\xi_\Gamma(\mathcal{M}_\Gamma)$.

Let

$$\partial \mathcal{M}_{g,P} = \overline{\mathcal{M}}_{g,P} \setminus \mathcal{M}_{g,P}.$$

We recall that $\partial\mathcal{N}_{g,p} = \cup_i \Delta_{\Gamma_i}$, where the Δ_{Γ_i} 's are the codimension 1 boundary components.

We denote by $\partial\partial\mathcal{N}_{g,p}$ the union of the codimension two boundary components, and write the homology exact sequence for the pair $(\partial\mathcal{N}_{g,p}, \partial\partial\mathcal{N}_{g,p})$:

$$\dots \rightarrow H_{2n-4}(\partial\partial\mathcal{N}_{g,p}) \xrightarrow{i_*} H_{2n-4}(\partial\mathcal{N}_{g,p}) \rightarrow H_{2n-4}(\partial\mathcal{N}_{g,p}, \partial\partial\mathcal{N}_{g,p}) \rightarrow \dots$$

Let us look at the relative term. By Lefschetz Theorem [23] we have:

$$H_{2n-4}(\partial\mathcal{N}_{g,p}, \partial\partial\mathcal{N}_{g,p}) \simeq H^2(\partial\mathcal{N}_{g,p} \setminus \partial\partial\mathcal{N}_{g,p}).$$

The space

$$\partial\mathcal{N}_{g,p} \setminus \partial\partial\mathcal{N}_{g,p}$$

consists of the disjoint union of the interior parts of the codimension 1 boundary components, the Δ_i^0 's.

We have a precise description of these Δ_i^0 's as quotients of moduli spaces of smooth curves:

$$\partial\mathcal{N}_{g,p} \setminus \partial\partial\mathcal{N}_{g,p} \simeq \sqcup_{a,A} (\mathcal{N}_{a,A \cup \{s\}} \times \mathcal{N}_{g-a,A^c \cup \{t\}}) / \text{Aut } \Gamma_{a,A} \sqcup \mathcal{N}_{g-1,P \cup \{qr\}} / \text{Aut } \Gamma_{irr}.$$

The rational cohomology of such quotients satisfies:

$$H^k(\mathcal{N}_{\Gamma_i} / \text{Aut } \Gamma_i, \mathbb{Q}) \cong H^k(\mathcal{N}_{\Gamma_i} \mathbb{Q})^{\text{Aut } \Gamma_i}$$

where we denote by $H^k(\mathcal{N}_{\Gamma_i})^{\text{Aut } \Gamma_i}$ the invariants with respect to the induced $\text{Aut } \Gamma_i$ action on the cohomology. In the case $k=2$, these invariants can be precisely described. The cohomology group $H^2(\mathcal{N}_{\Gamma_i})$ is generated by Mumford classes of degree 2. The class κ_1 is fixed by the automorphism group of any graph, whereas the ψ_i classes, for i a special point, are permuted by the group action in the obvious way.

We then get

$$H^2(\partial\mathcal{N}_{g,p} \setminus \partial\partial\mathcal{N}_{g,p}) \simeq \bigoplus_{a,A} H^2(\mathcal{N}_{a,A \cup \{s\}} \times \mathcal{N}_{g-a,A^c \cup \{t\}})^{\text{Aut } \Gamma_{a,A}} \oplus H^2(\mathcal{N}_{g-1,P \cup \{q,r\}})^{\text{Aut } \Gamma_{irr}}.$$

At this point, the bound for the dimension of the cohomology group is:

$$\begin{aligned} \dim H^4(\overline{\mathcal{M}}_{g,p}) \leq & + \sum_{a,A} \dim H^2(\mathcal{N}_{a,A \cup \{s\}} \times \mathcal{N}_{g-a,A^c \cup \{t\}})^{\text{Aut } \Gamma_{a,A}} + \\ & + \dim H^2(\mathcal{N}_{g-1,P \cup \{q,r\}})^{\text{Aut } \Gamma_{irr}} + \dim H^4(\mathcal{N}_{g,p}) + \dim i_* j_* H_{2n-4}(\partial\partial\mathcal{N}_{g,p}). \end{aligned}$$

The space $\partial\partial\mathcal{N}_{g,p}$ is the union of the codimension two boundary components, which we will call Θ_i 's. Their complex dimension is $n-2$. An easy application of the Maier-Vietoris exact sequence, shows that the obvious map

$$k : \sqcup_i \Theta_i \rightarrow \cup_i \Theta_i = \partial\partial\mathcal{N}_{g,p}$$

from the disjoint union into the union of these components induces the following iso-

morphism in homology:

$$\bigoplus_i H_{2n-4}(\Theta_i) \simeq H_{2n-4}(\partial\partial\mathcal{M}_{g,P}).$$

Observe that for dimension reasons, $\dim H_{2n-4}(\Theta_i) = 1$.

We claim that

$$\dim i_* j_* H_{2n-4}(\partial\partial\mathcal{M}_{g,P}) \leq r$$

where r equals the number of essential pure boundary classes. This number differs from the number of codimension two boundary components because of the presence of Keel's relations in genus 0. These relations live in the second homology group of $\overline{\mathcal{M}}_{0,n}$.

The push-forward induced by the map

$$\overline{\mathcal{M}}_{0,A \cup \{s\}} \rightarrow \overline{\mathcal{M}}_{g,P}$$

determines homological equivalences among codimension 2 boundary components of $\overline{\mathcal{M}}_{g,P}$.

Let

$$\phi : \sqcup_i \Theta_i \rightarrow \overline{\mathcal{M}}_{g,P}$$

be the collection of the inclusion maps of the codimension 2 boundary components. By what we said above, the image of the map

$$\phi_* : H_{2n-4}(\sqcup_i \Theta_i) \rightarrow H_4(\overline{\mathcal{M}}_{g,P})$$

has dimension less or equal than r . Since $\phi = k \circ i \circ j$, and k_* is an isomorphism, this implies that

$$\dim i_* j_* H_{2n-4}(\partial\partial\mathcal{M}_{g,P}) \leq r.$$

Our final bound is:

$$(5) \quad \dim H^4(\overline{\mathcal{M}}_{g,P}) \leq \sum_{a,A} \dim H^2(\mathcal{M}_{a,A \cup \{s\}} \times \mathcal{M}_{g-a,A^c \cup \{t\}})^{Aut \Gamma_{a,A}} + \dim H^2(\mathcal{M}_{g-1,P \cup \{gr\}})^{Aut \Gamma_{ir}} + \dim H^4(\mathcal{M}_{g,P}) + r.$$

By Ivanov [17], Harer [16], and Loojenga's [19] stability theorems for the homology of the mapping class group, $H^4(\mathcal{M}_{g,P})$ is freely generated by Mumford classes, for $g \geq 8$.

Instead of computing the dimension of all the cohomology groups involved in (5), we proceed more indirectly. We show that there is a bijection between the following two sets. On one hand, the set $\mathcal{B}_{g,P}^4$, on the other, the set whose elements are the pure boundary classes in $\mathcal{B}_{g,P}^4$ and the vectors belonging to the natural bases of the cohomology vector spaces appearing on the right hand side of the above inequality (5). The upper bound for the dimension of the cohomology group is therefore achieved, and consequently the tautological classes generate the cohomology group.

The bijection directly follows from the definition of essential tautological classes:

- pure Mumford classes in $\mathcal{B}_{g,P}^4$ correspond to a basis for

$$H^4(\mathfrak{M}_{g,P}),$$

- mixed boundary classes in $\mathcal{B}_{g,P}^4$ correspond to a basis for

$$\bigoplus_{a,A} H^2(\mathfrak{M}_{a,A \cup \{s\}} \times \mathfrak{M}_{g-a,A \cup \{t\}})^{Aut \Gamma_{a,A}} \oplus H^2(\mathfrak{M}_{g-1,P \cup \{qr\}})^{Aut \Gamma_{irr}},$$

- pure boundary classes in $\mathcal{B}_{g,P}^4$ are exactly r .

This completes the proof. \square

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