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$L^\infty - L^2$ weighted estimate for the wave equation with potential

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Analisi matematica. — *$L^\infty - L^2$ weighted estimate for the wave equation with potential.* Nota di VLADIMIR GEORGIEV e NICOLA VISCIGLIA, presentata (*) dal Socio S. Spagnolo.

ABSTRACT. — We consider a potential type perturbation of the three dimensional wave equation and we establish a dispersive estimate for the associated propagator. The main estimate is proved under the assumption that the potential $V \geq 0$ satisfies

$$|V(x)| \leq \frac{C}{(1+|x|)^{2+\varepsilon_0}},$$

where $\varepsilon_0 > 0$.

KEY WORDS: Perturbed wave equation; Resolvent estimates; Spectral theory; Fredholm theory.

RIASSUNTO. — *Stime $L^\infty - L^2$ pesate per l'equazione delle onde con potenziale.* Si considera l'equazione delle onde perturbata con un potenziale in dimensione tre e si provano delle stime dispersive per il propagatore associato. La stima principale è ottenuta sotto la condizione che il potenziale $V \geq 0$ soddisfi

$$|V(x)| \leq \frac{C}{(1+|x|)^{2+\varepsilon_0}},$$

dove $\varepsilon_0 > 0$.

1. INTRODUCTION

The classical models in quantum mechanics (see [11, Chap. X.2]) lead in a natural way to the study of potential type perturbations of the Laplace operator. The corresponding wave evolution problem is the following one

$$(\partial_t^2 - \Delta)\, u(t,x) + V(x)\, u(t,x) = 0, \qquad x \in \mathbb{R}^3. \tag{1.1}$$

For potential $V(x)$ satisfying the properties

$$V(x) \in L^\infty(\mathbb{R}^3) \tag{1.2}$$

the Kato-Rellich Theorem (see [11, Theorem X.12]) implies that

$$-\Delta + V : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

is a self-adjoint operator. In this work we study potentials satisfying the additional sign assumption

$$V(x) \geq 0, \qquad \text{almost everywhere in } \mathbb{R}^3. \tag{1.3}$$

In fact, operators of type $-\Delta + V$, such that (1.3) is not true, might have nontrivial kernels and this would be an obstacle to obtain dispersive estimates for (1.1).

The assumption (1.3) enables one to give a meaning to the operator $\sqrt{-\Delta + V}$ and to write an explicit representation of the solution to the Cauchy problem for (1.1) with

(*) Nella seduta del 14 febbraio 2003.

initial data

$$u(0, x) = 0, \; \partial_t u(0, x) = f \tag{1.4}$$

as follows

$$u(t, x) = \mathcal{U}_V(t)\, f, \tag{1.5}$$

where

$$\mathcal{U}_V(t) = \frac{\sin\left(t\sqrt{-\Delta + V}\right)}{\sqrt{-\Delta + V}}\,.$$

The main goal of the work is to obtain a dispersive estimate of the type

$$\| \mathcal{U}_V(t)\, f \|_{L^4} \leqslant \frac{C(f)}{\sqrt{t}}\,, \tag{1.6}$$

where $\|\cdot\|_{L^p}$ is the standard Lebesgue norm in $\mathbb{R}^3$ for $p \geqslant 1$.

This estimate for the case $V = 0$ is known as Strichartz estimate (see [15]) and it plays important role in the applications to nonlinear wave equations (see for example [13]).

The Strichartz estimate for the case of potential type perturbation is established in [4] and [7]. The assumptions in these works require that the potential decays very rapidly at infinity. For instance, the assumption $V(x) = O(|x|^{-3-\varepsilon_0})$, $\varepsilon_0 > 0$, is assumed in [7]. Here we shall relax the assumptions on the potential to the following one $V(x) = O(|x|^{-2-\varepsilon_0})$ as $|x| \to \infty$. We shall consider potentials that are not necessarily radial. The case $V(x) = a/|x|^2$ with radially symmetric data f is studied recently in [10], while in [6] is considered the same problem with general initial data.

The classical Strichartz estimate in dimension 3 is the following

$$\| \mathcal{U}_0(t)\, f \|_{L^4} \leqslant \frac{C}{\sqrt{t}} \| f \|_{L^{4/3}}, \tag{1.7}$$

where $f \in C_0^\infty(\mathbb{R}^3)$ and $\mathcal{U}_0(t)$ is the free propagator

$$\mathcal{U}_0(t) = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\,.$$

In this work we shall obtain the following variant of this estimate

$$\| \mathcal{U}_V(t)\, f \|_{L^4} \leqslant \frac{C}{\sqrt{t}} \| f \|_{3/2+\delta}, \tag{1.8}$$

where δ is arbitrary positive number and $\|\cdot\|_s$ denotes the following norm:

$$\| f \|_s^2 = \int_{\mathbb{R}^3} | f(x) |^2 \langle x \rangle^s \, dx,$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$. The weighted Lebesgue space with norm $\| f \|_s$ will be denoted by $L_s^2(\mathbb{R}^3)$.

In our case we shall assume that the measurable potential $V(x)$ satisfies (1.3) and for some $\varepsilon_0 > 0$ and $C > 0$ the following inequality is satisfied

$$|V(x)| \leq \frac{C}{(1+|x|)^{2+\varepsilon_0}} \tag{1.9}$$

The main result of this work is the following.

THEOREM 1.1. *If the potential* $V(x) \geq 0$ *satisfies the assumption* (1.9), *then the estimate* (1.8) *is fulfilled.*

The natural question that arises is the relation between the classical Strichartz norm

$$\|f\|_{L^{4/3}}$$

in the right side of (1.7) and the L^2 weighted norm

$$\|f\|_{3/2+\delta}$$

in our estimate (1.8). It is easy to see that

$$\|f\|_{L^{4/3}} \leq C_\delta \|f\|_{3/2+\delta}, \tag{1.10}$$

so the classical Strichartz estimate (1.7) seems to be stronger that the one in (1.8). However, the assumption (1.9) is a very weak one on the decay of the potential and we have met essential difficulties to apply the classical methods based on the study of the Lippman-Schwinger equation (see [7]). For this we have been forced to use different approach based on appropriate estimates of the resolvent operator for $\Delta - V$.

It is an open question the possible generalization of our result to the case of non-negative potentials satisfying the weaker estimate

$$|V(x)| \leq \frac{C}{|x|^2}. \tag{1.11}$$

The results in proved in [6] and [10] for the case of $V(x) = a/|x|^2$ suggest us that classical Strichartz estimate shall be satisfied, when the potential $V \geq 0$ has the form

$$V(x) = \frac{Y(x/|x|)}{|x|^2} + V_1(x),$$

where $Y(\omega) \geq 0$ is a measurable function on the sphere $\mathbb{S}^2$ and the remainder $V_1(x) \geq 0$ satisfies (1.9). The result in this work can be considered as a first step towards the proof of this conjecture.

The main idea of the proof is to use suitable a priori estimates for the resolvent of the perturbed Laplace operator $-\Delta + V$ that is

$$R_V(z) = (z + \Delta - V)^{-1}.$$

This operator is a well-defined bounded one in $L^2(\mathbb{R}^3)$ if $z \in \mathbb{C}\setminus\mathbb{R}$. Using suitable L^2 weighted estimates of $R_V(\lambda^2 \pm i\varepsilon)$ for $\varepsilon \in (0,1]$, $\lambda > 0$ it is possible to prove the existence of a natural limit operator $R_V(\lambda^2 \pm i0)$ defined as follows,

$$R_V(\lambda^2 \pm i0) f = \lim_{\varepsilon \to 0} R_V(\lambda^2 \pm i\varepsilon) f.$$

This limiting absorbtion principle is discussed in details in Corollary 6.2 in the Appendix.

The main step in the proof of Theorem 1.1 is to prove the estimate (1.8) under additional abstract assumption connected with the operator $R_0(\lambda^2 \pm i0)$ defined by

$$R_0(\lambda^2 \pm 0)\mathbb{U}(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{\pm i\lambda|x-y|}}{|x-y|} \mathbb{U}(y)\, dy. \tag{1.12}$$

It is clear that the kernel of the operator $[I - R_0(\lambda^2 \pm i0) V]$ consists of solutions $\mathbb{U}(x)$ of the integral equation

$$\mathbb{U}(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{\pm i\lambda|x-y|}}{|x-y|} V(y)\mathbb{U}(y)\, dy. \tag{1.13}$$

Our additional abstract assumption has the form

$$\begin{gathered} \textit{If } \lambda \geq 0 \textit{ and } \mathbb{U} \in L^2_{-3-\delta} \textit{ is a solution of } (1.13) \\ \textit{for } \delta > 0 \textit{ small}, \textit{ then } \mathbb{U}(x) = 0. \end{gathered} \tag{1.14}$$

The key point in the proof of Theorem 1.1 is the following.

THEOREM 1.2. *If the potential $V \geq 0$ satisfies the assumptions* (1.9) *and* (1.14), *then the estimate* (1.8) *is fulfilled.*

In the proof of Theorem 1.2 the following representation of the perturbed propagator

$$\mathcal{U}_V(t) f := \int_0^\infty \sin \lambda t [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)]\, f d\lambda, \tag{1.15}$$

will be very useful and will be combined with some L^2 weighted estimates for the limit resolvent operator $R_V(\lambda^2 \pm i0)$.

The idea to use the representation (1.15) for the proof of Strichartz type estimate as been used in [17], where a simplified proof of the classical Strichartz estimate is obtained in the case $V = 0$. More precisely, in the case $V = 0$ the solution to the free wave equation

$$(\partial_t^2 - \Delta)\, u(t, x) = 0, \qquad x \in \mathbb{R}^3, \tag{1.16}$$

with initial data $u(0) = 0$, $\partial_t u(0) = f$ is defined by

$$u(t, x) = \mathcal{U}_0(t) f, \tag{1.17}$$

where

$$\mathcal{U}_0(t) f := \int_0^\infty \sin \lambda t [R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)]\, f d\lambda. \tag{1.18}$$

The type of estimate that we prove for the resolvent limit

$$R_V(\lambda^2 \pm i0) = \lim_{\varepsilon \to 0} R_V(\lambda^2 \pm i\varepsilon)$$

have the general form

$$\|R_V(\lambda^2 \pm i0) f\|_s \leqslant \frac{C}{\lambda^A} \|f\|_\sigma \tag{1.19}$$

with $C = C(s, \sigma) > 0$, $A = A(s, \sigma) \geqslant 0$ suitably chosen.

These estimates have a semiclassical analogue in the study of the scattering poles [12, 14]. In some sense the high frequency case($\lambda \to \infty$) corresponds to the semiclassical estimates of the resolvent established in [12]. The corresponding estimates in the low frequency domain ($\lambda \sim 0$) have been studied in [16] and they play essential role in the proof of local energy decay.

In this paper the estimates of the type (1.19) will be combined with the representation (1.15), or more exactly they will be useful for the estimate of the following operators

$$\mathcal{U}_{V,\,\mathrm{low}}(t) f = \int_{\lambda \sim 0} \sin \lambda t [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)]\, f d\lambda$$

that represents the low frequency part of the propagator and

$$\mathcal{U}_{V,\,\mathrm{high}}(t) f = \int_{\lambda \sim \infty} \sin \lambda t [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)]\, f d\lambda$$

that corresponds to the high frequency part of the propagator. Note that the following identity is fulfilled trivially

$$\mathcal{U}_V(t) f = \mathcal{U}_{V,\,\mathrm{low}}(t) f + \mathcal{U}_{V,\,\mathrm{high}}(t) f.$$

For the estimate for the low frequency part we shall use a simple interpolation between a dispersive estimate and L^2 estimate. For the high frequency term we define an analytic family of operators and use the Stein interpolation Theorem.

Once the Theorem 1.2 is established it remains to show that the abstract assumption (1.14) follows from (1.9) and the fact that V is non-negative. Namely, we have the following.

THEOREM 1.3. *Suppose that the potential* $V \geqslant 0$ *satisfies* (1.9) *and* $\lambda \geqslant 0$. *If*

$$\mathbb{U} \in L^2_{-3-\delta}, \qquad 0 < \delta < \varepsilon_0$$

is a solution to the integral equation

$$\mathbb{U}(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{\pm i\lambda |x-y|}}{|x-y|} V(y)\, \mathbb{U}(y)\, dy, \tag{1.20}$$

then $\mathbb{U} = 0$.

In other words, the conclusion of this Theorem means that the kernel of the operator $[I - R_0(\lambda^2 \pm i0) V]$ is trivial for $\lambda \geqslant 0$.

We shall explain the main points in the proof of this Theorem.

The assumption (1.9) shows (see Section 3) that this operator

$$[I - R_0(\lambda^2 \pm i0)V]$$

is bounded in the weighted Lebesgue space $L^2_{-3-\delta}$, $0 < \delta < \varepsilon_0$.

From results due to Ikebe, Alsholm, Schmidt (see Lemma 4.4 in [8, p. 15], Proposition 1 in [2, p. 308] or Lemma 6.1 in the Appendix) one easily obtains the conclusion of Theorem 1.3 for $\lambda > 0$.

On the other hand, the operator $R_0(\lambda^2 \pm i0)$ has a natural limit in the uniform operator topology of $\mathcal{L}(L^2_{1+\delta}; L^2_{-3-\delta})$ (the Banach space of all bounded operators from $L^2_{1+\delta}$ in $L^2_{-3-\delta}$) as $\lambda \to 0$

$$\lim_{\lambda \to 0_+} R_0(\lambda^2 \pm i0) = R_0(0), \tag{1.21}$$

where

$$R_0(0) f(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy . \tag{1.22}$$

A detailed proof is given in Lemma 3.1 in Section 3 below.

Once we extended $R_0(\lambda^2 \pm i0)$ continuously for $\lambda \geqslant 0$ one can show (see Lemma 6.2 in the Appendix) that the conditions (1.9) and $V \geqslant 0$ imply that any solution to the integral equation (1.20) satisfies the estimate

$$|\mathbb{U}(x)| \leqslant \frac{C}{1+|x|}$$

with some constant $C > 0$, independent of $\lambda \geqslant 0$. Using this estimate we easily conclude the proof of Theorem 1.3.

The plan of the work is the following. In Section 2 we give various L^2 weighted estimate of the free resolvent and its square. To evaluate $R_V(\lambda^2 \pm i0)$ uniformly in λ we need an application of the Fredholm Theorem together with suitable estimates of the operator

$$[I - R_0(\lambda^2 \pm i0)V]^{-1}.$$

These estimates are discussed in Section 3 provided the abstract assumption (1.14) is satisfied. Applying the Fredholm Theorem, in Section 4 we extend the estimates of Section 2 for the case of perturbed resolvent. In Section 5 we prove Theorem 1.2. Finally, Section 6 is devoted to the verification of (1.14) when the potential $V \geqslant 0$ satisfies (1.9).

2. Resolvent and power resolvent estimates for the free Laplacian

The basic result of this section is a Theorem that unifies various weighted estimates for the free resolvent. In particular we are interested in the decay of the resolvent estimates for the high frequencies and its boundedness for the low frequencies.

Here and below the number ε_0 is the number related to the decay of the potential in (1.9).

THEOREM 2.1. *Given any real number $\delta > 0$ we have the following estimates*:

1) *if* $0 \leq a < 2 - \delta$, *then there exists* $C > 0$ *so that for any* $\lambda > 0$ *and any* $f \in C_c^\infty(\mathbb{R}^3)$ *we have*

$$\|R_0(\lambda^2 \pm i0) f\|_{-1-a-\delta} \leq C\|f\|_{3-a+\delta}; \tag{2.1}$$

2) *there exists* $C > 0$ *so that for any* $\lambda > 0$ *and any* $f \in C_c^\infty(\mathbb{R}^3)$ *we have*

$$\|R_0(\lambda^2 \pm i0) f\|_{-1-\delta} \leq \frac{C}{\lambda}\|f\|_{1+\delta}; \tag{2.2}$$

3) *if* $a \in [0, 2-\delta)$, $b \geq \delta a/(2-\delta)$ *and* $a + b \leq 2$ *then there exists* $C > 0$ *so that for any* $\lambda > 0$ *and any* $f \in C_c^\infty(\mathbb{R}^3)$ *we have*

$$\|R_0(\lambda^2 \pm i0) f\|_{-1-a-\delta} \leq \frac{C}{\lambda^{1-(a+b)/2}}\|f\|_{1+b+\delta}. \tag{2.3}$$

For the proof of the Theorem the following lemma will be useful.

LEMMA 2.1. *If* δ, a *are real numbers such that* $0 \leq a < 2 - \delta$, *then the functions*

$$\frac{1}{\langle y\rangle^{3+\delta-a}\langle x\rangle^{1+\delta+a}|x-y|^2}$$

belong to the Lebesgue space $L^1(\mathbb{R}^3_x \times \mathbb{R}^3_y)$.

PROOF. We start with the estimate of the following 3-dimensional integrals depending on the parameter y:

$$\int_{\mathbb{R}^3} \frac{1}{|x-y|^2\langle x\rangle^{1+\delta+a}}\,dx.$$

It is easy to prove that if $|y| < 1$, then these integrals are uniformly bounded. We now prove a decay estimate for $|y| > 1$. We split the integrals as follows:

$$\int_{\mathbb{R}^3} \frac{1}{|x-y|^2\langle x\rangle^{1+\delta+a}}\,dx = I + II + III, \tag{2.4}$$

where

$$I = \int_{|x| \geq 2|y|} \frac{1}{|x-y|^2\langle x\rangle^{1+\delta+a}}\,dx; \quad II = \int_{|x| \leq \frac{|y|}{2}} \frac{1}{|x-y|^2\langle x\rangle^{1+\delta+a}}\,dx;$$

$$III = \int_{\frac{|y|}{2} \leq |x| \leq 2|y|} \frac{1}{|x-y|^2\langle x\rangle^{1+\delta+a}}\,dx.$$

Estimate for I:

$$I \leqslant \int_{|x| \geqslant 2|y|} \frac{4}{|x|^2 \langle x \rangle^{1+\delta+a}} dx \leqslant \frac{C}{|y|^{\delta+a}}. \tag{2.5}$$

Estimate for II:

$$II \leqslant \frac{4}{|y|^2} \int_{B(0,\frac{|y|}{2})} \frac{1}{\langle x \rangle^{1+\delta+a}} dx \leqslant \frac{C}{|y|^{\delta+a}}. \tag{2.6}$$

Note that here we need the estimate $1+\delta+a<3$, *i.e.* $a<2-\delta$.

Estimate for III:

$$III \leqslant \int_{\frac{|y|}{2} \leqslant |x| \leqslant 2|y|} \frac{1}{|y|^{1+\delta+a}|x-y|^2} dx \leqslant \int_{B(y,4|y|)} \frac{1}{|y|^{1+\delta+a}|x-y|^2} dx \leqslant \frac{C}{|y|^{\delta+a}}, \tag{2.7}$$

then we have finally

$$\int_{\mathbb{R}^3} \frac{1}{|x-y|^2 \langle x \rangle^{1+\delta+a}} dx \leqslant \min\left\{C, \frac{C}{|y|^{\delta+a}}\right\}.$$

An easy application of the Fubini Theorem yields the desired estimate

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{\langle y \rangle^{3+\delta-a} \langle x \rangle^{1+\delta+a} |x-y|^2} dx\, dy \leqslant \int_{\mathbb{R}^3} \frac{C}{\langle y \rangle^{3+2\delta}} dy < \infty. \qquad \square$$

Proof of Theorem 2.1.

Proof of (2.1):

Using the explicit representation of $R_0(\lambda^2 \pm i0)$ we have

$$\|R_0(\lambda^2 \pm i0) f\|^2_{L^2_{-1-\delta-a}} \leqslant \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^{1+\delta+a}} \left| \int_{\mathbb{R}^3} |f(y)| \frac{1}{|x-y|} dy \right|^2 dx, \tag{2.8}$$

then the Cauchy inequality yields

$$\|R_0(\lambda^2 \pm i0) f\|^2_{L^2_{-1-\delta-a}} \leqslant \tag{2.9}$$

$$\leqslant \|f\|^2_{3+\delta-a} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^{1+\delta+a} \langle y \rangle^{3+\delta-a} |x-y|^2} dy\, dx \leqslant C\|f\|^2_{3+\delta-a},$$

where lemma 2.1 is used in the last inequality.

Proof of (2.2):

This estimate is contained in the paper [3].

An alternative proof can be obtained combining Theorem 2.3 and Theorem A.6 in [5].

Proof of (2.3):

The estimate (2.1) implies that the desired estimate is valid on the semi-closed segment

$$AB = \{(a, b);\ a + b = 2,\ 0 \leqslant a < 2 - \delta\},$$

where

$$A(2, 0), \qquad B(2 - \delta, \delta).$$

The estimate (2.2) shows that the desired estimate is valid at the point $C(0, 0)$. Making interpolation, we obtain the desired estimate in the triangle $\triangle ABC$ defined by the relations

$$0 \leqslant a < 2 - \delta, \qquad b \geqslant \frac{\delta a}{2 - \delta},\ a + b \leqslant 2.$$

This completes the proof. □

Our next step is to evaluate the square of the free resolvent. This type of estimates will be useful to perform an integration by parts in (1.15), since the derivative of the resolvent is related to its power.

THEOREM 2.2. *Given any real number* $\delta > 0$ *we have the following estimates*:
1) *there exists* $C > 0$ *so that for any* $\lambda > 0$ *and any* $f \in C_c^\infty(\mathbb{R}^3)$ *we have*

$$\|R_0^2(\lambda^2 \pm i0) f\|_{-3-\delta} \leqslant \frac{C}{\lambda} \|f\|_{3+\delta}; \tag{2.10}$$

2) *there exists* $C > 0$ *so that for any* $\lambda > 0$ *and any* $f \in C_c^\infty(\mathbb{R}^3)$ *we have*

$$\|R_0^2(\lambda^2 \pm i0) f\|_{-3-\delta} \leqslant \frac{C}{\lambda^2} \|f\|_{3+\delta}. \tag{2.11}$$

PROOF.

Proof of (2.10):

We have the following identity of operators,

$$2\lambda R_0^2(\lambda^2 \pm i0) = \frac{d}{d\lambda} R_0(\lambda^2 \pm i0)$$

then,

$$\lambda R_0^2(\lambda^2 \pm i0) f(x) = \frac{1}{2} \frac{d}{d\lambda} \frac{e^{i|\lambda||x|}}{|x|} * f = \frac{i}{2} e^{i|\lambda||x|} * f.$$

The desired estimate will be then a consequence of the following inequality

$$\|e^{i|\lambda||y|} * f\|_{-3-\delta} \leqslant C\|f\|_{3+\delta}.$$

that we can prove using the Hölder inequality:

$$\|e^{i|\lambda||y|} * f\|_{-3-\delta} \leqslant \int_{\mathbb{R}^3} \frac{1}{\langle x\rangle^{3+\delta}} \left| \int_{\mathbb{R}^3} \frac{|f(y)|\langle y\rangle^{\frac{3}{2}+\frac{\delta}{2}}}{\langle y\rangle^{\frac{3}{2}+\frac{\delta}{2}}} dy \right|^2 dx \tag{2.12}$$

$$\leqslant \|f\|^2_{3+\delta} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{\langle x\rangle^{3+\delta}\langle y\rangle^{3+\delta}} dx\, dy = C\|f\|^2_{3+\delta}. \tag{2.13}$$

Proof of (2.11):

The proof can be found in Theorem (11) of [9]. □

3. Fredholm theory for the Laplacian with a potential

The main result of this section concerns the existence of the inverse of a one-parameter family of operators and the estimates of its norms. The key point of the proof will be the Fredholm Theorem [11]. The result will be very useful to generalize the weighted estimates proved for the free Laplacian to the case of the perturbed Laplacian.

THEOREM 3.1. *Assume that the potential $V(x)$ satisfies the assumptions* (1.9) *and* (1.14) *and a is a real number satisfying*

$$1 \leqslant a \leqslant 3.$$

Then the operators $[I - R_0(\lambda^2 \pm i0)V]$ are invertible in $\mathcal{L}(L^2_{-a-\delta}, L^2_{-a-\delta})$. Moreover there exists a constant $C > 0$ such that for any $\delta < \varepsilon_0/2$ the following estimate holds

$$\|[I - R_0(\lambda^2 \pm i0)V]^{-1} f\|_{-a-\delta} \leqslant C\|f\|_{-a-\delta}, \qquad \forall \lambda \in \mathbb{R}.$$

To prove this Theorem first we shall consider the case $a = 3$. As a second step we shall consider the case $a = 1$. Applying an interpolation argument, we complete the proof.

LEMMA 3.1. *Let $\delta > 0$, $\delta_1 > 0$ and $0 \leqslant a \leqslant 2$. Then the one parameter family of operators $R_0(\lambda^2 \pm i0)$ is continuous in the space of operators $\mathcal{L}(L^2_{1+a+\delta_1}, L^2_{-3+a-\delta})$ for $\lambda \in [0, \infty)$.*

PROOF. We shall consider the cases $a = 0$, $a = 2$ separately and then we shall complete the proof by interpolation argument.

First, we start with the case $a = 0$.

We just prove the continuity in zero. Using explicit representation of the operators $R_0(\lambda^2 \pm i0)$ and making similar computations as in Theorem 2.1, we have

$$\|[R_0(\lambda^2 \pm i0) - R_0(0)]f\|^2_{-3-\delta} \leqslant \tag{3.1}$$

$$\leqslant \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|e^{i\lambda|x-y|} - 1|^2}{\langle x\rangle^{3+\delta}\langle y\rangle^{1+\delta_1}|x-y|^2} dy\, dx \|f\|^2_{1-\delta+2\varepsilon_0},$$

and for the Lebesgue dominated convergence Theorem the last integral converges to zero, when λ goes to zero.

The case $a = 2$ leads to the following integral

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|e^{i\lambda|x-y|} - 1|^2}{\langle x\rangle^{1+\delta}\langle y\rangle^{3+\delta_1}|x-y|^2}\,dy\,dx$$

and again the Lebesgue dominated convergence Theorem assures the convergence to 0.

This completes the proof. □

LEMMA 3.2. *Let σ, s be real numbers, such that $\sigma < s$, then the embedding*

$$H^1_{\mathrm{loc}}(\mathbb{R}^3) \cap L^2_s(\mathbb{R}^3) \hookrightarrow L^2_\sigma(\mathbb{R}^3)$$

is compact.

PROOF. Let $\{u_n\}$ be a bounded sequence in $H^1_{\mathrm{loc}}(\mathbb{R}^3) \cap L^2_s(\mathbb{R}^3)$. This means that there exists a sequence $C_k > 0$ of real constants, such that

$$\int_{|x|\le k} |\nabla u_n|^2 + |u_n|^2 \le C_k, \int_{\mathbb{R}^3} \langle x\rangle^s |u_n|^2\,dx \le C_0, \quad \forall n \in \mathbb{N}.$$

Using an elementary estimate we have

$$\int_{|x|\ge k} \langle x\rangle^\sigma |u_n|^2\,dx \le \frac{C_0}{\langle k\rangle^{s-\sigma}}. \tag{3.2}$$

Moreover for the compactness of the Sobolev embedding on bounded sets we have that for any $k \in \mathbb{N}$ there exists a subsequence of $\{u_n\}$, that we denote still by $\{u_n\}$, that has a strong limit in $L^2(|x| < k)$. It is now easy to deduce from this property and estimate (3.2) the result. □

PROOF OF THEOREM 3.1. First we take $a = 3$.

Using the hypothesis on the potential V we can prove that

$$V : L^2_{-3-\delta} \to L^2_{1-\delta+2\varepsilon_0}$$

and for Theorem 2.1 with $a = 2 + 2\delta - 2\varepsilon_0$ we have

$$R_0(\lambda^2 \pm i0) : L^2_{1-\delta+2\varepsilon_0} \to L^2_{-3-3\delta+2\varepsilon_0}.$$

Moreover with standard local elliptic regularity results one can prove that

$$R_0(\lambda^2 \pm i0) : L^2_{1-\delta+2\varepsilon_0} \to H^1_{\mathrm{loc}} \cap L^2_{-3-3\delta+2\varepsilon_0}$$

and using Lemma 3.2 we deduce that the linear applications

$$R_0(\lambda^2 \pm i0)V : L^2_{-3-\delta} \to L^2_{-3-\delta}$$

are compact.

The assumption (1.14) implies that the linear applications

$$[I - R_0(\lambda^2 \pm i0)]$$

are injective then for the Fredholm alternative Theorem [11] they are invertible operators.

To prove the uniform boundedness of the norm of the inverse operator we consider the low and the high frequencies case separately.

I case: $0 \leqslant \lambda \leqslant \lambda_0 < \infty$.

We remark that the family $[I - R_0(\lambda^2 \pm i0)]$ is continuous on the compact set $[0, \lambda_0]$ and their inverses have the same property hence they are uniformly bounded in $\mathcal{L}(L^2_{-3-\delta}, L^2_{-3-\delta})$.

II case: $\lambda > \lambda_0$ for λ_0 large enough and such that $\|R_0(\lambda^2 \pm i0) V\|_{-3-\delta,\,-3-\delta} \leqslant \frac{1}{2}$ for $\lambda > \lambda_0$. The existence of a λ_0 with this property can be deduced as follows. The decay assumption on the potential V guarantees that

$$V : L^2_{-3-\delta} \to L^2_{1-\delta+2\varepsilon_0}.$$

Moreover, we can apply (2.3) taking

$$a = 2 - 2\delta, \qquad b = \delta$$

in (2.3). It is not difficult to see that the conditions

$$0 \leqslant a < 2 - \delta,\ a + b \leqslant 2,\ b \geqslant \frac{\delta a}{(2-\delta)}$$

are satisfied with this choice of a, b. The operator

$$R_0(\lambda^2 \pm i0) : L^2_{1+2\delta} \to L^2_{-3+\delta}$$

has a norm $O(\lambda^{-\delta/2})$ according to (2.3) so this norm is small for λ_0 large enough. It remains to note that the embedding

$$L^2_{1-\delta+2\varepsilon_0} \hookrightarrow L^2_{1+2\delta}$$

for $\delta < 2\varepsilon_0/3$ has a norm 1. The same is valid for the embedding

$$L^2_{-3+\delta} \hookrightarrow L^2_{-3-\delta}$$

In conclusion

$$R_0(\lambda^2 \pm i0) : L^2_{1-\delta+2\varepsilon_0} \to L^2_{-3-\delta}$$

and its norm is $O(\lambda^{-\delta/2})$.

Hence, we conclude that

$$R_0(\lambda^2 \pm i0) V : L^2_{-3-\delta} \to L^2_{-3-\delta}$$

has a norm $< 1/2$ for $\lambda \geqslant \lambda_0$ and λ_0 large enough.

Then for $\lambda > \lambda_0$ we have $\|I - R_0(\lambda^2 \pm i0) V\| \geqslant \frac{1}{2}$ and consequently

$$\|[I - R_0(\lambda^2 \pm i0) V]^{-1}\| \leqslant C.$$

Next we consider the case $a = 1$. Then following the same argument we conclude that the operators $[I - R_0(\lambda^2 \pm i0) V]$ are invertible in $\mathcal{L}(L^2_{-1-\delta}, L^2_{-1-\delta})$. Moreover there exists a constant $C > 0$ such that for any $\delta < \varepsilon_0/2$ the following estimate holds

$$\|[I - R_0(\lambda^2 \pm i0) V]^{-1} f\|_{-1-\delta} \leqslant C\|f\|_{-1-\delta}, \qquad \forall \lambda \in \mathbb{R}.$$

Making interpolation between $a = 1$ and $a = 3$ we complete the proof of the Theorem. □

4. Resolvent and power resolvent estimates for the Laplacian with a potential

In this section we generalize the estimates proved for the free Laplacian to the perturbed Laplacian. The main tool that we use is the resolvent identity. The main result is the following.

Theorem 4.1. *Assume that the potential V satisfies the assumptions* (1.9) *and* (1.14) *then given any real number $\delta > 0$ we have the following estimates*:

1) *if $0 \leqslant a < 2 - \delta$, then there exists $C > 0$ so that for any $\lambda > 0$ and any $f \in C_c^\infty(\mathbb{R}^3)$ we have*

$$\|R_V(\lambda^2 \pm i0) f\|_{-1-a-\delta} \leqslant C\|f\|_{3-a+\delta}; \tag{4.1}$$

2) *there exists $C > 0$ so that for any $\lambda > 0$ and any $f \in L^2_{1+\delta}(\mathbb{R}^3)$ we have*

$$\|R_V(\lambda^2 \pm i0) f\|_{-1-\delta} \leqslant \frac{C}{\lambda}\|f\|_{1+\delta}; \tag{4.2}$$

3) *there exists a constant $C > 0$ such that for any $\lambda > 0$ and any $f \in L^2_{1+\delta}(\mathbb{R}^3)$*

$$\|(R_V(\lambda^2 \pm i0) f\|_{-3-\delta} \leqslant C\|f\|_{1+\delta}; \tag{4.3}$$

4) *if $a \in [0, 2-\delta)$, $b \geqslant \delta a/(2-\delta)$ and $a + b \leqslant 2$ then there exists $C > 0$ so that for any $\lambda > 0$ and any $f \in C_c^\infty(\mathbb{R}^3)$ we have*

$$\|R_V(\lambda^2 \pm i0) f\|_{-1-a-\delta} \leqslant \frac{C}{\lambda^{1-(a+b)/2}}\|f\|_{1+b+\delta}. \tag{4.4}$$

Proof.

Proof of (4.1):

The resolvent identity guarantees that we have,

$$R_V(\lambda^2 \pm i0) = R_0(\lambda^2 \pm i0) + R_0(\lambda^2 \pm i0) V R_V(\lambda^2 \pm i0) \tag{4.5}$$

or

$$[I - R_0(\lambda^2 \pm i0) V]\, R_V(\lambda^2 \pm i0) = R_0(\lambda^2 \pm i0).$$

From Theorem 3.1 we have,

$$R_V(\lambda^2 \pm i0) = [I - R_0(\lambda^2 \pm i0) V]^{-1} R_0(\lambda^2 \pm i0).$$

We conclude the proof applying Theorems (3.1) and (2.1).

Proof of (4.2):

It is similar to the Proof of (4.1).

Proof of (4.3):

It is obtained from (4.1) taking $a = 2 - 2\delta$. In this way we get

$$\|R_V(\lambda^2 \pm i0) f\|_{-3+\delta} \leq C\|f\|_{1+3\delta}.$$

Replacing 3δ by δ we obtain the desired estimate.

Proof of (4.4):

It is sufficient to combine (2.3) and Theorem 3.1. □

Next result generalizes the weighted estimates proved in Theorem 2.2 for the square of the free resolvent to the square of the perturbed resolvent.

THEOREM 4.2. *Assume that the potential V satisfies the assumptions* (1.9) *and* (1.14). *Then given any real number $\delta > 0$ we have the following estimates*:

1) *there exists $C > 0$ so that for any $\lambda > 0$ and any $f \in L^2_{3+\delta}(\mathbb{R}^3)$ we have*

$$\|R_V^2(\lambda^2 \pm i0) f\|_{-3-\delta} \leq \frac{C}{\lambda}\|f\|_{3+\delta}; \tag{4.6}$$

2) *there exists $C > 0$ so that for any $\lambda > 0$ and any $f \in L^2_{3+\delta}(\mathbb{R}^3)$ we have*

$$\|R_V^2(\lambda^2 \pm i0) f\|_{-3-\delta} \leq \frac{C}{\lambda^2}\|f\|_{3+\delta}; \tag{4.7}$$

3) *for any $\varepsilon \in (0, 1)$ there exists $C > 0$ so that for any $\lambda > 0$ and any $f \in L^2_{3+\delta}(\mathbb{R}^3)$ we have*

$$\|R_V^2(\lambda^2 \pm i0) f\|_{-3-\delta} \leq \frac{C}{\lambda^{1+\varepsilon}}\|f\|_{3+\delta}. \tag{4.8}$$

PROOF.

Proof of (4.6):

For the resolvent identity we have

$$R_V^2(\lambda^2 \pm i0) = [R_0(\lambda^2 \pm i0) + R_0(\lambda^2 \pm i0) V R_V(\lambda^2 \pm i0)]\, R_V(\lambda^2 \pm i0) \tag{4.9}$$

$$= R_0(\lambda^2 \pm i0)[R_0(\lambda^2 \pm i0) + R_0(\lambda^2 \pm i0) V R_V(\lambda^2 \pm i0)] \tag{4.10}$$

$$+ R_0(\lambda^2 \pm i0) V R_V^2(\lambda^2 \pm i0) \tag{4.11}$$

then

$$[I - R_0(\lambda^2 \pm i0) V]\, R_V^2(\lambda^2 \pm i0) = R_0^2(\lambda^2 \pm i0) + R_0^2(\lambda^2 \pm i0) V R_V(\lambda^2 \pm i0)$$

or equivalently

$$R_V^2(\lambda^2 \pm i0) = [I - R_0(\lambda^2 \pm i0) V]^{-1} R_0^2(\lambda^2 \pm i0)$$
$$+ [I - R_0(\lambda^2 \pm i0) V]^{-1} R_0^2(\lambda^2 \pm i0) V R_V(\lambda^2 \pm i0).$$

The estimate is an easy consequence of Theorem 3.1 and (2.10), (4.1).

Proof of (4.7):

It is similar to the proof of (4.6).

Proof of (4.8):

We make interpolation between previous two steps. □

5. Dispersive estimate

In this section we prove Theorem 1.2.

Take a Paley-Littlewood partition of unity

$$1 = \sum_{k=0}^{\infty} \phi_k(x),$$

such that $|x| \sim 2^k$ for $x \in \operatorname{supp} \phi_k$. Then the propagator

$$\mathcal{U}_V(t) f := \int_0^\infty \sin \lambda t [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] f d\lambda$$

can be decomposed as

$$\mathcal{U}_V(t) f = \sum_{k=0}^{\infty} \mathcal{U}_k(t) f, \tag{5.1}$$

where

$$\mathcal{U}_k(t) f = \mathcal{U}_V(t)(\phi_k f).$$

We can further decompose each operator $\mathcal{U}_k(t)$ as a sum of two terms: high and low frequency parts as follows

$$\mathcal{U}^k_{\text{low}}(t) f := \int_0^\infty \psi_1(\lambda 2^k) \sin \lambda t [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] \, \phi_k f d\lambda, \tag{5.2}$$

$$\mathcal{U}^k_{\text{high}}(t) f := \int_0^\infty \psi_2(\lambda 2^k) \sin \lambda t [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] \, \phi_k f d\lambda, \tag{5.3}$$

where ψ_1 is function satisfying the following condition

1) $\psi_1 \in C_0^\infty(\mathbb{R})$,
2) $\operatorname{supp} \psi_1 = [-2, 2]$,
3) $\psi_1 = 1$ for $x \in [-1, 1]$,

and $\psi_2 = 1 - \psi_1$.

The corresponding unperturbed operators are:

$$\mathcal{U}^{k,0}_{\text{low}}(t) f := \int_0^\infty \psi_1(\lambda 2^k) \sin \lambda t [R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)] \, \phi_k f d\lambda, \tag{5.4}$$

$$\mathcal{U}^{k,0}_{\text{high}}(t) f := \int_0^\infty \psi_2(\lambda 2^k) \sin \lambda t [R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)] \, \phi_k f d\lambda. \tag{5.5}$$

For the low frequency term we use the explicit integral representation in (5.2) and shall derive the following

LEMMA 5.1. *There exists a constant* $C > 0$ *such that for any integer* $k \geqslant 0$ *we have*

$$\| \mathcal{U}^k_{\mathrm{low}}(t) f \|_{L^\infty} \leqslant C \frac{2^{\frac{k}{2} + \delta k}}{t} \| f \|_{L^2}. \tag{5.6}$$

On the other hand, the functional calculus for $-\Delta + V$ leads to the following L^2 estimate of the low frequency part.

LEMMA 5.2. *There exists a real constant* $C > 0$ *such that the following estimates hold*

$$\| \mathcal{U}^k_{\mathrm{low}}(t) f \|_{L^2} \leqslant C 2^{k + \delta k} \| f \|_{L^2}. \tag{5.7}$$

Interpolation between Lemma 5.1 and Lemma 5.2 gives the following estimate.

COROLLARY 5.1. *There exists a real constant* $C > 0$ *such that the following estimate hold*

$$\| \mathcal{U}^k_{\mathrm{low}}(t)(f) \|_{L^4} \leqslant C \frac{2^{\frac{3k}{4} + \delta k}}{\sqrt{t}} \| f \phi_k \|_{L^2}. \tag{5.8}$$

For the high frequency term $U^k_{\mathrm{high}}(t)$ we introduce appropriate analytic family of operators

$$\mathcal{U}^k_{\mathrm{high},\,z}(t) f := \int_0^\infty \psi_2(\lambda 2^k) \sin \lambda t [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)] f \phi_k \lambda^{-z} d\lambda. \tag{5.9}$$

The unperturbed family has analogous definition

$$\mathcal{U}^{k,0}_{\mathrm{high},\,z}(t) f := \int_0^\infty \psi_2(\lambda 2^k) \sin \lambda t [R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)] f \phi_k \lambda^{-z} d\lambda. \tag{5.10}$$

This operator is well-defined for $\mathrm{Re}\, z \geqslant -1$. For $\mathrm{Re}\, z = 1$ we have the following estimate that we shall prove later on.

LEMMA 5.3. *There exists a real constant* $C > 0$ *such that for any integer* $k \geqslant 0$ *and* $\sigma \in \mathbb{R}$ *the following estimate holds*

$$\| \mathcal{U}^k_{\mathrm{high},\, 1 + i\sigma}(t) f \|_{L^\infty} \leqslant \frac{C}{t} 2^{\frac{3}{2}k + \delta k} \| f \|_{L^2}. \tag{5.11}$$

On the line $\mathrm{Re}\, z = -1$ we have

LEMMA 5.4. *There exists a constant* $C > 0$ *such that for any integer* $k \geqslant 0$ *we have*

$$\| \mathcal{U}^k_{\mathrm{high},\, -1 + i\sigma}(t) f \|_{L^2} \leqslant C \| f \|_{L^2}$$
$$\| \mathcal{U}^{k,0}_{\mathrm{high},\, -1 + i\sigma}(t) f \|_{L^2} \leqslant C \| f \|_{L^2}$$

for $\sigma \in \mathbb{R}$.

We will prove this Lemma later on. A complex interpolation between the estimates in the last two Lemmas imply.

COROLLARY 5.2. *There exists a real constant $C > 0$ such that the following estimates hold*

$$\|\mathcal{U}^k_{\text{high}}(t)(f)\|_{L^4} \leq C \frac{2^{\frac{3k}{4}+\delta k}}{\sqrt{t}} \|f\phi_k\|_{L^2}. \tag{5.12}$$

Unifying the estimates (5.8) for the low frequency part with the corresponding estimate (5.12) for the high frequency part, we get

$$\|\mathcal{U}_V(t)(\phi_k f)\|_{L^4} = \|\mathcal{U}_k(t)(f)\|_{L^4} \leq C \frac{2^{\frac{3k}{4}+\delta k}}{\sqrt{t}} \|f\phi_k\|_{L^2}. \tag{5.13}$$

Now we can complete the

PROOF OF THEOREM 1.2. We decompose f using Paley-Littlewood partition

$$f = \sum_{k=0}^{\infty} f\phi_k.$$

Applying the estimate (5.13) (substituting $\delta > 0$ with $\delta/4$), we find

$$\|\mathcal{U}_V(t) f\|_{L^4} \leq \left\| \sum_{k\geq 0} \mathcal{U}_V(t)(\phi_k f) \right\|_{L^4} \leq \sum_{k\geq 0} \|\mathcal{U}_k(t)(\phi_k f)\|_{L^4} \leq \frac{C}{\sqrt{t}} \sum_{k\geq 0} 2^{3k/4+\delta k/4} \|\phi_k f\|_{L^2}.$$

Using the Cauchy inequality, we get

$$\Big(\sum_{k\geq 0} 2^{3k/4+\delta k/4} \|\phi_k f\|_{L^2}\Big)^2 \leq C \sum_{k\geq 0} |2^{3k/4+\delta k/2} \|\phi_k f\|_{L^2}|^2.$$

The quantity in the right side is equivalent to $\|f\|^2_{3/2+\delta}$ so we obtain

$$\|\mathcal{U}_V(t) f\|_{L^4} \leq \frac{C}{\sqrt{t}} \|f\|_{3/2+\delta}. \qquad \square$$

The remaining part of this section is devoted to the proof of the above four Lemmas.

PROOF OF LEMMA 5.1. We use the resolvent identity to obtain the following identity of operators

$$\mathcal{U}^k_{\text{low}}(t) = \mathcal{U}^{k,0}_{\text{low}}(t) + r^k_{\text{low}}(t), \tag{5.14}$$

where

$$\mathcal{U}^{k,0}_{\text{low}}(t) f := \int_0^\infty \psi_1(\lambda 2^k) \sin \lambda t [R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)]\, \phi_k f d\lambda,$$

$r^k_{\text{low}}(t) f :=$

$$\int_0^\infty \psi_1(\lambda 2^k) \sin \lambda t [R_0(\lambda^2 + i0) V R_V(\lambda^2 + i0) - R_0(\lambda^2 - i0) V R_V(\lambda^2 - i0)]\, \phi_k f d\lambda.$$

Applying a standard stationary phase method (see [17]), we get

$$\|\mathcal{U}^{k,0}_{\mathrm{low}}(t)f\|_{L^\infty} \leq C\frac{2^{k/2+\delta k}}{t}\|f\|_{L^2}. \tag{5.15}$$

It is then sufficient to estimate $r^k_{\mathrm{low}}(t)$, or the operator

$$\tilde{r}^k_{\mathrm{low}}(t)f := \int_0^\infty \psi_1(\lambda 2^k)\sin\lambda t R_0(\lambda^2+i0)VR_V(\lambda^2+i0)\,\phi_k f d\lambda.$$

Using an explicit representation of the operators $R_0(\lambda^2+i0)$ we have the following identity

$$\tilde{r}^k_{\mathrm{low}}(t) = \int_0^\infty \psi_1(\lambda 2^k)\sin\lambda t\int_{\mathbb{R}^3}\frac{e^{i\lambda|x-y|}}{|x-y|}[VR_V(\lambda^2+i0)f\phi_k](y)\,d\lambda dy\,.$$

After an integration by parts with respect to λ we have

$$\tilde{r}^k_{\mathrm{low}}f = \frac{1}{t}(I+II+III) \tag{5.16}$$

where

$$I = \int_0^\infty 2^k\psi_1'(\lambda 2^k)\cos\lambda t\int_{\mathbb{R}^3}\frac{e^{i\lambda|x-y|}}{|x-y|}[VR_V(\lambda^2+i0)f\phi_k](y)\,dyd\lambda; \tag{5.17}$$

$$II = \int_0^\infty \psi_1(\lambda 2^k)\cos\lambda t\int_{\mathbb{R}^3}e^{i\lambda|x-y|}[VR_V(\lambda^2+i0)\,\phi_k f](y)\,dyd\lambda; \tag{5.18}$$

$$III = \int_0^\infty \psi_1(\lambda 2^k)\cos\lambda t\int_{\mathbb{R}^3}\frac{e^{i\lambda|x-y|}}{|x-y|}[V\partial_\lambda R_V(\lambda^2+i0)f\phi_k](y)\,dyd\lambda = \tag{5.19}$$

$$= -\int_0^\infty \psi_1(\lambda 2^k)\cos\lambda t\int_{\mathbb{R}^3}\frac{e^{i\lambda|x-y|}}{|x-y|}[V\lambda R_V^2(\lambda^2+i0)f\phi_k](y)\,dyd\lambda.$$

We estimate the integrals I, II, III separately.

The following inequality shall be used to evaluate these terms.

$$\int_{\mathbb{R}^3}\frac{V(y)\,|g(y)|}{|x-y|}dy \leq C\|g\|_{-3-\delta}, \qquad \delta < 2\varepsilon_0. \tag{5.20}$$

To verify this estimate we apply the Hölder inequality and get the estimate

$$\int_{\mathbb{R}^3}\frac{V(y)\,|g(y)|}{|x-y|}dy \leq C\|g\|_{-3-\delta}\|\langle\cdot\rangle^{2+\varepsilon_0}V\|_{L^\infty}\left(\int_{\mathbb{R}^3}\frac{\langle y\rangle^{-1-2\varepsilon_0+\delta}}{|x-y|^2}dy\right)^{1/2}.$$

The integral

$$\int_{\mathbb{R}^3} \frac{1}{|x-y|^2} \langle y \rangle^{-1-2\varepsilon_0+\delta} dy$$

is bounded for $\delta < 2\varepsilon_0$ (see the argument of the proof of Lemma 2.1). Thus, the decay assumption (1.9) leads to (5.20).

Estimate of the first term I.

Using the fact that $\lambda \sim 2^{-k}$ for $\lambda \in \mathrm{supp}_\lambda \psi_1'(\lambda 2^k)$, we see that $|I|$ is bounded from above by constant times

$$\sup_{\lambda \sim 2^{-k}} \int_{\mathbb{R}^3} \frac{1}{|x-y|} |[VR_V(\lambda^2 + i0) f\phi_k](y)| dy.$$

From (5.20) we derive

$$|I| \leqslant C \sup_{\lambda \sim 2^{-k}} \|R_V(\lambda^2 + i0) f\phi_k]\|_{-3-\delta}.$$

Now we are in situation to apply the estimate (4.3) and this gives

$$|I| \leqslant C2^{k/2+k\delta} \|f\phi_k\|_{L^2}. \tag{5.21}$$

Estimate of the second term II.

To evaluate II we follow similar idea. More precisely, $|II|$ is bounded from above by constant times

$$\int_{\lambda \leqslant 2^{-k}} \int_{\mathbb{R}^3} |[VR_V(\lambda^2 + i0) f\phi_k](y)| dy d\lambda.$$

Using the Cauchy inequality we have

$$\int_{\mathbb{R}^3} |[VR_V(\lambda^2 + i0) f\phi_k](y)| dy \leqslant C\|R_V(\lambda^2 + i0) f\phi_k\|_{-1-\delta}.$$

Now we use the estimate (4.1) and find

$$|II| \leqslant C2^{3k/2+\delta k} \int_{\lambda \leqslant 2^{-k}} d\lambda \|f\phi_k\|_{L^2} \leqslant C2^{k/2+\delta k} \|f\phi_k\|_{L^2}.$$

In conclusion, we get

$$|II| \leqslant C2^{k/2+k\delta} \|f\phi_k\|_{L^2}. \tag{5.22}$$

Estimate of the second term II.

In the estimate of III we use again (5.20) in combination with the estimate (4.6) and get

$$|III| \leqslant C \int_{\lambda \leqslant 2^{-k}} 2^{3k/2+\delta k} d\lambda \|f\phi_k\|_{L^2} = C2^{k/2+\delta k} \|f\phi_k\|_{L^2}.$$

From this estimate, (5.21) and (5.22) we obtain (5.16) and via (2.9) we arrive at the estimate (5.6). □

PROOF OF LEMMA 5.2. We have the representation formula (see (5.2))

$$\mathcal{U}^k_{\text{low}}(t)\, f = \psi_1(2^k\sqrt{-\Delta+V})\,\frac{\sin(t\sqrt{-\Delta+V})}{\sqrt{-\Delta+V}}\,(\phi_k f).$$

This relation and the fact that the operator $-\Delta+V$ is self-adjoint implies that

$$\|\mathcal{U}^k_{\text{low}}(t)\, f\|_{L^2} \leq \|(-\Delta+V)^{-1/2}(f\phi_k)\|_{L^2}. \tag{5.23}$$

To finish the proof it is sufficient to show the estimate

$$\|(-\Delta+V)^{-1/2} f\|_{L^2} \leq \||x|f\|_{L^2}. \tag{5.24}$$

To show this inequality we start with

$$\|(-\Delta+V)^{-1/2} f\|^2_{L^2} = \int u(x)\, f(x)\, dx, \tag{5.25}$$

where u is a solution of the equation

$$(-\Delta+V)\, u = f. \tag{5.26}$$

Multiplying this equation by u and using the fact that $V \geq 0$, we get

$$\|\nabla u\|^2_{L^2} \leq \int |u(x)\, f(x)|\, dx. \tag{5.27}$$

At this point we can use the Hardy inequality

$$\||x|^{-1} u\|_{L^2} \leq C\|\nabla u\|_{L^2}. \tag{5.28}$$

Combining the Hardy inequality, the estimate (5.27) and the Cauchy inequality, we find

$$\||x|^{-1} u\|^2_{L^2} \leq C\||x|f\|_{L^2}\||x|^{-1}u\|_{L^2}$$

so

$$\||x|^{-1} u\|_{L^2} \leq C\||x|f\|_{L^2}. \tag{5.29}$$

Turning back to (5.25), we find

$$\|(-\Delta+V)^{-1/2} f\|^2_{L^2} \leq C\||x|^{-1}u\|_{L^2}\||x|f\|_{L^2}$$

so applying (5.29), we get (5.24).

This completes the proof. □

PROOF OF LEMMA 5.3. We use the resolvent identity to obtain the following identity of operators

$$\mathcal{U}^k_{\text{high},\,1+i\sigma}(t) := \mathcal{U}^{k,\,0}_{\text{high},\,1+i\sigma} + s^k_{\text{high},\,1+i\sigma}$$

where

$$\mathcal{U}^{k,0}_{\mathrm{high},1+i\sigma} := \int_0^\infty \psi_2(\lambda 2^k)\sin\lambda t[(R_0(\lambda^2+i0)-R_0(\lambda^2-i0))\,\phi_k f](y)\,\lambda^{-1+i\sigma}d\lambda dy\,,$$

$$s^k_{\mathrm{high},1+i\sigma}(t) := \int_0^\infty \psi_2(\lambda 2^k)\sin\lambda t[R_0(\lambda^2+i0)VR_V(\lambda^2+i0)](\phi_k f)(y)\lambda^{-1+i\sigma}d\lambda dy -$$

$$-\int_0^\infty \psi_2(\lambda 2^k)\sin\lambda t[R_0(\lambda^2-i0)VR_V(\lambda^2-i0)](\phi_k f)(y)\,\lambda^{-1+i\sigma}d\lambda dy$$

The following estimate is a consequence of stationary phase method argument (see [17])

$$\|\mathcal{U}^{k,0}_{\mathrm{high},1+i\sigma}(t)f\|_\infty \leqslant \frac{C}{t}2^{\frac{3}{2}k+\delta k}\|f\|_2. \tag{5.30}$$

It is sufficient to prove the estimate for the operator

$$\tilde{s}^k_{\mathrm{high},1+i\sigma} := \int_0^\infty \psi_2(\lambda 2^k)\sin\lambda t[R_0(\lambda^2+i0)VR_V(\lambda^2+i0)\phi_k f](y)\lambda^{-1+i\sigma}d\lambda dy.$$

Using the explicit representation of the operator $R_0(\lambda^2+i0)$ we have the following identity

$$\tilde{s}^k_{\mathrm{high},-1+i\sigma} = \int_0^\infty \psi_2(\lambda 2^k)\sin\lambda t\int_{\mathbb{R}^3}\frac{e^{i\lambda|x-y|}}{|x-y|}[VR_V(\lambda^2+i0)\phi_k f](y)\lambda^{-1+i\sigma}dyd\lambda.$$

After an integration by parts we have

$$\tilde{s}^k_{\mathrm{high},1+i\sigma}f(x) = \frac{1}{t}(I+II+III+IV) \tag{5.31}$$

where

$$I=\int_0^\infty 2^k\psi_2'(\lambda 2^k)\cos\lambda t\int_{\mathbb{R}^3}\frac{e^{i\lambda|x-y|}}{|x-y|}[VR_V(\lambda^2+i0)\phi_k f](y)\lambda^{-1+i\sigma}dyd\lambda \tag{5.32}$$

$$II=\int_0^\infty \psi_2(\lambda 2^k)\cos\lambda t\int_{\mathbb{R}^3}e^{i\lambda|x-y|}[VR_V(\lambda^2+i0)\phi_k f](y)\lambda^{-1+i\sigma}dyd\lambda \tag{5.33}$$

$$III=\int_0^\infty \psi_2(\lambda 2^k)\cos\lambda t\int_{\mathbb{R}^3}\frac{e^{i\lambda|x-y|}}{|x-y|}[V\partial_\lambda R_V(\lambda^2+i0)\,\phi_k f](y)\lambda^{-1+i\sigma}dyd\lambda \tag{5.34}$$

$$IV=\int_0^\infty \psi_2(\lambda 2^k)\cos\lambda t\int_{\mathbb{R}^3}\frac{e^{i\lambda|x-y|}}{|x-y|}[VR_V(\lambda^2+i0)\phi_k f](y)\lambda^{-2+i\sigma}dyd\lambda. \tag{5.35}$$

We estimate the integrals I, II, III, IV separately.

Estimate of I.

We follow the argument of the estimate of the term I in the proof of Lemma 5.1. Thus, we get

$$|I| \leqslant \int_{\lambda \sim 2^{-k}} 2^{3k/2 + k\delta} \frac{d\lambda}{\lambda} \|f\phi_k\|_{L^2} \leqslant C 2^{3k/2 + k\delta} \|f\phi_k\|_{L^2}.$$

Estimate of II.

For the term II we follow the similar argument for the corresponding term II from the proof of Lemma 5.1 with the minor change: we use the estimate (4.4) and substitute δ with $\delta/2$. In this way we get

$$|II| \leqslant C 2^{3k/2 + \delta k/2} \int_{\lambda \geqslant 2^{-k}} \frac{d\lambda}{\lambda^{1+\varepsilon}} \|f\phi_k\|_{L^2} \leqslant C 2^{3k/2 + \delta k/2 + \varepsilon k} \|f\phi_k\|_{L^2}.$$

Taking $\varepsilon = \delta/2$ we obtain

$$|II| \leqslant C 2^{3k/2 + \delta k} \|f\phi_k\|_{L^2}.$$

Estimate of III.

We follow the same line, only we use the estimate (4.8), replace δ with $\delta/2$ and get

$$|III| \leqslant C \int_{\lambda \geqslant 2^{-k}} 2^{3k/2 + \delta k/2} \frac{d\lambda}{\lambda^{1+\varepsilon}} \|f\phi_k\|_{L^2} = C 2^{3k/2 + \delta k/2 + \varepsilon k} \|f\phi_k\|_{L^2}.$$

Taking $\varepsilon = \delta/2$ we obtain

$$|III| \leqslant C 2^{3k/2 + \delta k} \|f\phi_k\|_{L^2}.$$

Estimate for IV.

We have

$$|IV| \leqslant C \int_{\lambda > 2^{-k}} \|R_V(\lambda^2 + i0)\phi_k f\|_{-3-\delta} \frac{d\lambda}{\lambda^2}.$$

Applying the estimate (4.3), we get

$$|IV| \leqslant C \int_{\lambda > 2^{-k}} \|\phi_k f\|_{1+\delta} \frac{d\lambda}{\lambda^2} \leqslant C 2^{3k/2 + \delta k} \|\phi_k f\|_{L^2}.$$

Then using (5.30), (5.31) and the above estimates of I, II, III, IV we arrive at the estimate of the Lemma. □

Proof of Lemma 5.4. We have the following identities

$$\mathcal{U}^k_{\text{high}, -1+i\sigma}(t) = \psi_2(2^k\sqrt{-\Delta + V})\left(\sin t\sqrt{-\Delta + V}\right)(-\Delta + V)^{i\sigma}\phi_k(x)$$

$$\mathcal{U}^{k,0}_{\text{high}, -1+i\sigma}(t) = \psi_2(2^k\sqrt{-\Delta})\left(\sin t\sqrt{-\Delta}\right)(-\Delta)^{i\sigma}\phi_k(x)$$

and then the Lemma is a consequence of the spectral Theorem and the uniform boundedness of the family of functions

$$\psi_2(2^k\lambda)\lambda^{i\sigma}\sin t\lambda. \qquad \square$$

6. Absence of point spectrum for potential perturbation of Laplacian

This section is devoted to the proof of Theorem 1.3. We split the proof in few steps.

Lemma 6.1. *Let the potential V satisfies the assumption* (1.9). *Let be given a function $\mathbb{U}$ such that*:

1) $\mathbb{U} \in L^2_{-3-\delta}(\mathbb{R}^3)$ *for* $\delta \in (0, \varepsilon_0)$;

2) *the function $\mathbb{U}$ is a solution of the integral equation* (1.13) *for* $\lambda \geqslant 0$, *then $\mathbb{U}$ belongs to the weighted Lebesgue space $L^2_{-1-\delta}(\mathbb{R}^3)$.*

Proof. From the decay assumption (1.19) on the potential we deduce that if $\mathbb{U}$ satisfies the assumption 1 then

$$V\mathbb{U} \in L^2_{1+2\varepsilon_0-\delta}.$$

Thus, using (2.1) with $a = 2 - 2\varepsilon_0 + 2\delta$ and $0 < \delta \ll \varepsilon_0$, we deduce

$$\mathbb{U} = R_0(\lambda^2 \pm i0)V\mathbb{U} \in L^2_{-3+2\varepsilon_0-3\delta}.$$

Setting $b_0 = 2\varepsilon_0 - 3\delta$ and taking $\delta < 2\varepsilon_0/3$, we see that $b_0 > 0$ and

$$\mathbb{U} = R_0(\lambda^2 \pm i0)V\mathbb{U} \in L^2_{-3+b_0}.$$

Further we repeat the above argument, starting with

$$\mathbb{U} \in L^2_{-3+b}$$

with some $b > 0$. Then the assumption (1.9) on the potential implies that

$$V\mathbb{U} \in L^2_{1+b+2\varepsilon_0}.$$

Now we want to apply (2.1) with $a = 2 - b - 2\varepsilon_0 + \delta$. To do this we need the assumption $a \geqslant 0$ and this condition implies

$$b \leqslant 2 - 2\varepsilon_0 + \delta. \tag{6.1}$$

Once this condition is true, we apply (2.1) with $a = 2 - b - 2\varepsilon_0 + \delta$ and find

$$\mathbb{U} = R_0(\lambda^2 \pm i0)V\mathbb{U} \in L^2_{-3+b+2\varepsilon_0-2\delta}.$$

Therefore, take $b_0 = 2\varepsilon_0 - 3\delta > 0$ and

$$b_k = b_0 + 2k(\varepsilon_0 - \delta), \qquad k = 1, \ldots, N$$

with N defined by (6.1) as follows

$$b_{N-1} \leqslant 2 - 2\varepsilon_0 + \delta < b_N.$$

Now an induction with respect to k, $1 \leqslant k \leqslant N$ leads to

$$\mathbb{U} = R_0(\lambda^2 \pm i0)V\mathbb{U} \in L^2_{-3+b_N} \subset L^2_{-3+\tilde{b}},$$

where $\tilde{b} = 2 - 2\varepsilon_0 + \delta$. Now we have

$$\mathbb{U} \in L^2_{-3+\tilde{b}} = L^2_{-1-2\varepsilon_0+\delta}$$

so

$$V\mathbb{U} \in L^2_{3+\delta}$$

so this time we can apply (2.1) with $a = 0$ and this gives

$$\mathbb{U} = R_0(\lambda^2 \pm i0)\,V\mathbb{U} \in L^2_{-3+\tilde{b}+2\varepsilon_0-2\delta} = L^2_{-1-\delta}.$$

In this way we derive the improved decay of $\mathbb{U}$

$$\mathbb{U} \in L^2_{-1-\delta}$$

and the corresponding improvement for $V\mathbb{U}$

$$V\mathbb{U} \in L^2_{3+\delta}. \qquad \square$$

LEMMA 6.2. *Let the potential V satisfies the assumption* (1.9). *Let be given a function $\mathbb{U}$ such that*:

1) $\mathbb{U} \in L^2_{-3-\delta}(\mathbb{R}^3)$ *for* $\delta \in (0, \varepsilon_0)$;

2) *the function $\mathbb{U}$ is a solution of the integral equation* (1.13) *for* $\lambda \geq 0$,

then there exists a constant $C > 0$ such that

$$|\mathbb{U}(x)| \leqslant \frac{C}{\langle x\rangle} \tag{6.2}$$

and

$$|\nabla\mathbb{U}(x)| \leqslant \frac{C}{\langle x\rangle^2}. \tag{6.3}$$

PROOF. For the hypothesis we have the following identity

$$\mathbb{U} = R_0(\lambda^2 \pm i0)\,V\mathbb{U}$$

and for Lemma 6.1 we have $V\mathbb{U} \in L^2_{3+\delta}$. First, we shall show the pointwise estimate (6.2).

Using the explicit representation of the operator $R_0(\lambda^2 \pm i0)$ and the Cauchy inequality we see that

$$|\mathbb{U}| \leqslant \int_{\mathbb{R}^3} |x-y|^{-1}\,|V(y)\mathbb{U}(y)|\,dy \leqslant \left(\int_{\mathbb{R}^3} |x-y|^{-2}\langle y\rangle^{-3-\delta}\,dy\right)^{1/2} \|V\mathbb{U}\|_{3+\delta}.$$

Applying the argument of the estimate (2.4), we get

$$\int_{\mathbb{R}^3} |x-y|^{-2}\langle y\rangle^{-3-\delta}\,dy \leqslant \frac{C}{\langle x\rangle^2}$$

then using this estimate and the property $V\mathbb{U} \in L^2_{3+\delta}$ we deduce

$$|\mathbb{U}(x)| \leqslant \frac{C}{\langle x\rangle}. \tag{6.4}$$

Using agin explicit representation of the operator $R_0(\lambda^2 \pm i0)$, the decay assumption on the potential V and the estimate (6.2) we get

$$|\nabla \mathbb{U}(x)| \leq \int_{\mathbb{R}^3} |x-y|^{-2} V(y) |\mathbb{U}(y)| dy \leq C \int_{\mathbb{R}^3} |x-y|^{-2} \langle y \rangle^{-3-\varepsilon_0} dy.$$

It is easy to prove that the last integral is bounded by constant times $\langle x \rangle^{-2}$ and this proves (6.3).

The proof is complete. □

COROLLARY 6.1. *Suppose that the potential V satisfies* (1.9). *Let be given a function $\mathbb{U}$ such that*:

1) $\mathbb{U} \in L^2_{-3-\delta}(\mathbb{R}^3)$ *for* $\delta \in (0, \varepsilon_0)$;

2) *the function $\mathbb{U}$ is a solution of the integral equation* (1.13) *for* $\lambda > 0$,

then necessarily $\mathbb{U} = 0$.

PROOF. It is sufficient to apply Lemma 4.4 from [8] and use the fact that $\mathbb{U}$ satisfies the estimate (6.2) established in the previous Lemma. □

REMARK. Using the Proposition 1 in [2, p. 308] one can obtain the same conclusion ($\mathbb{U} = 0$) using only the boundedness of $\mathbb{U}$. Note that in the above corollary we did not use the fact that $V \geqslant 0$.

COROLLARY 6.2. *If the potential $V(x)$ satisfies the conditions* (1.9), *then for any $\lambda > 0$ the following limit exists*

$$\lim_{\delta \to 0_+} R_V(\lambda^2 \pm i\delta) = R_V(\lambda^2 \pm i0) \tag{6.5}$$

in the uniform topology of the operators space $\mathcal{L}(L^2_{1+\delta}; L^2_{-1-\delta})$.

PROOF. It follows from the limiting absorbtion principle established by Agmon [1] (see Theorem 4.2 in [1, p. 166]) and the fact that the previous corollary assures that the point spectrum of $-\Delta + V$ on $(0, +\infty)$ is empty. □

PROOF OF THEOREM 1.3. It is sufficient to consider only the case $\lambda = 0$, since the case $\lambda > 0$ has been treated in Corollary 6.1.

If $\mathbb{U} \in L^2_{-3-\delta}(\mathbb{R}^3)$ is solution of the integral equation (1.20) with $\lambda = 0$, then necessarily for the elliptic regularity we have $\mathbb{U}$ is a solution to the following elliptic equation equation

$$(-\Delta + V)\mathbb{U} = 0, \tag{6.6}$$

then for the elliptic regularity theory we have $\mathbb{U} \in H^2_{\text{loc}}(\mathbb{R}^3)$. Note that for any $f \in H^1(\mathbb{R}^3)$ having a compact support we can write the relation

$$\int_{\mathbb{R}^3} (\Delta \mathbb{U}(x)) f(x)\, dx = -\int_{\mathbb{R}^3} \langle \nabla \mathbb{U}(x), \nabla f(x) \rangle dx. \tag{6.7}$$

Take any smooth function $\varphi(x)$, such that $\varphi(x) = 1$ for $|x| \leqslant 1$ and the support of φ is contained in the ball of radius 2. Given any real $R > 1$ set

$$f(x) = \varphi\left(\frac{x}{R}\right) \mathbb{U}(x).$$

Then $f \in H^2(\mathbb{R}^3)$ and we can multiply the equation (6.6) by f and integrate over $\mathbb{R}^3$. Then we use (6.7) and obtain

$$\int_{\mathbb{R}^3} (|\nabla \mathbb{U}(x)|^2 + V(x)|\mathbb{U}(x)|^2)\, \varphi\left(\frac{x}{R}\right) dx + \tag{6.8}$$
$$+ \int_{\mathbb{R}^3} \left\langle \nabla \mathbb{U}(x), \nabla \varphi\left(\frac{x}{R}\right)\right\rangle \mathbb{U}(x) \frac{dx}{R} = 0.$$

Since $\varphi\left(\frac{x}{R}\right) = 1$ on $B(0, R)$ and $supp \nabla \varphi\left(\frac{x}{R}\right) \subset B(0, 2R) \backslash B(0, R)$ we have

$$\int_{B(0,2R)} \left(|\nabla \mathbb{U}(x)|^2 + V(x)|\mathbb{U}(x)|^2\right) dx \leqslant \frac{\|\nabla \varphi\|_{L^\infty}}{R} \int_{R<|x|<2R} |\nabla \mathbb{U}|\,|\mathbb{U}|\, dx$$

From Lemma 6.2 we know that

$$|\mathbb{U}(x)| \leqslant \frac{C}{\langle x \rangle}, \qquad |\nabla \mathbb{U}(x)| \leqslant \frac{C}{\langle x \rangle^2}.$$

These estimates show that there exists a real constant $C > 0$, independent of R, such that:

$$\int_{R<|x|<2R} |\nabla \mathbb{U}|\,|\mathbb{U}|\, dx < C$$

then taking the limit for $R \to +\infty$ and applying the Lebesgue convergence Theorem, we arrive at

$$\int_{\mathbb{R}^3} \left(|\nabla \mathbb{U}(x)|^2 + V(x)|\mathbb{U}(x)|^2\right) dx = 0 \tag{6.9}$$

so $\mathbb{U}(x) = 0$.

This completes the proof. □

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References

[1] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 2(2), 1975, 151-218.

[2] P. Alsholm - G. Schmidt, *Spectral and scattering theory for Schrödinger operators*. Arch. Rational Mech. Anal., 40, 1970/1971, 281-311.

[3] J.A. Barcelo - A. Ruiz - L. Vega, *Weighted estimates for the Helmholtz equation and some applications*. J. Funct. Anal., 150(2), 1997, 356-382.

[4] M. BEALS - W. STRAUSS, *L^pestimates for the wave equation with a potential*. Comm. Partial Differential Equations, 18(7-8), 1993, 1365-1397.

[5] M. BEN-ARTZI - A. DEVINATZ, *Resolvent estimates for a sum of tensor products with applications to the spectral theory of differential operators*. J. Analyse Math., 43, 1983/84, 215-250.

[6] N. BURQ - F. PLANCHON - J. STALKER - A. SHADI TAHVILDAR-ZADEH, *Strichartz estimates for the Wave and Schrödinger Equations with the Inverse-Square Potential*. Preprint, 2002.

[7] S. CUCCAGNA, *On the wave equation with a potential*. Comm. Partial Differential Equations, 25(7-8), 2000, 1549-1565.

[8] T. IKEBE, *Eigenfunction expansions associated with the Schroedinger operators and their applications to scattering theory*. Arch. Rational Mech. Anal., 5, 1960, 1-34.

[9] C. KERLER, *Perturbations of the Laplacian with variable coefficients in exterior domains and differentiability of the resolvent*. Asymptot. Anal., 19(3-4), 1999, 209-232.

[10] F. PLANCHON - J. STALKER - A. SHADI TAHVILDAR-ZADEH, *l^p estimates for the wave equation with the inverse-square potential*. Discrete Contin. Dynam. Systems, v. 9, n. 2, 2003, 427-442.

[11] M. REED - B. SIMON, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York 1975.

[12] D. ROBERT, *Autour de l'approximation semi-classique*. Progress in Mathematics, 68, Birkhäuser Boston Inc., Boston, MA, 1987.

[13] J. SHATAH - M. STRUWE, *Geometric wave equations*. Courant Lecture Notes in Mathematics, 2, New York University Courant Institute of Mathematical Sciences, New York 1998.

[14] J. SJÖSTRAND - M. ZWORSKI, *Asymptotic distribution of resonances for convex obstacles*. Acta Math., 183(2), 1999, 191-253.

[15] R.S. STRICHARTZ, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*. Duke Math. J., 44(3), 1977, 705-714.

[16] B.R. VAINBERG, *Asymptotic methods in equations of mathematical physics* (translated from the Russian by E. Primrose). Gordon & Breach Science Publishers, New York 1989.

[17] N. VISCIGLIA, *About the Strichartz estimate and the dispersive estimate*. C.R. Acad. Bulgare Sci., 55(5), 2002, 9-14.

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