Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei Matematica e Applicazioni

Henri Gouin, Tommaso Ruggeri

Hamiltonian principle in the binary mixtures of Euler fluids with applications to the second sound phenomena

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **14** (2003), n.1, p. 69–83.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2003_9_14_1_69_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2003.

Rend. Mat. Acc. Lincei
s. 9, v. 14:69-83 (2003)

Fisica matematica. — *Hamiltonian principle in the binary mixtures of Euler fluids with applications to the second sound phenomena.* Nota (*) di HENRI GOUIN e TOMMASO RUGGERI, presentata dal Socio T. Ruggeri.

ABSTRACT. — In the present paper we compare the theory of mixtures based on Rational Thermomechanics with the one obtained by Hamilton principle. We prove that the two theories coincide in the adiabatic case when the action is constructed with the *intrinsic Lagrangian.* In the complete thermodynamical case we show that we have also coincidence in the case of low temperature when the second sound phenomena arises for superfluid Helium and crystals.

KEY WORDS: Fluid mixtures; Superfluidity; Second sound; Hamiltonian principle.

RIASSUNTO. — *Principio di Hamilton in una miscela di fluidi di Eulero con applicazione ai fenomeni di secondo suono.* Nel presente lavoro noi confrontiamo la teoria delle miscele basata sulla Termomeccanica Razionale con quella ottenuta da un principio di Hamilton. Noi proviamo che le due teorie coincidono nel caso adiabatico quando l'azione è costruita mediante la *Lagrangiana intrinseca.* Nel caso termodinamico completo si dimostra la coincidenza delle due teorie se ci si limita nel range di basse temperature dove i fenomeni di secondo suono sono presenti per l'Elio superfluido e nei cristalli.

1. INTRODUCTION

The first mathematical model of homogeneous mixture of fluids in the context of Rational Thermodynamics was due to Truesdell [1]. The compatibility with the second principle of thermodynamics was well established by Müller in the framework of classical mechanics [2] and by Hutter and Müller in relativity [3].

In the framework of binary mixture of Euler fluids, [4, 5] was able to revisit the well known Landau model of superfluidity [6, 7]. The second sound phenomena in the case of liquid He II is now well explained from a macroscopic point of view. Recently Ruggeri [8] observed that a mixture of two Euler fluids can be regarded as a single heat conducting fluid. This result is advantageous to explain the second sound phenomena of crystals with the same model than for superfluid helium.

A different approach was given by Gavrilyuk *et al.* [9], Gavrilyuk and Gouin [10, 11]. They consider a variational approach to describe two-velocity effects in homogeneous mixtures: a Lagrangian of the system is chosen as a difference of the kinetic energy of the two constituents and a volumic potential which is Galilean invariant depending on the relative velocity of components. The equation of motions of the two components are not in balance form (in fact they are in balance form in Lagrangian variables associated with each component). Nevertheless, the momentum and the energy equations for the total mixture are in the classical balance form.

The present work compares the previous approaches and proves that the two theories coincide in the mechanical case when the Hamiltonian action is constructed with the *intrinsic Lagrangian*, *i.e.* does not depend on the relative velocity. Such is the case with

---

(*) Pervenuta in forma definitiva all'Accademia il 27 giugno 2002.

the Lagrangian considered by Gouin in [12]. In the thermodynamical case we prove also the coincidence in the case of low temperature and we obtain a complete agreement between the two approaches and the superfluid model considered first by Landau.

2. THE BINARY MIXTURES OF EULER FLUIDS

The thermodynamics of a homogeneous mixture of n constituents is well codified as a branch of Extended Thermodynamics [13]. It is based on the metaphysical principles of Truesdell [1] which postulates the same balance laws of a single fluid for simple mixtures.

2.1. *The balance system.*

The equations of balance of mass, momentum and energy of the constituents read as follows

$$\begin{aligned}
&\frac{\partial \varrho_a}{\partial t} + \operatorname{div}(\varrho_a \boldsymbol{v}_a) = \tau_a, \\
&\frac{\partial \varrho_a \boldsymbol{v}_a}{\partial t} + \operatorname{div}(\varrho_a \boldsymbol{v}_a \otimes \boldsymbol{v}_a - \boldsymbol{t}_a) = \boldsymbol{m}_a, \qquad (a = 1, 2, \dots n), \\
&\frac{\partial\left(\frac{1}{2}\varrho_a v_a^2 + \varrho_a \varepsilon_a\right)}{\partial t} + \operatorname{div}\left\{\left(\frac{1}{2}\varrho_a v_a^2 + \varrho_a \varepsilon_a\right)\boldsymbol{v}_a - \boldsymbol{t}_a \boldsymbol{v}_a + \boldsymbol{q}_a\right\} = e_a.
\end{aligned} \tag{1}$$

These equations have the same form as the balance equations for a single body, except for the non-zero right hand sides which represent the production of masses, momenta and energies. These productions are due to interaction between the different constituents. Of course, since the total mass, momentum and energy of the total mixture is conserved, we must have

$$\sum_{a=1}^{n} \tau_a = 0, \qquad \sum_{a=1}^{n} \boldsymbol{m}_a = 0, \qquad \sum_{a=1}^{n} e_a = 0,$$

where $\varrho_a, \boldsymbol{v}_a, \varepsilon_a, \boldsymbol{t}_a, \boldsymbol{q}_a$ are the mass density, velocity, internal energy, stress and heat flux respectively of the a-component of the mixture.

If we sum the equations (1) over all constituents and introduce

$$\text{the density } \varrho = \sum_{a=1}^{n} \varrho_a, \quad \text{the velocity } \boldsymbol{v} = \sum_{a=1}^{n} \frac{\varrho_a}{\varrho} \boldsymbol{v}^a, \tag{2}$$

$$\text{the diffusion velocity } \boldsymbol{u}_a = \boldsymbol{v}_a - \boldsymbol{v}, \tag{3}$$

$$\text{the stress tensor } \boldsymbol{t} = \sum_{a=1}^{n} (\boldsymbol{t}_a - \varrho_a \boldsymbol{u}_a \otimes \boldsymbol{u}_a), \tag{4}$$

$$\text{the intrinsic energy density } \varrho\varepsilon_I = \sum_{a=1}^{n} \varrho_a \varepsilon_a, \tag{5}$$

$$\text{the internal energy density } \varrho\varepsilon = \varrho\varepsilon_I + \frac{1}{2}\sum_{a=1}^{n} \varrho_a u_a^2, \tag{6}$$

$$\text{and the heat flux } \boldsymbol{q} = \sum_{a=1}^{n} \left\{\boldsymbol{q}_a + \varrho_a\left(\varepsilon_a + \frac{1}{2}u_a^2\right)\boldsymbol{u}_a - \boldsymbol{t}_a \boldsymbol{u}_a\right\}, \tag{7}$$

we obtain for the total mixture:

The balance mass

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0, \tag{8}$$

the balance equation of momentum

$$\frac{\partial \varrho \boldsymbol{v}}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{t}) = 0, \tag{9}$$

the balance of energy

$$\frac{\partial\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right)}{\partial t} + \operatorname{div}\left\{\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right)\boldsymbol{v} - \boldsymbol{t}\boldsymbol{v} + \boldsymbol{q}\right\} = 0. \tag{10}$$

Note that equations (8)-(10) have the same form as those for a single fluid. Moreover in equation (10) for the balance of energy we observe that the total kinetic energy is $\frac{1}{2}\varrho v^2$ is not the sum of the kinetic energy of the components. In fact we have

$$\frac{1}{2}\varrho v^2 = \frac{1}{2}\sum_{a=1}^{n} \varrho_a v_a^2 - \frac{1}{2}\sum_{a=1}^{n} \varrho_a u_a^2.$$

By analogy with the intrinsic internal energy we call *intrinsic kinetic energy* the expression

$$E_c = \frac{1}{2}\sum_{a=1}^{n} \varrho_a v_a^2.$$

As we consider a single absolute temperature T, the aim of extended thermodynamics for fluid mixtures is the determination of the $4n+1$ fields:

$$\begin{array}{lll} \text{mass densities} & \varrho_a & \\ \text{velocities} & \boldsymbol{v}_a & (a = 1, 2, \ldots n). \\ \text{temperature} & T & \end{array}$$

To determinate these fields we need an appropriate number of equations. They are based on the equations for each constituent of balance of mass $(1)_1$, momentum $(1)_2$ and conservation of energy of the total mixture (10).

2.2. *The equations of binary mixture of Euler fluids.*

We consider a binary mixture of Euler fluids, *i.e.* fluids that are neither viscous nor heat-conducting:

$$\boldsymbol{q}_a \equiv 0, \qquad \boldsymbol{t}_a = -p_a \boldsymbol{I}, \qquad (a = 1, 2).$$

Instead of the mass and momentum balance laws for the second component, we use the equivalent equations of total conservation for mass and momentum. Therefore, associated with the 9 unknown fields $(\varrho_1, \varrho_2, \boldsymbol{v}_1, \boldsymbol{v}_2, T)$, we have the 9 balance equations:

$$
\begin{aligned}
&\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0\\
&\frac{\partial \varrho_1}{\partial t} + \operatorname{div}(\varrho_1 \boldsymbol{v}_1) = \tau_1\\
&\frac{\partial \varrho \boldsymbol{v}}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{t}) = 0\\
&\frac{\partial \varrho_1 \boldsymbol{v}_1}{\partial t} + \operatorname{div}(\varrho_1 \boldsymbol{v}_1 \otimes \boldsymbol{v}_1 + p_1 \boldsymbol{I}) = \boldsymbol{m}_1\\
&\frac{\partial \left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right)}{\partial t} + \operatorname{div}\left\{\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right)\boldsymbol{v} - \boldsymbol{t}\boldsymbol{v} + \boldsymbol{q}\right\} = 0
\end{aligned}
\tag{11}
$$

with

$$
\begin{aligned}
&\boldsymbol{q} = \sum_{a=1}^{2}\left\{\varrho_a\left(\varepsilon_a + \frac{1}{2}u_a^2\right) + p_a\right\}\boldsymbol{u}_\alpha,\\
&\boldsymbol{t} = -\sum_{a=1}^{2}(p_a \boldsymbol{I} + \varrho_a \boldsymbol{u}_a \otimes \boldsymbol{u}_a),\\
&p = \sum_{a=1}^{2} p_\alpha .
\end{aligned}
\tag{12}
$$

2.3. *The entropy principle and thermodynamical restrictions.*

The compatibility between the system (1) and the entropy principle expresses in the form

$$
\frac{\partial \varrho S}{\partial t} + \operatorname{div}\{\varrho S \boldsymbol{v} + \boldsymbol{\Psi}\} \geqslant 0, \tag{13}
$$

which yields several restrictions on the constitutive equations [13]:

$$
\varrho S = \varrho_1 S_1 + \varrho_2 S_2 \tag{14}
$$

$$
p_1 \equiv p_1(\varrho_1, T);\ p_2 \equiv p_2(\varrho_2, T);\ \varepsilon_1 \equiv \varepsilon_1(\varrho_1, T);\ \varepsilon_2 \equiv \varepsilon_1(\varrho_2, T) \tag{15}
$$

such that

$$
T dS_1 = d\varepsilon_1 - \frac{p_1}{\varrho_1^2} d\varrho_1;\ T dS_2 = d\varepsilon_2 - \frac{p_2}{\varrho_2^2} d\varrho_2 \tag{16}
$$

$$
\boldsymbol{\Psi} = \frac{\boldsymbol{q}}{T} - \frac{1}{T}(\varrho_1 \mu_1 \boldsymbol{u}_1 + \varrho_2 \mu_2 \boldsymbol{u}_2), \tag{17}
$$

where $\mu_a \equiv \varepsilon_a + \dfrac{p_a}{\varrho_a} - TS_a$ is the chemical potential of constituent a.

2.4. *The mixture considered as a single heat conducting fluid.*

Ruggeri [8] proved that it is possible to write the velocities of the two constituents in terms of velocity and heat flux:

$$
\boldsymbol{v}_1 = \boldsymbol{v} + \frac{\alpha}{\varrho_1}\boldsymbol{q}, \qquad \boldsymbol{v}_2 = \boldsymbol{v} - \frac{\alpha}{\varrho_2}\boldsymbol{q}
$$

where

$$\frac{1}{\alpha} = \left(\varepsilon_1 + \frac{p_1}{\varrho_1} + \frac{1}{2}u_1^2\right) - \left(\varepsilon_2 + \frac{p_2}{\varrho_2} + \frac{1}{2}u_2^2\right). \tag{18}$$

Introducing the concentration $c = \dfrac{\varrho_1}{\varrho}$, equations $(11)_2$ and $(11)_4$ can be written in terms of ϱ, c, $\boldsymbol{v}$ and $\boldsymbol{q}$ and the system (11) becomes:

$$\begin{aligned}
&\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0\\
&\frac{\partial(\varrho c)}{\partial t} + \operatorname{div}(\varrho c \boldsymbol{v} + \alpha \boldsymbol{q}) = \tau\\
&\frac{\partial \varrho \boldsymbol{v}}{\partial t} + \operatorname{div}\left(\varrho \boldsymbol{v} \otimes \boldsymbol{v} + p\boldsymbol{I} + \frac{\alpha^2}{\varrho c(1-c)}\boldsymbol{q} \otimes \boldsymbol{q}\right) = 0\\
&\frac{\partial(\varrho c \boldsymbol{v} + \alpha \boldsymbol{q})}{\partial t} + \operatorname{div}\left\{\varrho c \boldsymbol{v} \otimes \boldsymbol{v} + \frac{\alpha^2}{\varrho c}\boldsymbol{q} \otimes \boldsymbol{q} + \alpha(\boldsymbol{v} \otimes \boldsymbol{q} + \boldsymbol{q} \otimes \boldsymbol{v}) + \nu \boldsymbol{I}\right\} = -b\boldsymbol{q}\\
&\frac{\partial\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right)}{\partial t} + \operatorname{div}\left\{\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon + p\right)\boldsymbol{v} + \left(\frac{\alpha^2 \boldsymbol{v}\cdot\boldsymbol{q}}{\varrho c(1-c)} + 1\right)\boldsymbol{q}\right\} = 0.
\end{aligned} \tag{19}$$

To eliminate the index 1, we write as in [8], $\nu = p_1$, $\tau = \tau_1$ and $\boldsymbol{m}_1 = -b\boldsymbol{q}$. In an extended thermodynamic model with 9 fields, the binary mixture can be considered as a single heat conducting fluid with a variable concentration.

Equation of evolution $(19)_4$ is a natural extension of the Cattaneo equation for the heat flux. Thermal inertia term α together with term ν have to be interpreted as new constitutive functions. The advantage of this procedure comes from the fact that the two functions are now understandable in the light of mixture theory: term ν plays the role of one-component pressure while the thermal inertia term α given in (18) is the inverse of the difference between the non-equilibrium enthalpies of the two constituents.

Moreover in [8] was proved that the previous system is symmetric hyperbolic in the main field components and therefore, for well known theorems, the local in time Cauchy problem is well posed in suitable Sobolev space.

2.5. *The superfluidity and second sound.*

Dreyer [4] proved that the Landau theory of superfluidity is a particular case of simple mixtures with the thermodynamical peculiarities:

$$S_s = 0; \qquad \mu_s - \mu_n + \frac{1}{2}(\boldsymbol{v}_s - \boldsymbol{v}_n)^2 = 0, \quad \boldsymbol{m}_s = \tau_s \boldsymbol{v}_s, \tag{20}$$

where the indexes n and s correspond to normal and the superfluid components.

By neglecting the quadratic term in the second equation, in the small diffusion case the two chemical potential μ_s and μ_n must be equal. Consequently, the relation

$\mu_s = \mu_n$ allows to obtain one field variable in terms of the others and it is possible to write

$$\varrho_s \equiv \varrho_s(\varrho, T).$$

In this case equation $(11)_2$ evaluates the mass production value τ_s and the superfluid helium framework becomes a theory with 8 fields (*i.e.* the system is formed by equations $(11)_1$, $(11)_3$, $(11)_4$, $(11)_5$ or equivalently equations $(19)_1$, $(19)_3$, $(19)_4$, $(19)_5$).

The condition $(20)_3$ is the most complex. In fact $(11)_4$ with $(11)_2$ can be rewritten (see [5] for details):

$$\frac{\partial \boldsymbol{v}_s}{\partial t} + \nabla\left(\frac{1}{2}v_s^2 + \mu_s\right) + \operatorname{curl} \boldsymbol{v}_s \times \boldsymbol{v}_s = 0 .$$

This equation is in balance form only when the involutive constraint curl $\boldsymbol{v}_s = 0$ holds. In this case the system (19) coincides with the Landau model [6]:

$$\begin{aligned}
&\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0 ,\\
&\frac{\partial \varrho \boldsymbol{v}}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{t}) = 0 ,\\
&\frac{\partial \boldsymbol{v}_s}{\partial t} + \nabla\left(\frac{1}{2}v_s^2 + \mu_s\right) = 0 ,\\
&\frac{\partial\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right)}{\partial t} + \operatorname{div}\left\{\left(\frac{1}{2}\varrho v^2 + \varrho\varepsilon\right)\boldsymbol{v} - \boldsymbol{t}v + \boldsymbol{q}\right\} = 0 .
\end{aligned} \tag{21}$$

Taking into account (13), (17) and $(20)_1$, the entropy law reduces to the Clausius form:

$$\frac{\partial \varrho S}{\partial t} + \operatorname{div}\left(\varrho S\, \boldsymbol{v} + \frac{\boldsymbol{q}}{T}\right) = 0 , \tag{22}$$

where the heat flux $(12)_1$ is:

$$\boldsymbol{q} = \varrho T S\, \boldsymbol{u}_n + \frac{1}{2}(\varrho_s u_s^2 \boldsymbol{u}_s + \varrho_n u_n^2 \boldsymbol{u}_n) . \tag{23}$$

In the diffusion velocity we neglect the third order terms and we obtain the Landau entropy law for the heat flux [6]. The entropy flux becomes $\varrho S\, \boldsymbol{v}_n$ and the entropy is convected by the normal component

$$\frac{\partial \varrho S}{\partial t} + \operatorname{div}(\varrho S\, \boldsymbol{v}_n) = 0 . \tag{24}$$

To focus on the thermal wave associated with the second sound we consider a rigid body at rest with constant density. For the superfluid component, the system of energy and momentum equations is:

$$\frac{\partial \varrho\varepsilon}{\partial t} + \operatorname{div} \boldsymbol{q} = 0\,,$$

$$\frac{\partial \boldsymbol{v}_s}{\partial t} + \nabla\left(\frac{1}{2} v_s^2 + \mu_s\right) = 0\,,$$

with $\boldsymbol{q} = \varrho T S\, \boldsymbol{v}_n$. Such a system is in the form (19) for a single fluid:

$$\frac{\partial \varrho\varepsilon}{\partial t} + \operatorname{div} \boldsymbol{q} = 0$$

$$\frac{\partial(\alpha \boldsymbol{q})}{\partial t} + \nabla\nu = -b\boldsymbol{q}\,.$$

The system coincides with the one deduced by Ruggeri and coworkers for the model of second sound in crystals [14]. Such a model explains the change of form of the initial square thermal waves both in crystals [14-16] and in the superfluid helium [17].

3. THE HAMILTONIAN PROCEDURE FOR TWO-FLUID MIXTURES

To obtain the equations of motion and energy, the procedure is the following:

Let us suppose that the mixture of two miscible fluids is well described by the two-component velocities $\boldsymbol{v}_1, \boldsymbol{v}_2$, the densities ϱ_1, ϱ_2 and the *intrinsic internal energy* $\beta = \varrho\varepsilon_I$.

The intrinsic internal energy is a Galilean invariant and does not depend on the reference frame. We consider the general case where β depends on ϱ_1, ϱ_2 but also of the relative velocity $\boldsymbol{w} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ through the norm $\omega = |\boldsymbol{v}_1 - \boldsymbol{v}_2|$ [9]. The *intrinsic kinetic energy* is $E_c = \frac{1}{2}(\varrho_1 v_1^2 + \varrho_2 v_2^2)$.

Without dissipative effects, chemical reactions and with conservation of masses of the two components, an extended form of Hamilton principle of least action is used in the form

$$\delta I = 0 \quad \text{with} \quad I = \int_{\mathcal{W}_0} L\, dx\, dt\,,$$

where the Lagrangian is $L = E_c - \beta(\varrho_1, \varrho_2, \omega)$, $\mathcal{W} = [t_0, t_1] \times D$ is a time-space cylinder and the variations must vanish on the boundary of $\mathcal{W}$. The virtual motions of the mixture are defined in [9, 10].

From the variations of Hamilton action, we obtain the equations of motions in the form

$$\frac{\partial \boldsymbol{k}_a}{\partial t} + \operatorname{curl} \boldsymbol{k}_a \times \boldsymbol{v}_a + \nabla\left(\frac{\partial \beta}{\partial \varrho_a} - \frac{1}{2} v_a^2 + \boldsymbol{k}_a \boldsymbol{v}_a\right) = 0 \qquad (a = 1, 2) \tag{25}$$

where

$$\boldsymbol{k}_a = \boldsymbol{v}_a - (-1)^a \frac{1}{\varrho_a} \frac{\partial \beta}{\partial \omega} \frac{\boldsymbol{w}}{\omega}\,.$$

The momentum conservation law is obtained by summing on $a = 1, 2$ equation (25) multiplied by ϱ_a:

$$\frac{\partial(\varrho_1 \boldsymbol{v}_1 + \varrho_2 \boldsymbol{v}_2)}{\partial t} + \nabla\left(\varrho_1 \frac{\partial \beta}{\partial \varrho_1} + \varrho_2 \frac{\partial \beta}{\partial \varrho_2} - \beta\right) + \\ + \operatorname{div}\left(\varrho_1 \boldsymbol{v}_1 \otimes \boldsymbol{v}_1 + \varrho_2 \boldsymbol{v}_2 \otimes \boldsymbol{v}_2 - \frac{\partial \beta}{\partial \omega} \frac{\boldsymbol{w} \otimes \boldsymbol{w}}{\omega}\right) = 0. \tag{26}$$

Additive terms come from the dependance of β in ω and in the mechanical case $\varrho_1 \dfrac{\partial \beta}{\partial \varrho_1} + \varrho_2 \dfrac{\partial \beta}{\partial \varrho_2} - \beta$ represents the total pressure p.

The conservation of energy is obtained by summing on $a = 1, 2$ equation (26) multiplied by $\varrho_a \boldsymbol{v}_a$:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho_1 v_1^2 + \frac{1}{2}\varrho_2 v_1^2 + \beta + \omega \frac{\partial \beta}{\partial \omega}\right) + \\ + \operatorname{div}\left(\varrho_1 \boldsymbol{v}_1 \frac{\partial \beta}{\partial \varrho_1} + \boldsymbol{k}_1 \boldsymbol{v}_1 + \varrho_2 \boldsymbol{v}_2 \frac{\partial W}{\partial \varrho_2} + \boldsymbol{k}_2 \boldsymbol{v}_2\right) = 0. \tag{27}$$

In the following, we consider the case where β is independent of ω and the entropy principle (15) presented in [13] yields $\beta = \varrho_1 \varepsilon_1(\varrho_1) + \varrho_2 \varepsilon_2(\varrho_2)$. Then, equation (25) writes

$$\frac{\partial \boldsymbol{v}_a}{\partial t} + \operatorname{curl} \boldsymbol{v}_a \times \boldsymbol{v}_a + \nabla\left(\frac{1}{2} v_a^2 + \mu_a\right) = 0, \qquad (a = 1, 2). \tag{28}$$

Multiplying equation (28) by ϱ_a straightforward calculations yield equation $(11)_4$ with $\boldsymbol{m}_a = 0$. Equations (26, 27) yield equations $(11)_3$, $(11)_5$ and balance of mass equations correspond to $\tau_a = 0$ $(a = 1, 2)$.

A purely mechanical case is the adiabatic one and we have verified the following results:

In the adiabatic case with intrinsic Lagrangian $L = E_c - \varrho\varepsilon_I$ *difference between the intrinsic kinetic energy and the intrinsic internal energy* $\varrho\varepsilon_I = \varrho_1 \varepsilon_1(\varrho_1) + \varrho_2 \varepsilon_2(\varrho_2)$, *the system deduced from Hamilton principle coincides with the system coming from Rational Thermomechanics.*

4. The Hamiltonian procedure for superfluid helium

In the case of a binary mixture some change must be done in the definition of *virtual motions* presented by Serrin in [18]. Let us consider the motion of Helium II as two diffeomorphisms

$$\boldsymbol{z} = M(\boldsymbol{Z}), \qquad \boldsymbol{z} = M_n(\boldsymbol{Z}_n)$$

where $\boldsymbol{z} = \begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix}$ corresponds to the Eulerian variables in time-space and $\boldsymbol{Z} = \begin{pmatrix} \lambda \\ \boldsymbol{X} \end{pmatrix}$, $\boldsymbol{Z}_n = \begin{pmatrix} \lambda_n \\ \boldsymbol{X}_n \end{pmatrix}$ correspond to the Lagrangian variables associated with the barycentric motion and the normal component motion of Helium II. In coordinate form,

$$M(\boldsymbol{Z}) = \begin{pmatrix} g(\lambda, \boldsymbol{X}) \\ \boldsymbol{\phi}(\lambda, \boldsymbol{X}) \end{pmatrix}, \qquad M_n(\boldsymbol{Z}_n) = \begin{pmatrix} g(\lambda_n, \boldsymbol{X}_n) \\ \boldsymbol{\phi}_n(\lambda_n, \boldsymbol{X}_n) \end{pmatrix}.$$

We consider three *one-parameter families of virtual motions* which are sufficient to obtain the governing equations:

$$(\mathcal{F}) \begin{cases} t = g(\lambda, \boldsymbol{X}) = g_n(\lambda_n, \boldsymbol{X}_n) \\ \boldsymbol{x} = \boldsymbol{\Phi}(\lambda, \boldsymbol{X}, \varepsilon) \\ \boldsymbol{x} = \boldsymbol{\phi}_n(\lambda_n, \boldsymbol{X}_n) \end{cases}$$

with $\boldsymbol{\Phi}(\lambda, \boldsymbol{X}, 0) = \boldsymbol{\phi}(\lambda, \boldsymbol{X})$,

$$(\mathcal{F}_n) \begin{cases} t = g(\lambda, \boldsymbol{X}) = g_n(\lambda_n, \boldsymbol{X}_n) \\ \boldsymbol{x} = \boldsymbol{\phi}(\lambda, \boldsymbol{X}) \\ \boldsymbol{x} = \boldsymbol{\Phi}_n(\lambda_n, \boldsymbol{X}_n, \varepsilon) \end{cases}$$

with $\boldsymbol{\Phi}_n(\lambda_n, \boldsymbol{X}_n, 0) = \boldsymbol{\phi}_n(\lambda_n, \boldsymbol{X}_n)$,

$$(\mathcal{F}_t) \begin{cases} t = G(\lambda, \boldsymbol{X}, \varepsilon) = G_n(\lambda_n, \boldsymbol{X}_n, \varepsilon) \\ \boldsymbol{x} = \boldsymbol{\phi}(\lambda, \boldsymbol{X}) \\ \boldsymbol{x} = \boldsymbol{\phi}_n(\lambda_n, \boldsymbol{X}_n) \end{cases}$$

with $G(\lambda, \boldsymbol{X}, 0) = G_n(\lambda_n, \boldsymbol{X}_n, 0) = g(\lambda, \boldsymbol{X}) = g_n(\lambda_n, \boldsymbol{X}_n)$.

The three families generate the virtual displacements

$$\boldsymbol{\zeta} = \begin{pmatrix} 0 \\ \boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{\partial \boldsymbol{\Phi}}{\partial \varepsilon} \end{pmatrix}\Bigg|_{\varepsilon = 0}, \qquad \boldsymbol{\zeta}_n = \begin{pmatrix} 0 \\ \boldsymbol{\xi}_n \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{\partial \boldsymbol{\Phi}_n}{\partial \varepsilon} \end{pmatrix}\Bigg|_{\varepsilon = 0}, \qquad \boldsymbol{\zeta}_t = \begin{pmatrix} \tau \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial G}{\partial \varepsilon} \\ \boldsymbol{0} \end{pmatrix}\Bigg|_{\varepsilon = 0}.$$

The virtual motion $(\mathcal{F})$ generates an associated displacement $\delta \boldsymbol{Z}_n$ of the normal component. Indeed, the relations

$$g(\lambda, \boldsymbol{X}) = g_n(\lambda_n, \boldsymbol{X}_1) \;\; \boldsymbol{\phi}_n(\lambda, \boldsymbol{X}_n) = \boldsymbol{\Phi}(\lambda, \boldsymbol{X}, \varepsilon)$$

imply

$$\boldsymbol{\zeta} = \begin{pmatrix} \dfrac{\partial g_n}{\partial \lambda_1}, & \dfrac{\partial g_n}{\partial \boldsymbol{X}_n} \\ \dfrac{\partial \boldsymbol{\phi}_n}{\partial \lambda_n}, & \dfrac{\partial \boldsymbol{\phi}_n}{\partial \boldsymbol{X}_n} \end{pmatrix} \delta \boldsymbol{Z}_n.$$

By using the definition of the deformation gradient $\boldsymbol{F}$ proposed in Appendix we get

$$\delta \boldsymbol{Z}_n = \boldsymbol{C}_n \boldsymbol{\zeta} \quad \text{with} \quad \boldsymbol{C}_n = \begin{pmatrix} 0 & , & \boldsymbol{0} \\ -\boldsymbol{F}_n^{-1} \boldsymbol{V}_n & , & \boldsymbol{F}_n^{-1} \end{pmatrix}. \tag{29}$$

In the same way, virtual motion $(\mathcal{F}_n)$ generates an associated displacement $\delta_n \boldsymbol{Z}$ of the barycentric motion

$$\delta_n \boldsymbol{Z} = \boldsymbol{C}\,\boldsymbol{\zeta}_n \qquad \text{with} \qquad \boldsymbol{C} = \begin{pmatrix} 0 & , \boldsymbol{0} \\ -\boldsymbol{F}^{-1}\boldsymbol{V} & ,\boldsymbol{F}^{-1} \end{pmatrix}.$$

Now, $H(\boldsymbol{Z}, \varepsilon)$ notes a perturbation of $h(\boldsymbol{Z})$, the variation of h is

$$\delta h = \frac{\partial H}{\partial \varepsilon}\bigg|_{\varepsilon=0}.$$

We can also introduce Lagrangian variations corresponding to the families $(\mathcal{F}_n)$ and $(\mathcal{F}_t)$:

$$\delta_n h_n = \frac{\partial H_n}{\partial \varepsilon}\bigg|_{\varepsilon=0} \qquad \text{and} \qquad \delta_t h_t = \frac{\partial H_t}{\partial \varepsilon}\bigg|_{\varepsilon=0}.$$

The variations of the entropy S is a main step of our model: we make the physical assumption that the entropy S is defined on the $\boldsymbol{Z}_n$-space. This result corresponds to equation (24) proposed by Landau. Consequently, we deduce $\delta_n S = 0$ and $\delta_t S = 0$. From relation (29) we obtain

$$\delta S = \frac{\partial S}{\partial \boldsymbol{Z}_n}\,\delta \boldsymbol{Z}_n = \frac{\partial S}{\partial \boldsymbol{x}}\,\boldsymbol{\xi}.$$

Following the Hamiltonian procedure presented in paragraph 3, we consider the Lagrangian L as a function of $\varrho, \boldsymbol{v}, \varrho_n, \boldsymbol{v}_n, S$ $(L = L(\varrho, \boldsymbol{v}, \varrho_n, \boldsymbol{v}_n, S))$. Such is the case for the intrinsic Lagrangian $L = \frac{1}{2}(\varrho_n v_n^2 + \varrho_s v_s^2) - \beta(\varrho, \varrho_n, S)$ where ϱ_s and $\boldsymbol{v}_s$ are given by the relations:

$$\varrho_s = \varrho - \varrho_n \qquad \text{and} \qquad \boldsymbol{v}_s = \frac{\varrho \boldsymbol{v} - \varrho_n \boldsymbol{v}_n}{\varrho - \varrho_n}. \tag{30}$$

Consequently,

$$\frac{\partial \boldsymbol{v}_s}{\partial \varrho} = \frac{1}{\varrho_s}(\boldsymbol{v} - \boldsymbol{v}_s), \qquad \frac{\partial \boldsymbol{v}_s}{\partial \varrho_n} = \frac{1}{\varrho_s}(\boldsymbol{v}_s - \boldsymbol{v}_n), \qquad \frac{\partial \boldsymbol{v}_s}{\partial \boldsymbol{v}} = \frac{\varrho}{\varrho_s}\boldsymbol{I}, \qquad \frac{\partial \boldsymbol{v}_s}{\partial \boldsymbol{v}_n} = -\frac{\varrho_n}{\varrho_s}\boldsymbol{I}.$$

The variation of the Hamilton action corresponding to the first family is:

$$\delta I = \int_{\mathcal{W}_0} \delta(L \det \boldsymbol{B})\, dw_0$$

where $\boldsymbol{B} = \dfrac{\partial z}{\partial \boldsymbol{Z}}$ is the Jacobian of M and $\mathcal{W}_0$ is the associated Lagrangian domain in the $\begin{pmatrix}\lambda \\ \boldsymbol{X}\end{pmatrix}$-space. Consequently,

$$\delta I = \int_{\mathcal{W}_0} (\delta L + L\,\mathrm{Div}\,\boldsymbol{\zeta}) \det \boldsymbol{B}\, dw_0.$$

Variations of L come from

$$\delta L = \frac{\partial L}{\partial \boldsymbol{v}}\delta \boldsymbol{v} + \frac{\partial L}{\partial \boldsymbol{v}_n}\delta \boldsymbol{v}_n + \frac{\partial L}{\partial \varrho}\delta \varrho + \frac{\partial L}{\partial \varrho_n}\delta \varrho_n + \frac{\partial L}{\partial S}\delta S.$$

with,

$$
\begin{aligned}
&\frac{\partial L}{\partial \boldsymbol{v}} = \varrho \boldsymbol{v}_s, \qquad \frac{\partial L}{\partial \boldsymbol{v}_n} = \varrho_n(\boldsymbol{v}_n - \boldsymbol{v}_s)\,,\\
&R = \frac{\partial L}{\partial \varrho} = -\frac{1}{2}v_s^2 + \boldsymbol{v}_s\boldsymbol{v} - \beta'_{\varrho_s}(\varrho_n, \varrho_s, S),\\
&R_n = \frac{\partial L}{\partial \varrho_n} = \frac{1}{2}v^2 + \frac{1}{2}v_s^2 - \boldsymbol{v}_s\boldsymbol{v}_n - \beta'_{\varrho_n}(\varrho_n, \varrho_s, S) + \beta'_{\varrho_s}(\varrho_n, \varrho_s, S),\\
&\varrho T = -\frac{\partial L}{\partial S}\,.
\end{aligned}
\tag{31}
$$

Moreover we have,

$$
\delta \boldsymbol{v}_n = \frac{\partial \boldsymbol{v}_n}{\partial \boldsymbol{Z}_n}\,\boldsymbol{C}_n\,\boldsymbol{\zeta} = \frac{\partial \boldsymbol{v}_n}{\partial \boldsymbol{x}}\,\boldsymbol{\xi} \quad \text{and} \quad \delta\varrho_n = \frac{\partial \varrho_n}{\partial \boldsymbol{Z}_n}\,\boldsymbol{C}_n\,\boldsymbol{\zeta} = \frac{\partial \varrho_n}{\partial \boldsymbol{x}}\,\boldsymbol{\xi}\,.
$$

Since $\boldsymbol{\zeta} = \begin{pmatrix} 0 \\ \boldsymbol{\xi} \end{pmatrix}$, we get (see Appendix for the variations $\delta\varrho$ and $\delta\boldsymbol{v}$ variations),

$$
\begin{aligned}
\delta L + L\,\mathrm{Div}\,\boldsymbol{\zeta} &= \frac{\partial L}{\partial \boldsymbol{v}}\,\frac{d\boldsymbol{\xi}}{dt} + \frac{\partial L}{\partial \boldsymbol{v}_n}\,\frac{\partial \boldsymbol{v}_n}{\partial \boldsymbol{x}}\,\boldsymbol{\xi} - \varrho\,\frac{\partial L}{\partial \varrho}\,\mathrm{div}\,\boldsymbol{\xi} + \frac{\partial L}{\partial \varrho_n}\,\frac{\partial \varrho_n}{\partial \boldsymbol{x}}\,\boldsymbol{\xi} + L\,\mathrm{div}\,\boldsymbol{\xi} + \frac{\partial L}{\partial S}\,\frac{\partial S}{\partial \boldsymbol{x}}\,\boldsymbol{\xi} =\\
&= \varrho\boldsymbol{v}_s\,\frac{d\boldsymbol{\xi}}{dt} + \varrho_n(\boldsymbol{v}_n - \boldsymbol{v}_s)\,\frac{\partial \boldsymbol{v}_n}{\partial \boldsymbol{x}}\,\boldsymbol{\xi} - \varrho R\,\mathrm{div}\,\boldsymbol{\xi} + R_n\,\frac{\partial \varrho_n}{\partial \boldsymbol{x}}\,\boldsymbol{\xi} + L\,\mathrm{div}\,\boldsymbol{\xi} + \frac{\partial L}{\partial S}\,\frac{\partial S}{\partial \boldsymbol{x}}\,\boldsymbol{\xi}\,.
\end{aligned}
$$

By using the expression

$$
\varrho\boldsymbol{v}_s\,\frac{d\boldsymbol{\xi}}{dt} = \frac{\partial}{\partial t}(\varrho\boldsymbol{v}_s\,\boldsymbol{\xi}) - \frac{\partial}{\partial t}(\varrho\boldsymbol{v}_s)\,\boldsymbol{\xi} + \mathrm{div}\,(\varrho(\boldsymbol{v}\otimes\boldsymbol{v}_s)\,\boldsymbol{\xi}) - \mathrm{div}\,(\varrho\boldsymbol{v}\otimes\boldsymbol{v}_s)\,\boldsymbol{\xi}
$$

we get

$$
\begin{aligned}
\delta L + L\,\mathrm{div}\,\boldsymbol{\xi} = {}& \frac{\partial}{\partial t}(\varrho\boldsymbol{v}_s\,\boldsymbol{\xi}) - \frac{\partial}{\partial t}(\varrho\boldsymbol{v}_s)\,\boldsymbol{\xi} + \mathrm{div}\,(\varrho(\boldsymbol{v}\otimes\boldsymbol{v}_s)\,\boldsymbol{\xi}) - \mathrm{div}\,(\varrho\boldsymbol{v}\otimes\boldsymbol{v}_s)\,\boldsymbol{\xi} +\\
& + \varrho_n\boldsymbol{v}_n\,\frac{\partial \boldsymbol{v}_n}{\partial \boldsymbol{x}}\,\boldsymbol{\xi} - \mathrm{div}\,(\varrho R\,\boldsymbol{\xi}) + \nabla(\varrho R)\,\boldsymbol{\xi} + R_n\,\frac{\partial \varrho_n}{\partial \boldsymbol{x}}\,\boldsymbol{\xi} + \mathrm{div}\,(L\boldsymbol{\xi}) + \frac{\partial L}{\partial S}\,\nabla S\,\boldsymbol{\xi} -\\
& - \left(\frac{\partial L}{\partial \varrho}\,\nabla\varrho + \frac{\partial L}{\partial \boldsymbol{v}}\,\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} + \frac{\partial L}{\partial \varrho_n}\,\nabla\varrho_n + \frac{\partial L}{\partial \boldsymbol{v}_n}\,\frac{\partial \boldsymbol{v}_n}{\partial \boldsymbol{x}} + \frac{\partial L}{\partial S}\,\nabla S\right)\boldsymbol{\xi}
\end{aligned}
$$

and from equations (31),

$$
\begin{aligned}
\delta L + L\,\mathrm{Div}\,\boldsymbol{\zeta} = {}& \left(-\frac{\partial}{\partial t}(\varrho\boldsymbol{v}_s) - \mathrm{div}\,(\varrho\boldsymbol{v}\otimes\boldsymbol{v}_s) + \nabla(\varrho R) - R\nabla\varrho - \varrho\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)^{*}\boldsymbol{v}_s\right)\boldsymbol{\xi} +\\
& + \frac{\partial}{\partial t}(\varrho\boldsymbol{v}_s\,\boldsymbol{\xi}) + \mathrm{div}\,(\varrho(\boldsymbol{v}\otimes\boldsymbol{v}_s)\,\boldsymbol{\xi}) - \mathrm{div}\,(\varrho R\boldsymbol{\xi}) + \mathrm{div}\,(L\boldsymbol{\xi})\,,
\end{aligned}
$$

where * denotes the transposition. Consequently, the first equation of momentum is

$$\frac{\partial \boldsymbol{v}_s}{\partial t} + \nabla\left(\frac{1}{2}v_s^2 + \beta'_{\varrho_s}\right) = \boldsymbol{v}_s \times \operatorname{curl} \boldsymbol{v}_s . \tag{32}$$

If we note $\mu_s = \beta'_{\varrho_s}$, when $\boldsymbol{v} \approx 0$, equation (32) yields

$$\frac{\partial \boldsymbol{v}_s}{\partial t} + \nabla\left(\frac{1}{2}v_s^2 + \mu_s\right) = 0 \tag{33}$$

which is the Landau equation for the superfluid component. In fact Landau pointed out that Helium II lose its superfluidity when the velocity is not small enough and the supplementary term curl $\boldsymbol{v}_s \times \boldsymbol{v} \approx 0$ corresponds to this experimental evidence.

Variations of the Hamilton action are closely the same for the second family. The variation of the entropy is $\delta_n S = 0$ and consequently an entropy term is now appearing in the equations of motion. The second equation of momentum is

$$\frac{\partial}{\partial t}\left(\varrho_n(\boldsymbol{v}_n - \boldsymbol{v}_s)\right) + \operatorname{div}\left(\varrho_n \boldsymbol{v}_n \otimes (\boldsymbol{v}_n - \boldsymbol{v}_s)\right) + \tag{34}$$

$$+\varrho_n\left(\frac{\partial \boldsymbol{u}_n}{\partial \boldsymbol{x}}\right)^{*}(\boldsymbol{v}_n - \boldsymbol{v}_s) - \varrho_n \nabla R_n - \varrho T \nabla S = 0\ .$$

By summing equations (32) and (34), equation (34) can be replaced by the balance of total momentum:

$$\frac{\partial}{\partial t}\left(\varrho \boldsymbol{v}_s + \varrho_n(\boldsymbol{v}_n - \boldsymbol{v}_s)\right) + \operatorname{div}\left(\varrho \boldsymbol{v} \otimes \boldsymbol{v}_s + \varrho_s \boldsymbol{v}_n \otimes (\boldsymbol{v}_n - \boldsymbol{v}_s) - \varrho\frac{\partial L}{\partial \varrho} - \varrho_n \frac{\partial L}{\partial \varrho_n} + L\right) = 0.$$

Straightforward calculations yield the equation of momentum

$$\frac{\partial \varrho \boldsymbol{v}}{\partial t} + \operatorname{div}\left(\varrho \boldsymbol{v}_n \otimes \boldsymbol{v}_n + \varrho \boldsymbol{v}_s \otimes \boldsymbol{v}_s + p\right) = 0, \tag{35}$$

where $p = \varrho_s \mu_s + \varrho_n \mu_n - \beta$ is the total pressure, with $\mu_n = \beta'_{\varrho_n}$.

Finally, the third family is associated with the vector displacement $\zeta_t = \binom{\tau}{0}$. The variations of basic variables are calculated in Appendix:

$$\delta_t \boldsymbol{v} = -\boldsymbol{v}\frac{d\tau}{dt}, \quad \delta_t \varrho = \varrho\, \nabla\tau\, \boldsymbol{v}, \quad \delta_t \boldsymbol{v}_n = -\boldsymbol{v}_n \frac{d_n \tau}{dt}, \quad \delta_t \varrho_n = \varrho_n \nabla\tau\, \boldsymbol{v}_n, \quad \delta_t S = 0.$$

The variation of the Hamilton action is

$$\delta_t I = \int_{\mathcal{W}_0}\left(\delta_t L + L\frac{\partial \tau}{\partial t}\right)\det \boldsymbol{B}\, dw_0$$

with

$$\delta_t L = \frac{\partial L}{\partial \boldsymbol{v}}\delta_t \boldsymbol{v} + \frac{\partial L}{\partial \boldsymbol{v}_n}\delta_t \boldsymbol{v}_n + \frac{\partial L}{\partial \varrho}\delta_t \varrho + \frac{\partial L}{\partial \varrho_n}\delta_t \varrho_n + \frac{\partial L}{\partial s}\delta_t s\ .$$

Hence,

$$\begin{aligned}
\delta_t L + L\frac{\partial \tau}{\partial t} &= -\varrho \boldsymbol{v}_s \boldsymbol{v}\left(\frac{\partial \tau}{\partial t} + \nabla\tau\, \boldsymbol{v}\right) - \varrho_n (\boldsymbol{v}_n - \boldsymbol{v}_s)\, \boldsymbol{v}_n \left(\frac{\partial \tau}{\partial t} + \nabla\tau\, \boldsymbol{v}_n\right) + \\
&\quad + \varrho R \nabla\tau\, \boldsymbol{v} + \varrho_n R_n \nabla\tau\, \boldsymbol{v}_n + \frac{\partial}{\partial t}(L\tau) - \frac{\partial L}{\partial t}\tau = \\
&\quad = -\frac{\partial}{\partial t}(\varrho \boldsymbol{v}_s \boldsymbol{v}\, \tau) + \frac{\partial}{\partial t}(\varrho \boldsymbol{v}_s \boldsymbol{v})\, \tau - \operatorname{div}(\varrho(\boldsymbol{v}_s \boldsymbol{v})\, \boldsymbol{v}\, \tau) + \operatorname{div}(\varrho(\boldsymbol{v}_s \boldsymbol{v})\, \boldsymbol{v})\, \tau - \\
&\quad - \frac{\partial}{\partial t}(\varrho_n (\boldsymbol{v}_n - \boldsymbol{v}_s)\, \boldsymbol{v}_n \tau) + \frac{\partial}{\partial t}(\varrho_n (\boldsymbol{v}_n - \boldsymbol{v}_s)\, \boldsymbol{v}_n)\, \tau - \operatorname{div}(\varrho_n \boldsymbol{v}_n (\boldsymbol{v}_n - \boldsymbol{v}_s)\, \boldsymbol{v}_n \tau) + \\
&\quad + \operatorname{div}(\varrho_n ((\boldsymbol{v}_n - \boldsymbol{v}_s)\, \boldsymbol{v}_n)\, \boldsymbol{v}_n)\, \tau + \operatorname{div}(\varrho R \boldsymbol{v}\, \tau) - \operatorname{div}(\varrho R\, \boldsymbol{v})\, \tau + \\
&\qquad + \operatorname{div}(\varrho_n R_n \boldsymbol{v}_n \tau) - \operatorname{div}(\varrho_n R_n \boldsymbol{v}_n)\, \tau + \frac{\partial}{\partial t}(L\tau) - \frac{\partial L}{\partial t}\tau .
\end{aligned}$$

Consequently,

$$\frac{\partial}{\partial t}(\varrho \boldsymbol{v}_s \boldsymbol{v} + \varrho_n (\boldsymbol{v}_n - \boldsymbol{v}_s)\, \boldsymbol{v}_n - L) + \operatorname{div}\{(\boldsymbol{v}_s \boldsymbol{v} - R)\, \varrho \boldsymbol{v} + [(\boldsymbol{v}_n - \boldsymbol{v}_s)\, \boldsymbol{v}_n - R_n]\, \varrho_n \boldsymbol{v}_n\} = 0 .$$

If we notice that

$$\varrho \boldsymbol{v}_s \boldsymbol{v} + \varrho_n (\boldsymbol{v}_n - \boldsymbol{v}_s)\, \boldsymbol{v}_n - L = \frac{1}{2}\varrho_n v_n^2 + \frac{1}{2}\varrho_s v_s^2 + \beta = \varrho\varepsilon$$

and

$$\begin{aligned}
&(\boldsymbol{v}_s \boldsymbol{v} - R)\, \varrho \boldsymbol{v} + ((\boldsymbol{v}_n - \boldsymbol{v}_s)\, \boldsymbol{v}_n - R_n)\, \varrho_n \boldsymbol{v}_n = \\
&\qquad = \left(\frac{1}{2}v_s^2 + \beta'_{\varrho_s}\right)\varrho_s \boldsymbol{v}_s + \left(\frac{1}{2}v_n^2 + \beta'_{\varrho_n}\right)\varrho_n \boldsymbol{v}_n = \boldsymbol{q} ,
\end{aligned}$$

we obtain the equation of balance of the total energy in the form:

$$\frac{\partial \varrho\varepsilon}{\partial t} + \operatorname{div} \boldsymbol{q} = 0 . \tag{36}$$

We notice that the specific entropy S does not appear explicitly anymore in equations (32), (35), (36) and we conclude: *In the case of superfluid Helium the* Hamilton *principle yields* the Landau *model.*

5. Appendix. Variation of basic tensorial quantities

Let $(\lambda, \boldsymbol{X})$ be any generalized Lagrangian coordinates and $(t, \boldsymbol{x})$ the associated Eulerian coordinates

$$\begin{cases} t = g(\lambda, \boldsymbol{X}) \\ \boldsymbol{x} = \boldsymbol{\phi}(\lambda, \boldsymbol{X}) . \end{cases} \tag{37}$$

The relation $d\boldsymbol{x} = \boldsymbol{v}\, dt + \boldsymbol{F}\, d\boldsymbol{X}$ defines simultaneously the velocity vector and the deformation gradient of motion (37):

$$\boldsymbol{v} = \frac{\partial \boldsymbol{\phi}}{\partial \lambda}\, \frac{1}{\dfrac{\partial g}{\partial \lambda}}\,, \qquad \boldsymbol{F} = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{X}} - \frac{\partial \boldsymbol{\phi}}{\partial \lambda}\, \frac{\partial g}{\partial \boldsymbol{X}}\, \frac{1}{\dfrac{\partial g}{\partial \lambda}}\,.$$

Let $\begin{cases} t = G(\lambda, \boldsymbol{X}, \varepsilon) \\ \boldsymbol{x} = \boldsymbol{\Phi}(\lambda, \boldsymbol{X}, \varepsilon) \end{cases}$ be a virtual motion. The associated perturbation of the velocity $\boldsymbol{v}$ is given by the formula:

$$\boldsymbol{u} = \frac{\partial \boldsymbol{\Phi}}{\partial \lambda}\, \frac{1}{\dfrac{\partial G}{\partial \lambda}}$$

and consequently,

$$\delta \boldsymbol{v} = \left.\frac{d\boldsymbol{u}}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\partial \boldsymbol{\xi}}{\partial \lambda}\, \frac{1}{\dfrac{\partial g}{\partial \lambda}} - \boldsymbol{v}\, \frac{\partial \tau}{\partial \lambda}\, \frac{1}{\dfrac{\partial g}{\partial \lambda}}\,.$$

For fixed values of Lagrangian coordinates the variation of $\boldsymbol{v}$ in Eulerian coordinates is:

$$\delta \boldsymbol{v} = \frac{d\boldsymbol{\xi}}{dt} - \boldsymbol{v}\, \frac{d\tau}{dt} \qquad \text{where} \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\, \frac{\partial}{\partial \boldsymbol{x}}\,.$$

Analogous calculation for $\boldsymbol{F}$ is:

$$\delta \boldsymbol{F} = \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}} - \boldsymbol{v}\, \frac{\partial \tau}{\partial \boldsymbol{x}} \right) \boldsymbol{F}\,.$$

Moreover, the Euler-Jacobi identity yields

$$\delta \det \boldsymbol{F} = \det \boldsymbol{F}\ \operatorname{tr}(\boldsymbol{F}^{-1}\delta \boldsymbol{F})\,.$$

Hence, the mass conservation law is: $\varrho \det \boldsymbol{F} = \varrho_0(\boldsymbol{X})$ and implies

$$\delta\varrho = -\varrho(\operatorname{div}\boldsymbol{\xi} - \nabla\tau\cdot\boldsymbol{v})\,.$$

Equation $(11)_2$ is the form of the mass balance for the normal component of Helium. If we assume

$$\varrho_n \det \boldsymbol{F}_n = \varrho_{0n}(\lambda_n, \boldsymbol{X}_n)\,,$$

which means that ϱ_n is defined on the Lagrangian space of the normal component, the variation of ϱ_n with respect to δ_n is always in the form:

$$\delta_n \varrho_n = -\varrho_n(\operatorname{div}\boldsymbol{\xi}_n - \nabla\tau\cdot\boldsymbol{v}_n)\,.$$

ACKNOWLEDGEMENTS

This paper was developed during the stay of Henri Gouin as visiting professor in C.I.R.A.M. of the University of Bologna with a fellowship of the Italian GNFM-INDAM and was supported in part (T.R.) by MIUR Progetto di interesse Nazionale *Problemi Matematici Non Lineari di Propagazione e Stabilità nei Modelli del Continuo* Coordinator T. Ruggeri, by the GNFM-INDAM, and by the Istituto Nazionale di Fisica Nucleare (INFN).

REFERENCES

[1] C. TRUESDELL, *Sulle basi della termomeccanica.* Nota II. Atti Acc. Lincei Rend. fis., s. 8, v. 21, 1957, 158-166.

[2] I. MÜLLER, *A new approach to thermodynamics of simple mixtures.* Zeitschrift für Naturforschung, 28a, 1973, 1801.

[3] K. HUTTER - I. MÜLLER, *On mixtures of relativistic fluids.* Helvetica physica Acta, 48, 1975, 675.

[4] W. DREYER, *Zur Thermodynamik von Helium II - Superfluides Helium mit und ohne Wirbellinien als binäre Mischung.* Dissertation Technische Universität, Berlin 1983.

[5] I. MÜLLER, *Thermodynamics.* Pitman, New York 1985.

[6] L. LANDAU - E. LIFCHSITZ, *Mécanique des Fluides.* Mir, Moscow 1971, 418 pp.

[7] S. PUTTERMAN, *Super Fluid Hydrodynamics.* Elsevier, New York 1974.

[8] T. RUGGERI, *The binary mixtures of Euler fluids: A unified theory of second sound phenomena.* In: B. STRAUGHAN - R. GREVE - H. EHRENTRAUT - Y. WANG (eds.), *Continuum Mechanics and Applications in Geophysics and the Environment.* Springer-Verlag, Berlin 2001, 79 pp.

[9] S. GAVRILYUK - H. GOUIN - YU. PEREPECHKO, *Hyperbolic models of homogeneous two-fluid mixtures.* Meccanica, 33, 1998, 161-175.

[10] H. GOUIN - S. GAVRILYUK, *Hamilton's principle and Rankine-Hugoniot conditions for general motions of mixtures.* Meccanica, 34, 1999, 39-47.

[11] S. GAVRILYUK - H. GOUIN, *A new form of governing equations of fluids arising from Hamilton's principle.* Int. J. Eng. Sci., 37 (12), 1999, 1485-1520.

[12] H. GOUIN, *Variational theory of mixtures in continuum mechanics.* Eur. J. Mech. B/Fluids, 9, 1990, 469.

[13] I. MÜLLER - T. RUGGERI, *Rational Extended Thermodynamics.* 2nd ed., Springer Tracts in Natural Philosophy, 37, Springer-Verlag, New York 1998.

[14] T. RUGGERI - A. MURACCHINI - L. SECCIA, *A Continuum Approach to Phonon Gas and Shape Changes of Second Sound via Shock Waves Theory.* Nuovo Cimento D, 16, 1994, 15.

[15] T. RUGGERI - A. MURACCHINI - L. SECCIA, *Shock Waves and Second Sound in a Rigid Heat Conductor: A Critical Temperature for NaF and Bi.* Phys. Rev. Lett., 64, 1990, 2640.

[16] T. RUGGERI - A. MURACCHINI - L. SECCIA, *Second Sound and Characteristic Temperature in Solids.* Phys. Rev. B, 54, 1996, 332-339.

[17] T. RUGGERI - A. MURACCHINI - L. SECCIA, *Second Sound Propagation in Superfluid Helium via Extended Thermodynamics.* Lecture notes WASCOM 2001.

[18] J. SERRIN, *Mathematical principles of classical fluid mechanics.* Encyclopedia of Physics, vol. VIII/I, Springer-Verlag, Berlin 1959, 125-263.

Pervenuta il 15 aprile 2002,
in forma definitiva il 27 giugno 2002.

H. Gouin:
Laboratoire de Modelisation en Mecanique et Thermodynamique
UPRES E.A. 2596, case 322 Avenue Escadrille Normandie-Niemen
13397 MARSEILLE CEDEX 20 (Francia)
henri.gouin@univ.u-3mrs.fr

T. Ruggeri:
Dipartimento di Matematica e
Centro Interdipartimentale di Ricerca per le Applicazioni della Matematica (CIRAM)
Università degli Studi di Bologna
Via Saragozza, 8 - 40123 BOLOGNA
ruggeri@ciram.ing.unibo.it