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On the G-convergence of Morrey operators

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Equazioni a derivate parziali. — *On the G-convergence of Morrey operators.* Nota (*)
di Maria Rosaria Formica e Carlo Sbordone, presentata dal Socio E. Magenes.

Abstract. — Following Morrey [14] we associate to any measurable symmetric 2×2 matrix valued function $A(x)$ such that

$$\frac{|\xi|^2}{K} \leq \langle A(x)\,\xi,\,\xi\rangle \leq K|\xi|^2 \quad \text{a.e.} \quad x \in \Omega,\ \forall \xi \in \mathbb{R}^2,$$

$\Omega \subset \mathbb{R}^2$ and to any $u \in W^{1,2}(\Omega)$ another symmetric 2×2 matrix valued function $\mathcal{A} = \mathcal{A}(A, u)$ with $det\ \mathcal{A} = 1$ and satisfying

$$\frac{|\xi|^2}{K} \leq \langle \mathcal{A}(x)\,\xi,\,\xi\rangle \leq K|\xi|^2 \quad \text{a.e.} \quad x \in \Omega,\ \forall \xi \in \mathbb{R}^2.$$

The crucial property of $\mathcal{A}$ is that $\mathcal{A}\nabla u = A\nabla u$, if $\nabla u \neq 0$. We study the properties of $\mathcal{A}$ as a function of A and u. In particular, we show that, if $A_h \xrightarrow{G} A$, $u_h \rightharpoonup u$, $\nabla u \neq 0$ and $div A_h \nabla u_h = 0$ then $\mathcal{A}(A_h, u_h) \xrightarrow{G} \mathcal{A}(A, u)$

Key words: Elliptic equations; *G*-convergence; Morrey matrices.

Riassunto. — *Sulla G-convergenza degli operatori di Morrey.* Seguendo Morrey [14], ad ogni matrice simmetrica $A(x)$ a coefficienti misurabili, tale che

$$\frac{|\xi|^2}{K} \leq \langle A(x)\,\xi,\,\xi\rangle \leq K|\xi|^2 \quad \text{per q.o.} \quad x \in \Omega,\ \forall \xi \in \mathbb{R}^2,$$

$\Omega \subset \mathbb{R}^2$ e ad ogni $u \in W^{1,2}(\Omega)$ si può associare un'altra matrice simmetrica $\mathcal{A} = \mathcal{A}(A, u)$ con $det\ \mathcal{A} = 1$ e soddisfacente

$$\frac{|\xi|^2}{K} \leq \langle \mathcal{A}(x)\,\xi,\,\xi\rangle \leq K|\xi|^2 \quad \text{per q.o.} \quad x \in \Omega,\ \forall \xi \in \mathbb{R}^2.$$

La principale proprietà di $\mathcal{A}$ è che $\mathcal{A}\nabla u = A\nabla u$, se $\nabla u \neq 0$. Si studiano le proprietà di $\mathcal{A}$ come funzione di A e di u. In particolare, si dimostra che, se $A_h \xrightarrow{G} A$, $u_h \rightharpoonup u$, $\nabla u \neq 0$ e $div A_h \nabla u_h = 0$, allora $\mathcal{A}(A_h, u_h) \xrightarrow{G} \mathcal{A}(A, u)$.

1. Introduction

Let Ω be a simply connected bounded open set in $\mathbb{R}^2$ and $K \geq 1$ a fixed real number. Denote by $\mathcal{E}(K)$ the set of measurable symmetric matrix-valued functions

$$A : \Omega \to \mathbb{R}^{2 \times 2}$$

verifying the ellipticity bounds

$$\frac{|\xi|^2}{K} \leq \langle A(x)\,\xi,\,\xi\rangle \leq K|\xi|^2 \tag{1.1}$$

for a.e. $x \in \Omega$ and $\forall \xi \in \mathbb{R}^2$.

(*) Pervenuta in forma definitiva all'Accademia il 15 luglio 2002.

For $A \in \mathcal{E}(K)$, $A = (a_{ij})$ and $u \in W^{1,2}(\Omega)$ define a new matrix valued function $\mathcal{A} = \mathcal{A}(A, u)$

$$\mathcal{A} : \Omega \to \mathbb{R}^{2\times 2}$$

whose entries (α_{ij}) are given by

$$\alpha_{11}(x) = \frac{u_{x_2}^2 + (a_{11} u_{x_1} + a_{12} u_{x_2})^2}{a_{11} u_{x_1}^2 + 2 a_{12} u_{x_1} u_{x_2} + a_{22} u_{x_2}^2}$$

$$\alpha_{12}(x) = \alpha_{21}(x) = \frac{(a_{11} u_{x_1} + a_{12} u_{x_2})(a_{12} u_{x_1} + a_{22} u_{x_2}) - u_{x_1} u_{x_2}}{a_{11} u_{x_1}^2 + 2 a_{12} u_{x_1} u_{x_2} + a_{22} u_{x_2}^2}, \tag{1.2}$$

$$\alpha_{22}(x) = \frac{(a_{12} u_{x_1} + a_{22} u_{x_2})^2 + u_{x_1}^2}{a_{11} u_{x_1}^2 + 2 a_{12} u_{x_1} u_{x_2} + a_{22} u_{x_2}^2}$$

if $\langle A\nabla u(x), \nabla u(x)\rangle \neq 0$ otherwise, let $\alpha_{11} = \alpha_{22} = 1$, $\alpha_{12} = 0$ (see [14]).

It is interesting to note that $\mathcal{A}$ belongs to $\mathcal{E}(K)$ and enjoys the special property

$$det\, \mathcal{A} = 1 .$$

A main relation between matrices A and $\mathcal{A}(A, u)$ is the following: if $u \in W^{1,2}(\Omega)$, then

$$div\,(A\nabla u) = 0$$

if and only if

$$div\,(\mathcal{A}(A, u)\, \nabla u) = 0 .$$

Actually

$$\mathcal{A}(A, u)\, \nabla u = A\nabla u .$$

(See also [1, 10]). We will call $div\,(\mathcal{A}(A, u)\, \nabla)$ the *Morrey operator* associated to A and u.

A similar device to construct, for a given function u, elliptic systems to which u is a solution was adopted by De Giorgi [2] as clarified by J. Soucek [17], (see also [8, 11-13]) and it revealed useful to give examples of irregular solutions.

In this work our purpose is to illustrate some properties of $\mathcal{A}(A, u)$ as a function of A and u; in particular we prove the following

THEOREM 1. *Let* $A_h \in \mathcal{E}(K)$ *and assume* $A_h \xrightarrow{G} A$. *Let* $u_h, u \in W^{1,2}(\Omega)$ *satisfy* $\nabla u \neq 0$,

$$u_h \rightharpoonup u \quad in \quad W^{1,2}(\Omega)$$

and

$$div\,(A_h \nabla u_h) = 0 .$$

Then

$$\mathcal{A}(A_h, u_h) \xrightarrow{G} \mathcal{A}(A, u).$$

(For related results, see [4, 3, 17]).

Let us recall [18] that, by the definition of G-convergence for a sequence of matrix valued functions $A_h \in \mathcal{E}(K)$, if $A_h \xrightarrow{G} A$ and $u_h \in W^{1,2}(\Omega)$ verify

$$div\,(A_h \nabla u_h) = 0 \tag{1.3}$$

and

$$u_h \rightharpoonup u \quad \text{in} \quad W^{1,2}(\Omega) \tag{1.4}$$

then

$$div\,(A\nabla u) = 0\,.$$

REMARK 1.1. It is well known [15, 20] that from (1.3), (1.4) it follows that

$$A_h \nabla u_h \rightharpoonup A\nabla u \quad \text{in} \quad L^2(\Omega, \mathbb{R}^2).$$

REMARK 1.2. Let us observe that in general the condition $det\,\mathcal{A}_h = 1$ is not preserved under weak convergence of $\mathcal{A}_h$ to $\mathcal{A}$ in $\sigma(L^\infty, L^1)$.

On the contrary G-convergence enjoys this property (see [7] *e.g.*) and so the class of Morrey operators is G-closed.

2. SOME PRELIMINARY RESULTS

Let W be a 2×2 matrix

$$W = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} \tag{2.1}$$

with $det\,W \neq 0$ and set

$$\mathcal{A} = \left[\frac{W^t W}{det\,W} \right]^{-1}. \tag{2.2}$$

We have the following elementary result

LEMMA 2.1. *If W and $\mathcal{A}$ are defined as above, then $\mathcal{A}^t = \mathcal{A}$ and*

$$det\,\mathcal{A} = 1\,.$$

PROOF. It is easy to check that $\mathcal{A}$ has the following expression

$$\mathcal{A} = \frac{1}{det\,W} \begin{pmatrix} w_2^2 + w_4^2 & -(w_1 w_2 + w_3 w_4) \\ -(w_1 w_2 + w_3 w_4) & w_1^2 + w_3^2 \end{pmatrix}.$$

Then it is obvious that $\mathcal{A}^t = \mathcal{A}$ and

$$det\,\mathcal{A} = \frac{(w_1 w_4 - w_2 w_3)^2}{(det\,W)^2} = 1\,.$$

For $u \in W^{1,2}(\Omega)$ and $A \in \mathcal{E}(K)$, define $\mathcal{A} = \mathcal{A}(A, u) = (\alpha_{ij})$ given by (1.2) if $\nabla u \neq 0$. Otherwise we set $\mathcal{A} = (\delta_{ij})$.

PROPOSITION 2.1. *If $A = (a_{ij})$ and $\mathcal{A} = \mathcal{A}(A, u)$ is defined as above, then $\mathcal{A}^t = \mathcal{A}$ and $det\, \mathcal{A} = 1$.*

PROOF. If we set

$$w_1 = u_{x_1}, \qquad w_2 = u_{x_2}$$
$$w_3 = -\sum_j a_{j2} u_{x_j}, \quad w_4 = \sum_j a_{1j} u_{x_j}$$

and

$$W = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$$

then $\langle A\nabla u, \nabla u\rangle \neq 0$ if and only if $det\, W \neq 0$ and we have

$$\mathcal{A} = \left[\frac{W^t W}{det\, W} \right]^{-1}.$$

So, the result follows from Lemma 2.1.

We note the following useful lemmas

LEMMA 2.2. *Let $A, B \in \mathcal{E}(K)$. Assume that $det A = det B > 0$ and*

$$det\,(A - B) = 0\,. \tag{2.3}$$

Then

$$A = B\,.$$

PROOF. Set $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ where we have denoted with $A^{-\frac{1}{2}}$ the inverse matrix of the square root $A^{\frac{1}{2}}$ of matrix A. Then C is a symmetric matrix and obviously $det\, C = (det\, B)/(det\, A) = 1$. Moreover we have

$$A - B = A^{\frac{1}{2}} (I - C)\, A^{\frac{1}{2}} \tag{2.4}$$

and so our assumptions imply

$$0 = det\,(A - B) = (det\, A)^{\frac{1}{2}}\, det\,(I - C)(det\, A)^{\frac{1}{2}},$$

therefore we arrive at

$$0 = det\,(I - C) = 1 - tr\, C + det\, C = 2 - tr\, C. \tag{2.5}$$

If we indicate the entries of C as

$$C = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

we obtain explicitely

$$(a + c)^2 = (tr\, C)^2 = 4 = 4\; det\, C = 4ac - 4b^2$$

and so

$$(a - c)^2 = -4b^2$$

that is $a = c$, $b = 0$. Since by (2.5) $tr\, C = 2$ we deduce $a = c = 1$.

For the sake of completeness we give also the following

LEMMA 2.3 [1]. *Let* $E = (E_1, E_2)$, $F = (F_1, F_2)$ *be two vectors of* $\mathbb{R}^2$ *such that* $\langle E, F \rangle > 0$. *Then there exists a unique matrix* $\mathcal{A} \in \mathcal{E}(K)$ *with* $det\, \mathcal{A} = 1$ *such that*

$$\mathcal{A} E = F . \tag{2.6}$$

PROOF. The definition of $\mathcal{A}$ is suggested by (1.2) choose

$$\mathcal{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}$$

with

$$\alpha_{11} = \frac{E_2^2 + F_1^2}{\langle E, F \rangle}, \qquad \alpha_{12} = \frac{F_1 F_2 - E_1 E_2}{\langle E, F \rangle}, \qquad \alpha_{22} = \frac{F_2^2 + E_1^2}{\langle E, F \rangle}.$$

Then it is immediate to check that $\mathcal{A} \in \mathcal{E}(K)$, $det\, \mathcal{A} = 1$ and that (2.6) holds. Assume that $\mathcal{B} \in \mathcal{E}(K)$ is another matrix satisfying $det\, \mathcal{B} = 1$ and

$$\mathcal{A} E = F = \mathcal{B} E.$$

Then E is a non zero vector such that

$$(\mathcal{A} - \mathcal{B})\, E = 0;$$

so $det\, (\mathcal{A} - \mathcal{B}) = 0$ and by previous lemma we deduce $\mathcal{A} = \mathcal{B}$.

PROPOSITION 2.2. *Let* $A \in \mathcal{E}(K)$ *and* $u \in W^{1,2}(\Omega)$, *with* $\nabla u \neq 0$, *then the following properties hold true.*

(*i*) $\mathcal{A}(A, u) = A$ *if and only if* $det A = 1$.
(*ii*) $\mathcal{A}(A, u) \in \mathcal{E}(K)$.
(*iii*) $\mathcal{A}(A, u)\, \nabla u = A \nabla u$ *a.e in* Ω.
(*iv*) $B \in \mathcal{E}(K)$, $det B = 1$, $B \nabla u = A \nabla u \Rightarrow B = \mathcal{A}(A, u)$.

PROOF OF (*i*). If $\mathcal{A}(A, u) = A$, Proposition 2.1 implies $det\, A(x) = det\, \mathcal{A}(x) = 1$.

Conversely, if $det\, A(x) = 1$, we have $a_{11} a_{22} - a_{12}^2 = 1$, so by replacing $a_{11} a_{22} = 1 + a_{12}^2$ into the expression of α_{ij}, $i, j = 1, 2$, it is easily seen that

$$\alpha_{ij} = a_{ij}.$$

PROOF OF (*ii*). This property shows that passing from A to $\mathcal{A}(A, u)$ the coercivity and boundness constants are exactly preserved.

We use the fact that we already know by Proposition 2 that $det\, \mathcal{A} = 1$. As a consequence the eigenvalues of $\mathcal{A}$ are $\lambda_1 \geq 1 \geq \lambda_2 = \frac{1}{\lambda_1}$. We want to prove that $\frac{1}{K} \leq \lambda_i \leq K$ and this is equivalent to $\lambda_1 + \frac{1}{\lambda_1} \leq K + \frac{1}{K}$.

Recall that

$$\lambda_1 + \frac{1}{\lambda_1} = tr\, \mathcal{A}$$

and that by the definition (1.2)

$$tr\,\mathcal{A} = \frac{u_{x_1}^2 + u_{x_2}^2 + (a_{11}\,u_{x_1} + a_{12}\,u_{x_2})^2 + (a_{12}\,u_{x_1} + a_{22}\,u_{x_2})^2}{\langle A\nabla u,\, \nabla u\rangle}\,.$$

By $A \in \mathcal{E}(K)$ we deduce (see [10] *e.g.*)

$$|\xi|^2 + |A\xi|^2 \leqslant \left(K + \frac{1}{K}\right)\langle A\xi,\, \xi\rangle \quad \forall\, \xi \in \mathbb{R}^2.$$

Choosing $\xi = \nabla u$ we obtain $tr\,\mathcal{A} \leqslant K + \dfrac{1}{K}$.

PROOF OF (*iii*). We can assume $\langle A\nabla u,\, \nabla u\rangle \neq 0$, otherwise (*iii*) is obvious. So, we have to prove that

$$\begin{cases} a_{11}\,u_{x_1} + a_{12}\,u_{x_2} = \alpha_{11}\,u_{x_1} + \alpha_{12}\,u_{x_2} \\ a_{12}\,u_{x_1} + a_{22}\,u_{x_2} = \alpha_{12}\,u_{x_1} + \alpha_{22}\,u_{x_2}\,. \end{cases}$$

By the definition of $\mathcal{A} = \mathcal{A}(A,\, u)$, we have

$$\begin{aligned} \alpha_{11}\,u_{x_1} + \alpha_{12}\,u_{x_2} &= \frac{u_{x_2}^2 + a_{11}^2\,u_{x_1}^2 + 2\,a_{11}\,a_{12}\,u_{x_1}\,u_{x_2} + a_{12}^2\,u_{x_2}^2}{\langle A\nabla u,\, \nabla u\rangle}\,u_{x_1} + \\ &+ \frac{-\,u_{x_1}\,u_{x_2} + a_{11}\,a_{12}\,u_{x_1}^2 + a_{11}\,a_{22}\,u_{x_1}\,u_{x_2} + a_{12}^2\,u_{x_1}\,u_{x_2} + a_{12}\,a_{22}\,u_{x_2}^2}{\langle A\nabla u,\, \nabla u\rangle}\,u_{x_2} = \\ &= \frac{1}{\langle A\nabla u,\, \nabla u\rangle}\{a_{11}\,u_{x_1}[a_{11}\,u_{x_1}^2 + 2\,a_{12}\,u_{x_1}\,u_{x_2} + a_{22}\,u_{x_2}^2] + \\ &+ a_{12}\,u_{x_2}[a_{11}\,u_{x_1}^2 + 2\,a_{12}\,u_{x_1}\,u_{x_2} + a_{22}\,u_{x_2}^2]\} = a_{11}\,u_{x_1} + a_{12}\,u_{x_2}\,. \end{aligned}$$

Similarly we have also

$$\begin{aligned} \alpha_{12}\,u_{x_1} + \alpha_{22}\,u_{x_2} &= \frac{-\,u_{x_1}\,u_{x_2} + a_{11}\,a_{12}\,u_{x_1}^2 + a_{11}\,a_{22}\,u_{x_1}\,u_{x_2} + a_{12}^2\,u_{x_1}\,u_{x_2} + a_{12}\,a_{22}\,u_{x_2}^2}{\langle A\nabla u,\, \nabla u\rangle}\,u_{x_1} + \\ &+ \frac{u_{x_1}^2 + a_{12}^2\,u_{x_1}^2 + 2\,a_{12}\,a_{22}\,u_{x_1}\,u_{x_2} + a_{22}^2\,u_{x_2}^2}{\langle A\nabla u,\, \nabla u\rangle}\,u_{x_2} = \\ &= \frac{1}{\langle A\nabla u,\, \nabla u\rangle}\{a_{12}\,u_{x_1}[a_{11}\,u_{x_1}^2 + 2\,a_{12}\,u_{x_1}\,u_{x_2} + a_{22}\,u_{x_2}^2] + \\ &+ a_{22}\,u_{x_2}[a_{11}\,u_{x_1}^2 + 2\,a_{12}\,u_{x_1}\,u_{x_2} + a_{22}\,u_{x_2}^2]\} = a_{12}\,u_{x_1} + a_{22}\,u_{x_2}\,. \end{aligned}$$

PROOF OF (*iv*). It is sufficient to note that according to Lemma 2.3 the equality $det\,B = 1$ uniquely defines B under the specified conditions.

For later purpose it is worth recalling that, if Ω is simply connected and $u \in W^{1,2}(\Omega)$ is a weak solution to

$$div\, A(x)\, \nabla u = 0$$

with $A \in \mathcal{E}(K)$, then there exists, and it is uniquely determined up to an additive constant, a function $v \in W^{1,2}(\Omega)$ such that

$$\begin{cases} v_{x_1} = -(a_{12}\, u_{x_1} + a_{22}\, u_{x_2}) \\ v_{x_2} = a_{11}\, u_{x_1} + a_{12}\, u_{x_2} \end{cases} \tag{2.7}$$

and we have

$$div\left(\frac{A}{det\, A}\, \nabla v\right) = 0\,.$$

The following proposition is of particular relevance.

PROPOSITION 2.3. *Let* $A = (a_{ij}) \in \mathcal{E}(K)$, $u \in W^{1,2}(\Omega)$ *and* $v \in W^{1,2}(\Omega)$ *verify* (2.7); *then*

$$\mathcal{A}(A, u) = \mathcal{A}\left(\frac{A}{det\, A}\,, v\right). \tag{2.8}$$

PROOF. Let us show, first of all, that if u and v satisfy (2.7) then their corresponding energies coincide:

$$\langle A \nabla u, \nabla u\rangle = \left\langle \frac{A}{det\, A} \nabla v, \nabla v\right\rangle. \tag{2.9}$$

If R denotes the standard complex structure of $\mathbb{R}^2$:

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and R^t its adjoint, system (2.7) takes the form

$$\nabla v = RA \nabla u$$

and it is equivalent to

$$\nabla u = R^t \frac{A}{det\, A} \nabla v\,.$$

Now the proof of (2.9) follows by mean of the elementary equality

$$\langle E, R^t F\rangle = \langle F, RE\rangle$$

valid for E, F arbitrary vectors of $\mathbb{R}^2$.

For notational simplicity set now $B = \dfrac{A}{\det A} = (b_{ij})$ and $\mathcal{A}\left(\dfrac{A}{\det A}, v\right) = (\beta_{ij})$. By the definition (1.2) β_{ij} assume the following expressions

$$\beta_{11} = \frac{(1 + b_{12}^2)\, v_{x_2}^2 + b_{11}^2 v_{x_1}^2 + 2 b_{11} b_{12} v_{x_1} v_{x_2}}{\langle B\nabla v, \nabla v\rangle},$$

$$\beta_{12} = \beta_{21} = \frac{b_{11} b_{12} v_{x_1}^2 + (b_{11} b_{22} + b_{12}^2 - 1)\, v_{x_1} v_{x_2} + b_{12} b_{22} v_{x_2}^2}{\langle B\nabla v, \nabla v\rangle},$$

$$\beta_{22} = \frac{(1 + b_{12}^2)\, v_{x_1}^2 + 2 b_{12} b_{22} v_{x_1} v_{x_2} + b_{22}^2 v_{x_2}^2}{\langle B\nabla v, \nabla v\rangle}.$$

By mean of (2.7), (2.9) we have

$$\beta_{11} = \frac{((\det A)^2 + a_{12}^2)\, v_{x_2}^2 + a_{11}^2 v_{x_1}^2 + 2 a_{11} a_{12} v_{x_1} v_{x_2}}{(\det A)^2 \langle A\nabla u, \nabla u\rangle} =$$

$$= \frac{1}{D}\Big[A_{11}^{(11)} u_{x_1}^2 + 2 A_{12}^{(11)} u_{x_1} u_{x_2} + A_{22}^{(11)} u_{x_2}^2\Big]$$

where

$$D = (\det A)^2 \langle A\nabla u, \nabla u\rangle$$

$$A_{11}^{(11)} = [(\det A)^2 + a_{12}^2]\, a_{11}^2 + a_{11}^2 a_{12}^2 - 2 a_{11}^2 a_{12}^2 = (\det A)^2 a_{11}^2$$

$$A_{12}^{(11)} = [(\det A)^2 + a_{12}^2]\, a_{11} a_{12} + a_{11}^2 a_{12} a_{22} - a_{11} a_{12} (a_{12}^2 + a_{11} a_{22}) = (\det A)^2 a_{11} a_{12}$$

$$A_{22}^{(11)} = [(\det A)^2 + a_{12}^2]\, a_{12}^2 + a_{11}^2 a_{22}^2 - 2 a_{11} a_{12}^2 a_{22} = (\det A)^2 (1 + a_{12}^2).$$

Then

$$\beta_{11} = \frac{1}{D} (\det A)^2 [a_{11}^2 u_{x_1}^2 + 2 a_{11} a_{12} u_{x_1} u_{x_2} + (1 + a_{12}^2) u_{x_2}^2] = \alpha_{11}.$$

Similarly we have

$$\beta_{12} = \beta_{21} = \frac{a_{11} a_{12} v_{x_1}^2 + (a_{11} a_{22} + a_{12}^2 - (\det A)^2)\, v_{x_1} v_{x_2} + a_{12} a_{22} v_{x_2}^2}{(\det A)^2 \langle A\nabla u, \nabla u\rangle} =$$

$$= \frac{1}{D}\Big[A_{11}^{(12)} u_{x_1}^2 + A_{12}^{(12)} u_{x_1} u_{x_2} + A_{22}^{(12)} u_{x_2}^2\Big]$$

where

$$A_{11}^{(12)} = a_{11}\,a_{12}\,[a_{12}^2 - (a_{11}\,a_{22} + a_{12}^2 - (\det A)^2) + a_{11}\,a_{22}] = (\det A)^2\,a_{11}\,a_{12}$$

$$A_{12}^{(12)} = 2\,a_{11}\,a_{12}^2\,a_{22} - (a_{11}\,a_{22} + a_{12}^2 - (\det A)^2)(a_{11}\,a_{22} + a_{12}^2) + 2\,a_{11}\,a_{12}^2\,a_{22} = \\ = (\det A)^2(a_{11}\,a_{22} + a_{12}^2 - 1)$$

$$A_{22}^{(12)} = a_{12}\,a_{22}\,[a_{11}\,a_{22} - (a_{11}\,a_{22} + a_{12}^2 - (\det A)^2) + a_{12}^2] = (\det A)^2\,a_{12}\,a_{22}.$$

Then

$$\beta_{12} = \beta_{21} = \frac{1}{D}(\det A)^2[a_{11}\,a_{12}\,u_{x_1}^2 + (a_{11}\,a_{22} + a_{12}^2 - 1)\,u_{x_1}u_{x_2} + a_{11}\,a_{22}\,u_{x_2}^2] = \alpha_{12} = \alpha_{21}.$$

Finally we also have

$$\beta_{22} = \frac{(a_{12}^2 + (\det A)^2)\,v_{x_1}^2 + 2\,a_{12}\,a_{22}\,v_{x_1}v_{x_2} + a_{22}^2\,v_{x_2}^2}{(\det A)^2\langle A\nabla u,\,\nabla u\rangle} = \\ = \frac{1}{D}\Big[A_{11}^{(22)}\,u_{x_1}^2 + 2A_{12}^{(22)}\,u_{x_1}u_{x_2} + A_{22}^{(22)}\,u_{x_2}^2\Big]$$

where

$$A_{11}^{(22)} = (a_{12}^2 + (\det A)^2)\,a_{12}^2 - 2\,a_{11}\,a_{12}^2\,a_{22} + a_{11}^2\,a_{22}^2 = (\det A)^2(a_{12}^2 + 1)$$

$$A_{12}^{(22)} = (a_{12}^2 + (\det A)^2)\,a_{12}\,a_{22} - a_{12}\,a_{22}(a_{11}\,a_{22} + a_{12}^2) + a_{11}\,a_{12}\,a_{22}^2 = (\det A)^2\,a_{12}\,a_{22}$$

$$A_{22}^{(22)} = (a^2 + (\det A)^2)\,a_{22}^2 - 2\,a_{12}^2\,a_{22}^2 + a_{12}^2\,a_{22}^2 = (\det A)^2\,a_{22}^2.$$

Then

$$\beta_{22} = \frac{1}{D}(\det A)^2[(a_{12}^2 + 1)\,u_{x_1}^2 + 2\,a_{11}\,a_{22}\,u_{x_1}u_{x_2} + a_{22}^2\,u_{x_2}^2] = \alpha_{22}.$$

The following proposition indicates in particular that every constant matrix with determinant one can be obtained as the Morrey matrix of an isotropic matrix.

PROPOSITION 2.4. *For any $\mathcal{A} = \mathcal{A}(x) \in \mathcal{E}(K)$ such that $\det \mathcal{A} = 1$, if $\mathcal{A}(x)$ has an eigenvector independent of x, then there exists $A = A(x) \in \mathcal{E}(K)$ of the isotropic form*

$$A(x) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

such that

$$\mathcal{A}(A,\,u) = \mathcal{A}$$

with $u = \lambda_1 x_1 + \lambda_2 x_2$, for λ_1 and λ_2 suitable.

Proof. According to an observation in [1] a matrix $\mathcal{A} \in \mathcal{E}(K)$ has determinant one if and only if there exist $a = a(x) \in L^\infty$ with $aI \in \mathcal{E}(K)$ and $e = e(x) \in L^\infty(\Omega; \mathbb{R}^2)$ with $|e(x)|^2 = 1$ a.e. such that

$$\mathcal{A}(x) = a(x)\, I + \left(\frac{1}{a(x)} - a(x)\right) e(x) \otimes e(x). \tag{2.10}$$

On the other hand, choosing $u(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2$, $a_{ij}(x) = a(x)\, \delta_{ij}$ in the definition (1.2) we obtain the entries $\alpha_{ij}(x)$ of $\mathcal{A}(A, u)$ as

$$\alpha_{11} = a(x)\, \theta + \frac{1}{a(x)}(1-\theta)$$

$$\alpha_{12} = \left(a(x) - \frac{1}{a(x)}\right)\sqrt{\theta(1-\theta)}$$

$$\alpha_{22} = \frac{1}{a(x)}\theta + a(x)(1-\theta)$$

with $\theta = \dfrac{\lambda_1^2}{\lambda_1^2 + \lambda_2^2} \in [0, 1]$. If we define $e_1 = \sqrt{1-\theta}$, $e_2 = \sqrt{\theta}$ we obtain

$$\mathcal{A}(A, u) = \mathcal{A}(x).$$

Let us point out here that the result of Proposition 2.4 cannot be true for a general matrix $\mathcal{A} \in \mathcal{E}(K)$ with $\det \mathcal{A} = 1$ and arbitrary u.

Proposition 2.5. *Let $\Omega = \{|x| < 1\}$ and $\mathcal{A} \in \mathcal{E}(K)$ be the matrix (of Serrin)*

$$\mathcal{A}(x) = \frac{1}{K} I + \left(K - \frac{1}{K}\right)\frac{x}{|x|} \otimes \frac{x}{|x|}. \tag{2.11}$$

Let $u \in W^{1,2}_{loc}(\Omega)$ be any local solution to the equation

$$\operatorname{div}(\mathcal{A}(x)\, \nabla u) = 0 \tag{2.12}$$

such that $\nabla u \neq 0$ and

$$\int_{|x|=1} u\, x_1\, ds \neq 0. \tag{2.13}$$

Then no isotropic matrix $A \in \mathcal{E}(K)$, $A(x) = a(x) I$ can satisfy the equality

$$\mathcal{A}(A, u) = \mathcal{A}. \tag{2.14}$$

Proof. Fix $u \in W^{1,2}_{loc}(\Omega)$ verifying (2.12), (2.13). Assume, by contradiction, that there exists an isotropic matrix $A = a(x)\, I$ such that (2.14) holds for a certain u. Then, by (*iii*) of Proposition 2.2 we obtain

$$\operatorname{div}(a(x)\, I \nabla u) = 0.$$

By a theorem on the precise Hölder continuity of solutions to planar elliptic equations of isotropic type due to Piccinini and Spagnolo [16], u should enjoy the regularity

$$u \in C^{0,\alpha}_{loc}(\Omega)$$

with $\alpha = \frac{4}{\pi} Arctan \frac{1}{K} > \frac{1}{K}$.

On the contrary, by a result of [9] any such solution u belongs to $C^{0,\frac{1}{K}}_{loc}$ but not to $C^{0,\beta}_{loc}$ for any $\beta > \frac{1}{K}$. This concludes the proof.

We conclude the present section with the following

PROPOSITION 2.6. *Let $A, B \in \mathcal{E}(K)$ and $u \in W^{1,2}(\Omega)$ with $\nabla u \neq 0$. Then $\mathcal{A}(A, u) = \mathcal{A}(B, u)$ if and only if $det(A - B) = 0$.*

PROOF. It is an immediate consequence of (*iii*) of Proposition 2.2.

3. THE G-CONVERGENCE

Let us recall the definition of G-convergence. A sequence A_h of elements of $\mathcal{E}(K)$ G-converges to an element A of $\mathcal{E}(K)$ if for any $f \in H^{-1}(\Omega)$ the solutions u_h of

$$\begin{cases} -div(A_h \nabla u_h) = f \\ u_h \in H^1_0(\Omega) \end{cases}$$

converge weakly in $H^1_0(\Omega)$ to the solution u of

$$\begin{cases} -div(A \nabla u) = f \\ u \in H^1_0(\Omega). \end{cases}$$

Now we can pass to the proof of Theorem 1.

PROOF OF THEOREM 1. By a well known [19] compactness theorem with respect to G-convergence we may assume that the sequence $\mathcal{A}_h = \mathcal{A}(A_h, u_h)$ G-converges to $\mathcal{A}_0 \in \mathcal{E}(K)$.

Let us first show that $det\, \mathcal{A}_0 = 1$. This is a particular case of a result of [7] but we present here a simple proof of it for the sake of completeness (see also [1]).

Let us define $v_h \in W^{1,2}(\Omega)$ as a solution to the system

$$\nabla v_h = R A_h \nabla u_h = R \mathcal{A}_h \nabla u_h. \tag{3.1}$$

Then note that, by the condition $det\, \mathcal{A}_h = 1$, (3.1) is equivalent to

$$\nabla u_h = R^t \mathcal{A}_h \nabla v_h. \tag{3.2}$$

By Remark 1.1, using the G-convergence of $\mathcal{A}_h$ to $\mathcal{A}_0$ and of A_h to A we infer the weak convergence in $L^2(\Omega, \mathbb{R}^2)$:

$$\mathcal{A}_h \nabla u_h = A_h \nabla u_h \rightharpoonup \mathcal{A}_0 \nabla u = A \nabla u. \tag{3.3}$$

Then by (3.1) and (3.3) we obtain $\nabla v_h \rightharpoonup \nabla v$ where

$$\nabla v = R\,\mathcal{A}_0\,\nabla u \tag{3.4}$$

and $\nabla v \neq 0$ a.e.

On the other hand (3.1) implies

$$div\ \mathcal{A}_h \nabla v_h = 0 \tag{3.5}$$

and by Remark 1.1 we deduce

$$\mathcal{A}_h \nabla v_h \rightharpoonup \mathcal{A}_0 \nabla v. \tag{3.6}$$

Passing to the limit in (3.2) we obtain

$$\nabla u = R^t \mathcal{A}_0 \nabla v. \tag{3.7}$$

We want to prove that (3.4) and (3.7) together imply $det\ \mathcal{A}_0 = 1$.

Solving (3.4) with respect to ∇u we obtain

$$\nabla u = R^t \frac{\mathcal{A}_0}{det\ \mathcal{A}_0} \nabla v\ . \tag{3.8}$$

By (3.7) and (3.8) we deduce

$$det\left(\mathcal{A}_0 - \frac{\mathcal{A}_0}{det\ \mathcal{A}_0}\right) = 0$$

and this forces A_0 to have determinant equal to one, as one can immediately verify. So we have proved that $det\ \mathcal{A}_0 = 1$. In conclusion, the two matrices $\mathcal{A} = \mathcal{A}(A, u)$ and $\mathcal{A}_0$ have determinant one and satisfy

$$\mathcal{A}_0 \nabla u = A \nabla u = \mathcal{A} \nabla u.$$

This implies $\mathcal{A}_0 = \mathcal{A}$, since $\nabla u \neq 0$ a.e., by Proposition 2.2.

4. Change of variable properties of Morrey operators

In this section we will prove some results which clarify the role of $\mathcal{A}(A, u)$ with respect to change of variables.

Let u, v be a solution to the system

$$\begin{cases} v_{x_1} = -(a_{12} u_{x_1} + a_{22} u_{x_2}) \\ v_{x_2} = a_{11} u_{x_1} + a_{12} u_{x_2} \end{cases} \tag{4.1}$$

and let (ξ, η) be a solution to the system

$$\begin{cases} \eta_{x_1} = -(\alpha_{12} \xi_{x_1} + \alpha_{22} \xi_{x_2}) \\ \eta_{x_2} = \alpha_{11} \xi_{x_1} + \alpha_{12} \xi_{x_2} \end{cases} \tag{4.2}$$

where $(\alpha_{ij}) = \mathcal{A}(A, u)$, $A = (a_{ij})$.

THEOREM 4.1. *If* (u, v) *and* (ξ, η) *are defined as above and there exist U and V such that*

$$U(\xi(x, y), \eta(x, y)) = u(x, y) \tag{4.3}$$

$$V(\xi(x, y), \eta(x, y)) = v(x, y), \tag{4.4}$$

then V and U are conjugate harmonic functions:

$$\begin{cases} V_\xi = -U_\eta \\ V_\eta = U_\xi . \end{cases} \tag{4.5}$$

PROOF. Differentiating (4.3), (4.4), we get by *(iii)* in Proposition 2.2

$$V_\xi \xi_{x_1} + V_\eta \eta_{x_1} = -\alpha_{12}(U_\xi \xi_{x_1} + U_\eta \eta_{x_1}) - \alpha_{22}(U_\xi \xi_{x_2} + U_\eta \eta_{x_2})$$

and

$$V_\xi \xi_{x_2} + V_\eta \eta_{x_2} = \alpha_{11}(U_\xi \xi_{x_1} + U_\eta \eta_{x_1}) + \alpha_{12}(U_\xi \xi_{x_2} + U_\eta \eta_{x_2}).$$

Solving with respect to V_ξ, V_η we obtain

$$V_\xi = -\frac{1}{J}[(\alpha_{12}\xi_{x_1}\eta_{x_2} + \alpha_{22}\xi_{x_2}\eta_{x_2} + \alpha_{11}\xi_{x_1}\eta_{x_1} + \alpha_{12}\xi_{x_2}\eta_{x_1})\, U_\xi + \\ + (\alpha_{12}\eta_{x_1}\eta_{x_2} + \alpha_{22}\eta_{x_2}^2 + \alpha_{11}\eta_{x_1}^2 + \alpha_{12}\eta_{x_1}\eta_{x_2})\, U_\eta]$$

$$V_\eta = \frac{1}{J}[(\alpha_{11}\xi_{x_1}^2 + \alpha_{12}\xi_{x_1}\xi_{x_2} + \alpha_{12}\xi_{x_1}\xi_{x_2} + \alpha_{22}\xi_{x_2}^2)\, U_\xi + \\ + (\alpha_{11}\xi_{x_1}\eta_{x_1} + \alpha_{12}\xi_{x_1}\eta_{x_2} + \alpha_{12}\eta_{x_1}\xi_{x_2} + \alpha_{22}\xi_{x_2}\eta_{x_2})\, U_\eta]$$

where $J = \xi_{x_1}\eta_{x_2} - \xi_{x_2}\eta_{x_1}$.

In order to arrive at (4.5) we impose the restrictions

$$\alpha_{12}\xi_{x_1}\eta_{x_2} + \alpha_{22}\xi_{x_2}\eta_{x_2} + \alpha_{11}\xi_{x_1}\eta_{x_1} + \alpha_{12}\xi_{x_2}\eta_{x_1} = 0$$

$$\alpha_{11}\eta_{x_1}^2 + 2\alpha_{12}\eta_{x_1}\eta_{x_2} + \alpha_{22}\eta_{x_2}^2 = J$$

$$\alpha_{11}\xi_{x_1}^2 + 2\alpha_{12}\xi_{x_1}\xi_{x_2} + \alpha_{22}\xi_{x_2}^2 = J$$

which are easily checked by mean of (4.2).

The following theorem gives a sufficient condition under which the Morrey matrices corresponding to two different matrices $A, B \in \mathcal{E}(K)$ agree on suitable functions.

THEOREM 4.2. *Let* $A, B \in \mathcal{E}(K)$ *and* $f = (u, v)$, $g = (\xi, \eta)$ *be solutions to the systems*

$$\nabla v = RA\nabla u \tag{4.6}$$

$$\nabla \eta = RB\nabla \xi . \tag{4.7}$$

If there exists a conformal mapping $H=(U,V)$ *on* $g(\Omega)$ *such that*

$$f = H \circ g \tag{4.8}$$

then

$$\mathcal{A}(A, u) = \mathcal{A}(B, \xi) \tag{4.9}$$

and

$$\mathcal{A}\left(\frac{A}{\det A}, v\right) = \mathcal{A}\left(\frac{B}{\det B}, \eta\right). \tag{4.10}$$

PROOF. It is worth noting that

$$\mathcal{A}(A, u) = \frac{1}{J_f}\begin{pmatrix} u_{x_2}^2 + v_{x_2}^2 & -(u_{x_1}u_{x_2} + v_{x_1}v_{x_2}) \\ -(u_{x_1}u_{x_2} + v_{x_1}v_{x_2}) & u_{x_1}^2 + v_{x_1}^2 \end{pmatrix} \tag{4.11}$$

$$\mathcal{A}(B, \xi) = \frac{1}{J_g}\begin{pmatrix} \xi_{x_2}^2 + \eta_{x_2}^2 & -(\xi_{x_1}\xi_{x_2} + \eta_{x_1}\eta_{x_2}) \\ -(\xi_{x_1}\xi_{x_2} + \eta_{x_1}\eta_{x_2}) & \xi_{x_1}^2 + \eta_{x_1}^2 \end{pmatrix} \tag{4.12}$$

where J_f and J_g denote the jacobian determinants of f and g respectively. By a straightforward calculation it turns out that

$$J_f = \det H \cdot J_g. \tag{4.13}$$

Now let us prove that, if $J_f \neq 0$

$$\frac{u_{x_2}^2 + v_{x_2}^2}{J_f} = \frac{\xi_{x_2}^2 + \eta_{x_2}^2}{J_g}. \tag{4.14}$$

In fact, we have

$$\begin{aligned} u_{x_2}^2 + v_{x_2}^2 &= (U_\xi \xi_{x_2} + U_\eta \eta_{x_2})^2 + (V_\xi \xi_{x_2} + V_\eta \eta_{x_2})^2 = \\ &= (U_\xi^2 + V_\xi^2)\,\xi_{x_2}^2 + 2(U_\xi U_\eta + V_\xi V_\eta)\,\xi_{x_2}\eta_{x_2} + (U_\eta^2 + V_\eta^2)\,\eta_{x_2}^2. \end{aligned} \tag{4.15}$$

Taking into account the assumption that H is a conformal mapping, *i.e.*

$$\begin{cases} U_\xi^2 + V_\xi^2 = U_\eta^2 + V_\eta^2 = \det H \\ U_\xi U_\eta + V_\xi V_\eta = 0 \end{cases} \tag{4.16}$$

we deduce (4.14) by (4.13) and (4.15).

Let us now prove that

$$\frac{u_{x_1}u_{x_2} + v_{x_1}v_{x_2}}{J_f} = \frac{\xi_{x_1}\xi_{x_2} + \eta_{x_1}\eta_{x_2}}{J_g}. \tag{4.17}$$

Infact we have

$$u_{x_1}u_{x_2}+v_{x_1}v_{x_2}=(U_\xi^2+V_\xi^2)\,\xi_{x_1}\xi_{x_2}+(U_\eta^2+V_\eta^2)\,\eta_{x_1}\eta_{x_2}+ \\ +(U_\xi U_\eta+V_\xi V_\eta)(\xi_{x_1}\eta_{x_2}+\eta_{x_1}\xi_{x_2}). \tag{4.18}$$

By (4.16) we obtain

$$u_{x_1}u_{x_2}+v_{x_1}v_{x_2}=det\,H(\xi_{x_1}\xi_{x_2}+\eta_{x_1}\eta_{x_2})=\frac{J_f}{J_g}(\xi_{x_1}\xi_{x_2}+\eta_{x_1}\eta_{x_2})$$

i.e. (4.17). Similarly one can prove

$$\frac{u_{x_1}^2+v_{x_1}^2}{J_f}=\frac{\xi_{x_1}^2+\eta_{\xi_1}^2}{J_g},$$

establishing (4.9). The equality (4.10) follows then by Proposition 2.3.

5. Examples

Example 5.1. If

$$A=\begin{pmatrix} a_{11}(x) & a_{12}(x)\\ a_{12}(x) & a_{22}(x)\end{pmatrix}$$

then choosing the functions depending only on one variable, we obtain

$$\mathcal{A}(A,u(x_1))=\begin{pmatrix} a_{11}(x) & a_{12}(x)\\ a_{12}(x) & \dfrac{1+(a_{12}(x))^2}{a_{11}(x)}\end{pmatrix}$$

$$\mathcal{A}(A,v(x_2))=\begin{pmatrix} \dfrac{1+(a_{12}(x))^2}{a_{22}(x)} & a_{12}(x)\\ a_{12}(x) & a_{22}(x)\end{pmatrix}.$$

Example 5.2. If the matrix A is isotropic, *i.e.*

$$A=\begin{pmatrix} a(x) & 0\\ 0 & a(x)\end{pmatrix}$$

and $\nabla u\neq 0$ then

$$\mathcal{A}(A,u)=\frac{1}{|\nabla u|^2}\begin{pmatrix} a(x)\,u_{x_1}^2+\dfrac{1}{a(x)}u_{x_2}^2 & \left(a(x)-\dfrac{1}{a(x)}\right)u_{x_1}u_{x_2}\\ \left(a(x)-\dfrac{1}{a(x)}\right)u_{x_1}u_{x_2} & \dfrac{1}{a(x)}u_{x_1}^2+a(x)\,u_{x_2}^2\end{pmatrix}.$$

EXAMPLE 5.3. If $A = (a_{ij})$ and $a_{12} = 0$ then, assuming $u_{x_i} \neq 0$ for $i = 1, 2$, the matrix

$$\mathcal{A} = \mathcal{A}(A, u) = (\alpha_{ij})$$

verifies

$$\alpha_{12} = 0$$

if and only if

$$A = \mathcal{A}.$$

Under our assumptions, we deduce $a_{11} u_{x_1}^2 + a_{22} u_{x_2}^2 \neq 0$ and

$$\alpha_{12} = \frac{(a_{11} a_{22} - 1)\, u_{x_1} u_{x_2}}{a_{11} u_{x_1}^2 + a_{22} u_{x_2}^2}. \tag{5.1}$$

Therefore, if $\alpha_{12} = 0$, by (5.1) we deduce $a_{11} a_{22} - 1 = 0$, *i.e.* $\det A = 1$. Using (*i*) in Proposition 2.2 we get $\mathcal{A} = A$.

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