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Gustavo Garrigós

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POISSON-LIKE KERNELS IN TUBE DOMAINS OVER LIGHT-CONES

Abstract. — A family of holomorphic function spaces can be defined with reproducing kernels $B_\alpha(z, w)$, obtained as real powers of the Cauchy-Szegö kernel. In this paper we study properties of the associated Poisson-like kernels: $P_\alpha(z, w) = |B_\alpha(z, w)|^2 / B_\alpha(z, z)$. In particular, we show boundedness of associated maximal operators, and obtain formulas for the limit of Poisson integrals in the topological boundary of the cone.

Key words: Tube domain; Poisson kernel; Maximal function; Bergman space.

1. Introduction

Let $\Omega = \{y = (y_1, y') \in \mathbb{R}^n \mid y_1 > |y'|\}$ denote the *forward light-cone* in $\mathbb{R}^n$, $n \geq 3$, and $T_\Omega = \mathbb{R}^n + i\Omega$ the corresponding tube domain in $\mathbb{C}^n$. We also denote by

$$\Delta(y) := y_1^2 - |y'|^2 = y_1^2 - (y_2^2 + \ldots + y_n^2), \quad y \in \mathbb{R}^n,$$

the *Lorentz form* (or *determinant*) associated with Ω. When $0 < p < \infty$ the *classical Hardy space* is defined as:

$$H^p(T_\Omega) = \left\{ F \in \mathcal{H}(T_\Omega) \mid \sup_{y \in \Omega} \|F(\cdot + iy)\|_{L^p(\mathbb{R}^n)} < \infty \right\}.$$

There are two well-known kernels related to these spaces:

1. *The Cauchy-Szegö kernel*:

$$S(z, u) = c_0 \, \Delta^{-\frac{n}{2}}((z-u)/i), \quad z \in T_\Omega, \; u \in \mathbb{R}^n.$$

2. *The Poisson-Szegö kernel*:

$$P(z, u) = |S(z, u)|^2 / S(z, z) = c_0 \frac{\Delta^{\frac{n}{2}}(2y)}{|\Delta(x - u + iy)|^n}, \quad z = x + iy, \, u \in \mathbb{R}^n.$$

The first one is the reproducing kernel of H^2, and thus naturally related to the complex geometry of Hardy spaces. Its behavior for real analysis is however somewhat pathological, since the associated orthogonal projector does not admit bounded extensions to L^p for any $p \neq 2$ [10, 6]. The second kernel is derived from $S(z, u)$ in such a way that reproduces functions in all Hardy spaces H^p, when $p \geq 1$ (see, *e.g.*, [12, Chapter 3]). Moreover, it is better suited for real analysis since the family of functions $\{P_y(x) = P(x + iy, 0)\}_{y \in \Omega}$ is an *approximation of the identity in* $\mathbb{R}^n$. That is, when $p \geq 1$

every function $f \in L^p(\mathbb{R}^n)$ can be extended to the tube domain by its *Poisson integral*: $\mathcal{P}f(x+iy) = P_y * f(x)$, $x+iy \in T_\Omega$, so that it holds the continuity property:

$$\mathcal{P}f(x+iy) = P_y * f(x) \longrightarrow f(x)\,, \quad \text{as } y \to 0 \ (y \in \Omega)\,, \tag{1.1}$$

with convergence in the norm of $L^p(\mathbb{R}^n)$. There is also pointwise convergence for *a.e.* $x \in \mathbb{R}^n$, provided the values of y are restricted to a proper subcone of Ω [11, p. 449]. A second pathological behavior appears in relation with the unrestricted pointwise convergence in (1.1). This can be shown to fail using the following remarkable identity from [13]: if $t \in \partial\Omega$ then

$$\lim_{\substack{y \to t \\ (y \in \Omega)}} \mathcal{P}f(x+iy) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x-\lambda t)}{1+\lambda^2}\, d\lambda\,, \tag{1.2}$$

with convergence in $L^p(\mathbb{R}^n)$. In fact, a combination of (1.2) and the Besicovitch counterexample shows that the maximal operator

$$\mathcal{P}^* f(x) = \sup_{y \in \Omega} |P_y * f|(x)\,, \quad x \in \mathbb{R}^n\,,$$

is never bounded in $L^p(\mathbb{R}^n)$ for any finite p (see [11, pp. 449, 458]). These two pathological behaviors of the Cauchy-Szegö kernel in tube domains over higher rank cones motivate a further investigation of other related operators arising from reproducing kernels in T_Ω.

In this paper we shall mainly concentrate in Poisson-like kernels associated with two families of holomorphic function spaces in tube domains over light-cones. The simplest case is the family of (*weighted*) *Bergman spaces*, defined for $0 < p < \infty$ and $\alpha > \frac{n}{2} - 1$ as:

$$A^p_\alpha = \left\{ F \in \mathcal{H}(T_\Omega) \,\Big|\, \int_\Omega \int_{\mathbb{R}^n} |F(x+iy)|^p \, dx\, \Delta(y)^{\alpha - \frac{n}{2}}\, dy < \infty \right\}.$$

The reproducing kernel for A^2_α is now

$$B_\alpha(z,w) = c_\alpha\, \Delta((z-\overline{w})/i)^{-(\alpha+\frac{n}{2})}\,, \quad z, w \in T_\Omega\,,$$

and the corresponding *Poisson-Bergman kernels*:

$$P_\alpha(z,w) = |B_\alpha(z,w)|^2 / B_\alpha(z,z) = c_\alpha \frac{\Delta^{\alpha+\frac{n}{2}}(2y)}{|\Delta(x-u+i(y+v))|^{2\alpha+n}}\,,$$

where $z = x+iy$, $w = u+iv \in T_\Omega$. In this paper we shall investigate the associated *Poisson-Bergman integrals*:

$$\mathcal{P}_\alpha f(z) = \int_\Omega \int_{\mathbb{R}^n} P_\alpha(z,w) f(w)\, \Delta(\Im m\, w)^{\alpha-\frac{n}{2}}\, dw\,, \quad z \in T_\Omega\,,$$

defined for $f \in L^p_\alpha = L^p(T_\Omega; dx \Delta^{\alpha-\frac{n}{2}}(y) dy)$ and $p \geq 1$. Our first result in this direction is the following.

THEOREM 1.3. *Let $\alpha > \frac{n}{2} - 1$. Then, $\mathcal{P}_\alpha$ is a bounded operator in L^p_α if and only if $p > p_\alpha := \frac{\alpha + \frac{n}{2} - 1}{\alpha}$. Moreover, if Ω_0 is a proper subcone of Ω, the restricted maximal operator:*

$$\mathcal{P}^*_{\alpha,\Omega_0} f(z) := \sup_{y \in \Omega_0} |\mathcal{P}_\alpha f(z + iy)| , \quad z \in T_\Omega , \tag{1.4}$$

is bounded in L^p_α if and only if $p > p_\alpha$.

We point out that the index p_α in the theorem coincides with the index q'_α in [4]; that is, $p_\alpha < p < p'_\alpha$ is precisely the range where the «positive» Bergman projector (with kernel $|B_\alpha(z, w)|$) is bounded in L^p_α. The boundedness (or unboundedness) of the unrestricted maximal operator is still an open question subject of current investigation. Likewise happens with the boundedness of the Bergman projector (with kernel $B_\alpha(z, w)$), for which the latest results give the region $1 + \frac{1}{p'_\alpha} < p < 1 + p'_\alpha$ and several equivalences with other geometric problems of Bergman spaces (see [3, 2] and the survey paper in this journal [4]).

The second family of holomorphic function spaces we shall consider is perhaps less known, and arises as a limit case of the weighted Bergman spaces when $\alpha \to \alpha_0 = \frac{n}{2} - 1$:

$$H^p_\mu := \left\{ F \in \mathcal{H}(T_\Omega) \;\Big|\; \sup_{y \in \Omega} \int_{\partial\Omega} \int_{\mathbb{R}^n} |F(x + i(y + v))|^p \, dx \, d\mu(v) < \infty \right\} .$$

In this definition μ denotes the measure:

$$\int_{\partial\Omega} f(v) \, d\mu(v) = \int_{\mathbb{R}^{n-1}} f(|v'| , v') \frac{dv'}{|v'|} , \quad f \in C_c(\mathbb{R}^n) ,$$

supported on the topological boundary of the cone $\partial\Omega$. Alternatively, μ can be seen as the «delta distribution» of the surface $\partial\Omega$: $\mu = \delta(\Delta)$, as defined in [8, Chapter 3]. It can be shown that μ is the distributional limit of the measures $(\alpha - \frac{n}{2} + 1) \Delta^{\alpha - \frac{n}{2}}(\xi) \, d\xi$, as $\alpha \searrow \frac{n}{2} - 1$ [8, 5], justifying the terminology of «limit space» we gave above to H^p_μ. The reproducing kernel of H^2_μ is now:

$$B_{\alpha_0}(z, w) = c_{\alpha_0} \Delta((z - \overline{w})/i)^{-(n+1)} , \quad z \in T_\Omega , \; w \in T_{\partial\Omega} ,$$

with the inner product of $L^2_\mu = L^2(\mathbb{R}^n + i\partial\Omega; dx \, d\mu(v))$. This space was first considered when $p = 2$ in [14], in connection with the representation theory of the group $G(T_\Omega)$. A more complete investigation of H^p_μ, for all $0 < p < \infty$, was recently presented in [7], establishing several properties analogous to those of classical Hardy spaces. The main question we wish to treat here concerns the «Poisson-like kernel» defined by:

$$P_{\alpha_0}(z, w) = |B_{\alpha_0}(z, w)|^2 / B_{\alpha_0}(z, z) = c_{\alpha_0} \frac{\Delta^{n-1}(2y)}{|\Delta(x - u + i(y + v))|^{2(n-1)}} , \tag{1.5}$$

when $z = x + iy \in T_\Omega$, $w = u + iv \in T_{\partial\Omega}$, and the associated «Poisson-like integrals»:

$$\mathcal{P}_{\alpha_0} f(z) = \int_{\partial\Omega} \int_{\mathbb{R}^n} P_{\alpha_0}(z, w) f(w) \, du \, d\mu(v) , \quad z \in T_\Omega , \tag{1.6}$$

defined for any $f \in L^p_\mu$ and $p \geq 1$. In this case, the behavior is more complicated than in the Poisson-Bergman situation, because of the strong singularities of the kernel when z is close to $\partial\Omega$. There are also fundamental differences with the Poisson-Szegö kernel, since $\mathcal{P}_{\alpha_0} f(z)$ is no longer of convolution type (there is an extra integral over $\partial\Omega$), and one cannot expect a behavior of approximate identity when $z \to \partial\Omega$. The results in this paper show that the Poisson integrals in (1.6) admit an extension to the topogical boundary of the tube similar to (1.2), and provide a sharp region of boundedness for the «pointwise operator» $\mathcal{P}_{\alpha_0,y} f := \mathcal{P}_{\alpha_0} f(\cdot + iy)$ in L^p_μ. These results are presented in the next two theorems.

THEOREM 1.7. *For every fixed $y \in \Omega$, the operator $f \mapsto \mathcal{P}_{\alpha_0,y} f$ is bounded in L^p_μ if and only if $p > 2$. In this case, the norm $\|\mathcal{P}_{\alpha_0,y}\|_{L^p_\mu \to L^p_\mu}$ is independent of $y \in \Omega$.*

THEOREM 1.8. *For every $f \in C_c(\mathbb{R}^n + i\partial\Omega)$ and $z = x + it \in T_{\partial\Omega}$, we have*

$$\lim_{\substack{y \to 0 \\ (y \in \Omega)}} \mathcal{P}_{\alpha_0} f(z + iy) = d_{\alpha_0} \int_0^\infty \int_{\mathbb{R}} \frac{f(x + (r + is)t)}{[r^2 + (1 + s)^2]^{\frac{n}{2}}} \, dr \, s^{\frac{n}{2} - 1} \frac{ds}{s} . \tag{1.9}$$

Moreover, if we denote by $\widetilde{\mathcal{P}}_{\alpha_0} f(x + it)$ the integral on the right of (1.9), *then the operator $\widetilde{\mathcal{P}}_{\alpha_0}$ in* (1.9) *is bounded in L^p_μ if and only if $p > 2$.*

As an interesting observation we point out that

$$\wp_\alpha(\rho + i\sigma, r + is) = \frac{d_\alpha \, \sigma^{\alpha+1}}{[(\rho - r)^2 + (\sigma + s)^2]^{\alpha+1}} , \; \rho + i\sigma , r + is \in \mathbb{H} = \mathbb{R} + i(0, \infty) ,$$

is the 1-dimensional «Poisson-Bergman kernel» corresponding to the holomorphic function space in the half-plane $A^2_\alpha(\mathbb{H}) = L^2(\mathbb{H}; dr \, s^{\alpha-1} ds)$. In this way we find an analogy between (1.2) and (1.9), where in the former case we obtained a 1-dimensional Poisson kernel, and in the latter the Poisson-Bergman $\wp_{\alpha_0}$. This should not be surprising since, as we showed in [7], every $F \in H^p_\mu$ has a boundary limit in $T_{\partial\Omega}$ which satisfies the associated *tangential Cauchy-Riemann equations*, and in particular belongs to 1-dimensional Bergman spaces when restricted to half-planes $x + \mathbb{H}t$. Finally, we point out that several questions remain open concerning the maximal operator $\mathcal{P}^*_{\alpha_0} f = \sup_{y \in \Omega} |P_{\alpha_0,y} f|$ in L^p_μ, even for the restricted case of a proper subcone of Ω. These are under current investigation and will be presented elsewhere.

2. THE POISSON-BERGMAN KERNELS

Throughout this section, we fix $\alpha > \alpha_0 = \frac{n}{2} - 1$. In [7] it was shown that $P_\alpha(z, w)$ is a reproducing kernel of A^p_α for every $p \geq 1$, meaning that $F = \mathcal{P}_\alpha F$, for all $F \in A^p_\alpha$. Further, when Ω_0 is a proper subcone of Ω we established the equivalence of norms:

$$\|F\|_{L^p_\alpha} \leq \Big\| \sup_{\eta \in \Omega_0} |F(\cdot + i\eta)| \Big\|_{L^p_\alpha} \leq C_{\Omega_0} \|F\|_{L^p_\alpha} , \quad F \in A^p_\alpha . \tag{2.1}$$

In this section we shall prove Theorem 1.3, which can be seen as a real analog of (2.1), where the holomorphic function F is replaced by the Poisson-Bergman integral $\mathcal{P}_\alpha f(z)$

of a function $f \in L^p_\alpha$. To begin with, we show the boundedness of the «pointwise» operator: $f \mapsto \mathcal{P}_\alpha f$. We shall use the following well-known formula:

$$\int_{\mathbb{R}^n} \frac{dx}{|\Delta(x+iy)|^\beta}\, dx = c_\beta\, \Delta(y)^{-(\beta-\frac{n}{2})}, \quad y \in \Omega, \tag{2.2}$$

valid for all $\beta > n-1$ (see [3, Lemma 3.4]). We shall also need the finer estimate

$$\int_{|x|\le\frac{1}{2}} \frac{dx}{|\Delta(x+iy)|^\beta}\, dx \ge C\, \Delta(y)^{-(\beta-\frac{n}{2})}, \quad \text{when } y \in \Omega \cap B_2(0) \tag{2.3}$$

(see [1, Proposition 2.3] or [4]).

Proposition 2.4. *Let $\alpha > \frac{n}{2}-1$. Then, $\mathcal{P}_\alpha$ is a bounded operator in L^p_α if and only if $p > p_\alpha := \frac{\alpha+\frac{n}{2}-1}{\alpha}$.*

Proof. We prove first the sufficiency. Given $f \in L^p_\alpha$, an elementary estimate using (2.2) gives:

$$\|\mathcal{P}_\alpha f\|_{L^p_\alpha} \le \left\| \int_\Omega \int_{\mathbb{R}^n} P_\alpha(iy, u+iv) \|f(\cdot+iv)\|_{L^p(\mathbb{R}^n)} du \Delta(v)^{\alpha-\frac{n}{2}}\, dv \right\|_{L^p_\alpha(dy)} =$$

$$= C \left\| \int_\Omega \frac{\Delta(y)^{\alpha+\frac{n}{2}}}{\Delta(y+v)^{2\alpha+\frac{n}{2}}} \|f(\cdot+iv)\|_{L^p}\, \Delta(v)^{\alpha-\frac{n}{2}}\, dv \right\|_{L^p_\alpha(dy)}.$$

Thus, we have reduced matters to study a positive operator T_α in the space $L^p_\alpha(dy) = L^p(\Omega; \Delta(y)^{\alpha-\frac{n}{2}} dy)$ defined by the kernel

$$K_\alpha(y, v) = \frac{\Delta(y)^{\alpha+\frac{n}{2}}}{\Delta(y+v)^{2\alpha+\frac{n}{2}}}, \quad y, v \in \Omega.$$

Now, the boundedness of this operator when $p > \frac{\alpha+\frac{n}{2}-1}{\alpha}$ follows from Schur's lemma, after testing with the functions $\varphi(v) = (v_1 - v_2)^{\theta_1}(v_1^2 - |v'|^2)^{\theta_2}$, $v \in \Omega$, for suitable $\theta_1, \theta_2 \in \mathbb{R}$. The computations are similar to those in [1, p. 89], so details are left to the reader.

For the sharpness, we shall actually show that $\mathcal{P}_\alpha$ is bounded in L^p_α if and only if the operator T_α is bounded in $L^p_\alpha(dy)$. For the last operator is now easy to obtain a counterexample, since testing with $g = \chi_{B_{\frac{1}{4}}(\mathbf{e})}$, where $\mathbf{e} = (1, 0, \dots, 0)$, we see that

$$T_\alpha g(y) \ge c\, \frac{\Delta(y)^{\alpha+\frac{n}{2}}}{\Delta(y+\mathbf{e})^{2\alpha+\frac{n}{2}}}, \quad y \in \Omega \tag{2.5}$$

(because $\Delta(y+v) \sim \Delta(y+\mathbf{e})$ if $v \in B_{\frac{1}{4}}(\mathbf{e})$, see [3, Corollary 2.3]), and the function in (2.5) only belongs to $L^p_\alpha(dy)$ when $p > p_\alpha$ (see [3, Lemma 3.3]). We still need to show that boundedness of $\mathcal{P}_\alpha$ implies boundedness of T_α, for which we follow similar ideas from [1, Theorem 2]. Given a non-negative function $g \in C_c(\Omega \cap B_1(0))$, we test

$\mathcal{P}_\alpha$ on $f(u+iv)=\chi_{B_1(0)}(u)g(v)$, and using (2.3) we obtain

$$\|\mathcal{P}_\alpha f\|^p_{L^p_\alpha} \geq \int_{|y|\leq 1}\int_{|x|\leq\frac{1}{2}}\left|\int_\Omega\int_{|u|\leq 1} P_\alpha(x+iy,u+iv)g(v)du\Delta(v)^{\alpha-\frac{n}{2}}\,dv\right|^p dx\Delta^{\alpha-\frac{n}{2}}(y)dy \geq$$

$$\geq C\int_{|y|\leq 1}\left|\int_\Omega K_\alpha(y,v)g(v)\Delta(v)^{\alpha-\frac{n}{2}}\,dv\right|^p \Delta^{\alpha-\frac{n}{2}}(y)dy\,.$$

Now, if g is supported on $\Omega\cap B_R(0)$, a scaling argument using $g_R(v)=g(Rv)$ leads to:

$$\int_{|y|\leq R}\left|\,T_\alpha g(y)\right|^p\,\Delta^{\alpha-\frac{n}{2}}(y)dy \leq\ C\,\|\mathcal{P}_\alpha\|^p\int_\Omega |g(y)|^p\,\Delta^{\alpha-\frac{n}{2}}(y)dy\ ,$$

which letting $R\to\infty$ establishes our claim. $\square$

To study the maximal function we shall need the following elementary lemma:

Lemma 2.6. *If $y, y'\in\Omega$ and $x\in\mathbb{R}^n$ then*

$$\Delta(y+y')\geq\Delta(y)\ \textit{and}\ |\Delta(x+i(y+y'))|\geq|\Delta(x+iy')|\geq\Delta(y')\,.$$

Proof. The first and last inequalities are well known (see, *e.g.*, [7, §7] or [3, 3.1]). We prove here the middle one. By the action of the group $G=\mathbb{R}_+SO(n-1,1)$, we may assume that $y'=\mathbf{e}=(1,0,\dots,0)$, and further, by rotating we can also suppose that $y=(y_1,y_2,0)$. Now we use the explicit formula

$$|\Delta(x+i(\mathbf{e}+y))|^2=\Delta(x)^2+\Delta(\mathbf{e}+y)^2+(x_1+x_2)^2(y_1+1-y_2)^2+$$
$$+(x_1-x_2)^2(y_1+1+y_2)^2+2\Delta(\mathbf{e}+y)|x_3|^2\geq|\Delta(x+i\mathbf{e})|^2. \quad\square$$

Let us now turn to the proof of Theorem 1.3. We split the maximal function in (1.4) into two parts:

$$\mathcal{P}^{*,1}_\alpha f(z)=\sup_{\substack{\eta<y\\ \eta\in\Omega_0}}|\mathcal{P}_\alpha f(z+i\eta)|\,,\quad z=x+iy\in T_\Omega$$

$$\mathcal{P}^{*,2}_\alpha f(z)=\sup_{\substack{\eta>y\\ \eta\in\Omega_0}}|\mathcal{P}_\alpha f(z+i\eta)|\,,\quad z=x+iy\in T_\Omega\ ,$$

where the inequality $\eta<y$ means $y-\eta\in\Omega$. For the first operator, elementary estimates involving Lemma 2.6 give

$$\frac{\Delta(y+\eta)}{|\Delta(x+i(y+\eta+v))|^2}\leq\frac{\Delta(2y)}{|\Delta(x+i(y+v))|^2}\,,\quad\text{when }\eta<y\ ,$$

and therefore we conclude $\mathcal{P}^{*,1}_\alpha f(z)\leq C\mathcal{P}_\alpha|f|(z)$, from which the boundedness in L^p_α is a consequence of Proposition 2.4. Observe that in this case the restriction of η to a proper subcone plays no role at all.

For the second maximal operator we use the following estimate valid for all $y < \eta$:

$$\frac{\Delta(y+\eta)^{\alpha+\frac{n}{2}}}{|\Delta(x+i(y+\eta+v))|^{2\alpha+n}} \leq \frac{\Delta(y+\eta)^{\frac{n}{2}}}{|\Delta(x+i(y+\eta))|^{n}} \frac{\Delta(y+\eta)^{\alpha}}{|\Delta(y+\eta+v)|^{2\alpha}} \leq \tag{2.7}$$
$$\leq \frac{\Delta(2\eta)^{\frac{n}{2}}}{|\Delta(x+i\eta)|^{n}} \frac{\Delta(2\eta)^{\alpha}}{|\Delta(\eta+v)|^{2\alpha}} .$$

Now, in this last expression, the first quotient is precisely the Poisson-Szegö kernel, and therefore,

$$\mathcal{P}_{\alpha}^{*,2} f(z) \lesssim \sup_{\substack{\eta>y \\ \eta\in\Omega_0}} \int_{\Omega} \mathcal{P}_{\Omega_0}^{*} f_v(x)\, K_{\eta}(v)\Delta(v)^{\alpha-\frac{n}{2}}\, dv , \quad z = x+iy \in T_{\Omega} ,$$

where

$$\mathcal{P}_{\Omega_0}^{*} f_v(x) = \sup_{y\in\Omega_0} \int_{\mathbb{R}^n} |f(x-u+iv)| \frac{\Delta(y)^{\frac{n}{2}}}{|\Delta(x+iy)|^{n}}\, du ,$$

and $K_{\eta}(v)$ is the positive kernel in Ω given by the last quotient in (2.7). Since $\mathcal{P}_{\Omega_0}^{*}$ is bounded in $L^p(\mathbb{R}^n)$, matters are reduced to study the maximal operator in Ω:

$$T^* f(y) = \sup_{\substack{\eta>y \\ \eta\in\Omega_0}} \int_{\Omega} K_{\eta}(v) f(v)\, \Delta(v)^{\alpha-\frac{n}{2}}\, dv , \quad y \in \Omega . \tag{2.8}$$

The proof of Theorem 1.3 will be complete with the following proposition:

Proposition 2.9. *Let $\alpha > \frac{n}{2} - 1$ and $1 < p < \infty$. Then, the maximal operator T^* in* (2.8) *is bounded in $L^p(\Delta(v)^{\alpha-\frac{n}{2}}\, dv)$.*

Proof. We write $\eta = \varepsilon(1, \eta_0)$, $y = \sigma(1, \nu)$, $v = \rho(1, \omega)$, where $|\eta_0|, |\nu|, |\omega| \leq 1$ and $\varepsilon, \sigma, \rho > 0$. Since $\eta \in \Omega_0$, we must have $0 \leq |\eta_0| < 1-\delta$, for some $\delta > 0$. Now, an explicit computation gives

$$\Delta(\eta) = (1-|\eta_0|^2)\varepsilon^2 \leq \varepsilon^2$$
$$\Delta(\eta+v) = (1-|\eta_0|^2)\varepsilon^2 + 2\varepsilon\rho(1-\eta_0\cdot\omega) + \rho^2(1-|\omega|^2) \geq \delta(\varepsilon^2+2\varepsilon\rho) .$$

Thus, since $\Delta(v)^{\alpha-\frac{n}{2}}\, dv = \rho^{n-1}(\rho^2(1-|\omega|^2))^{\alpha-\frac{n}{2}}\, d\rho d\omega$, we have

$$T^* f(\sigma, \nu) \lesssim \sup_{\varepsilon>\sigma} \int_0^{\infty} \int_{|\omega|\leq 1} |f(\rho, \omega)| \frac{\varepsilon^{2\alpha}\rho^{2\alpha-1}(1-|\omega|^2)^{\alpha-\frac{n}{2}}}{(\varepsilon^2+\varepsilon\rho)^{2\alpha}}\, d\omega\, d\rho \leq$$
$$\leq \sup_{\varepsilon>\sigma} \int_0^{\infty} \frac{1}{(\varepsilon+\rho)^{2\alpha}} \left[\int_{|\omega|\leq 1} |f(\rho, \omega)|(1-|\omega|^2)^{\alpha-\frac{n}{2}} d\omega\right] \rho^{2\alpha-1} d\rho .$$

Now,

$$\sup_{\varepsilon>\sigma} \frac{1}{(\varepsilon+\rho)^{2\alpha}} \leq \min\left\{\frac{1}{\sigma^{2\alpha}}, \frac{1}{\rho^{2\alpha}}\right\} = k_{\alpha}(\sigma, \rho) ,$$

and it is easy to verify that this last kernel produces a bounded operator in $L^p((0, \infty);$ $\rho^{2\alpha-1} d\rho)$, for all $1 < p < \infty$ and $\alpha > 0$ (by Schur's Lemma). On the other hand, since $\int_{|\omega|\leq 1}(1-|\omega|^2)^{\alpha-\frac{n}{2}}\,d\omega$ is a finite constant when $\alpha > \frac{n}{2}-1$, it follows that:

$$\begin{aligned}
&\int_0^\infty \int_{|\nu|\leq 1} |T^* f(\sigma\,,\nu)|^p (1-|\nu|^2)^{\alpha-\frac{n}{2}}\,d\nu\sigma^{2\alpha-1}\,d\sigma \lesssim \\
&\quad\lesssim \int_0^\infty \left| \int_0^\infty k_\alpha(\sigma\,,\rho)\left[\int_{|\omega|\leq 1} |f(\rho\,,\omega)|(1-|\omega|^2)^{\alpha-\frac{n}{2}}\,d\omega\right]\rho^{2\alpha-1}\,d\rho\right|^p \sigma^{2\alpha-1}\,d\sigma \lesssim \\
&\quad\lesssim \int_0^\infty \left|\int_{|\omega|\leq 1} |f(\sigma\,,\omega)|(1-|\omega|^2)^{\alpha-\frac{n}{2}}\,d\omega\right|^p \sigma^{2\alpha-1}\,d\sigma \lesssim \\
&\qquad\qquad \lesssim \int_0^\infty \int_{|\omega|\leq 1} |f(\sigma\,,\omega)|^p (1-|\omega|^2)^{\alpha-\frac{n}{2}}\,d\omega\sigma^{2\alpha-1}\,d\sigma = \|f\|^p_{L^p_\alpha(dy)}
\end{aligned}$$

establishing the proposition, and with it the theorem. □

3. Boundedness of H^p_μ-Poisson kernels

In this section we prove Theorem 1.7. That is, we shall show that for $p > 2$ it holds

$$\|\mathcal{P}_{\alpha_0} f(\cdot + i\eta)\|_{L^p_\mu} \leq C\,\|f\|_{L^p_\mu}\,, \quad \forall\,\eta\in\Omega\,. \tag{3.1}$$

The first observation is that, by the group action in Ω, we can assume $\eta = \mathbf{e} = (1\,,0\,,\ldots\,,0)$. Next, we reduce the problem to an inequality in the imaginary variable.

Proposition 3.2. *The inequality in* (3.1) *holds if and only if*

$$\left\|\int_{\partial\Omega} K(t,\,v)g(v)\,d\mu(v)\right\|_{L^p(d\mu(t))} \leq C\,\|g\|_{L^p(d\mu(t))}\,, \tag{3.3}$$

where

$$K(t,\,v) = \frac{\Delta(t+\mathbf{e})^{n-1}}{|\Delta(t+\mathbf{e}+v)|^{2(n-1)-\frac{n}{2}}}\,, \quad t,\,v\in\partial\Omega\,.$$

Proof. Analogous to the same equivalence in Proposition 2.4. □

A counterexample for $p\leq 2$ is now easy to obtain using the following estimates:

Lemma 3.4. *Let $t,\,v\in\partial\Omega$, then*

$$\Delta(t+\mathbf{e}) \leq \Delta(t+\mathbf{e}+v) \leq \Delta(v+\mathbf{e})\,\Delta(t+\mathbf{e})\,. \tag{3.5}$$

Proof. By definition we have

$$\Delta(t+\mathbf{e}+v) = \Delta(t+\mathbf{e}) + 2((1+t_1)v_1 - t'\cdot v')\,.$$

Now, using $t_1 = |t'|$ and $\Delta(t+\mathbf{e}) = 1+2t_1$ (and likewise for v), we conclude

$$0 \leq (1+t_1)v_1 - t'\cdot v' \leq v_1(1+2t_1) = v_1\,\Delta(t+\mathbf{e}). \quad \square$$

Now, testing in (3.3) with $g = \chi_{B_1(0)\cap\partial\Omega}$ it follows that

$$\int_{\partial\Omega} K(t, v) g(v)\, d\mu(v) \geq c \frac{1}{\Delta(t+\mathbf{e})^{\frac{n}{2}-1}}, \quad t \in \partial\Omega,$$

which belongs to $L^p(d\mu)$ iff $p > 2$.

Let us now show the boundedness in (3.3) for $p > 2$. The proof will be an application of Schur's lemma, for which we need the following estimate.

LEMMA 3.6. *Let* $\gamma, \delta \in \mathbb{R}$, $t \in \partial\Omega$, *and*

$$I_{\gamma,\delta}(t) = \int_{\partial\Omega} \Delta^{\gamma}(v+\mathbf{e})\, \Delta^{\delta}(v+\mathbf{e}+t)\, d\mu(v).$$

Then, if $\gamma + \delta < -(n-2)$ *and* $\gamma > -(\frac{n}{2}-1)$ *we have*

$$I_{\gamma,\delta}(t) \lesssim \Delta(\mathbf{e}+t)^{\gamma+\delta+\frac{n}{2}-1}, \quad t \in \partial\Omega.$$

PROOF. By rotating we can assume $t = \lambda c_1 = (\lambda, \lambda, 0)$, $\lambda > 0$. Now, observe that under the assumption $\lambda \leq \lambda_0$ we have $\Delta(\lambda c_1 + \mathbf{e} + v) \sim \Delta(v + \mathbf{e})$ (by Lemma 3.4), and therefore, $I_{\gamma,\delta}(\lambda c_1) \leq C_{\lambda_0} I_{\gamma,\delta}(0) = C'_{\lambda_0}$. Hence, it suffices to show that

$$L := \lim_{\lambda\to\infty} \frac{I_{\gamma,\delta}(\lambda c_1)}{\lambda^{\gamma+\delta+\frac{n}{2}-1}} < \infty. \tag{3.7}$$

For this, we write the integral $I_{\gamma,\delta}$ with the new coordinates

$$v = (s^2 + |\sigma|^2, s^2 - |\sigma|^2, 2s\sigma) \in \partial\Omega, \quad s > 0, \sigma \in \mathbb{R}^{n-2}, \tag{3.8}$$

so that

$$I_{\gamma,\delta}(\lambda c_1) = c_n \int_0^\infty s^{n-3} \int_{\mathbb{R}^{n-2}} (1 + 2(s^2+|\sigma|^2))^{\gamma}\, (1 + 2(s^2+|\sigma|^2) + 4\lambda|\sigma|^2 + 2\lambda)^{\delta}\, d\sigma\, ds$$

(see, *e.g.*, [5, p. 134, 11, p. 493]). Now, to calculate the limit in (3.7) we divide the above expression by the suitable power of λ and change variables in s, so that

$$L = \lim_{\lambda\to\infty} c_n \int_0^\infty s^{n-3} \int_{\mathbb{R}^{n-2}} \left(\frac{1}{\lambda} + 2\left(s^2 + \frac{|\sigma|^2}{\lambda}\right)\right)^{\gamma} \left(\frac{1}{\lambda} + 2\left(s^2 + \frac{|\sigma|^2}{\lambda}\right) + 4|\sigma|^2 + 2\right)^{\delta} d\sigma\, ds =$$

$$= c_n \int_0^\infty s^{n-3} \int_{\mathbb{R}^{n-2}} (2s^2)^{\gamma} (2s^2 + 4|\sigma|^2 + 2)^{\delta}\, d\sigma\, ds,$$

where the last step can be justified with the Dominated Convergence Theorem. Finally, an elementary computation shows that the last integral is a finite constant under the assumptions $\gamma + \delta < -(n-2)$, $\gamma > -(\frac{n}{2}-1)$, establishing the lemma. □

We are now in position of proving (3.3), for which we shall choose $\theta \in \mathbb{R}$ so that $\varphi(v) = \Delta^{\theta}(v+\mathbf{e})$ satisfies the conditions of Schur's lemma:

$$\int_{\partial\Omega} K(t, v) \varphi(v)^{p'}\, d\mu(v) \leq C\, \varphi(t)^{p'}, \quad t \in \partial\Omega,$$

$$\int_{\partial\Omega} K(t, v) \varphi(t)^{p}\, d\mu(t) \leq C\, \varphi(v)^{p}, \quad v \in \partial\Omega.$$

According to Lemma 3.6, these inequalities hold when

$$-\left(\frac{n}{2}-1\right) < \theta p' < \frac{n}{2} \quad \text{and} \quad -\left(\frac{3n}{2}-2\right) < \theta p < -\left(\frac{n}{2}-1\right) .$$

Solving the inequalities we see that such θ always exists when $p > 2$. The Schur Lemma then establishes (3.3), and with it Theorem 1.7. □

4. Boundary limits of H^p_μ-Poisson integrals

In this section we shall show the limiting formula in (1.9). Assuming it holds for the moment, the boundedness of $\widetilde{\mathcal{P}}_{\alpha_0}$ in L^p_μ, for $p > 2$, is just a consequence of Theorem 1.7 and Fatou's lemma (one can also prove it directly using Minkowski's inequality). A counterexample for $p \leq 2$ is also easy to construct, letting

$$f(x+iv) = \chi_{B_1(0)}(x)\,\chi_{B_1(0)\cap\partial\Omega}(v)\,\frac{1}{|v'|^{\frac{n}{2}-1}\log\frac{1}{|v'|}} , \qquad x+iv \in T_{\partial\Omega} .$$

Indeed, a simple computation shows that $f \in L^p_\mu$ for all $p \leq 2$, while letting $t \in$ $\in B_{\frac{1}{4}}(c_1) \cap \partial\Omega$ we have

$$\widetilde{\mathcal{P}}_{\alpha_0} f(x+it) \sim \int_0^\infty \int_{\mathbb{R}} \chi_{B_1(0)}(x-rt)\,\chi_{B_1(0)}(st)\,\frac{\wp_{\alpha_0}(r, is)}{s^{\frac{n}{2}-1}\log\frac{1}{s}}\,dr s^{\frac{n}{2}-1}\frac{ds}{s} \sim$$

$$\sim \int_{\mathbb{R}} \chi_{B_1(0)}(x-rt)\,\wp_{\alpha_0}(r, i)dr \int_0^1 \frac{1}{s\log\frac{1}{s}}\,ds = \infty .$$

We now turn to the proof of (1.9). Observe that, by translation invariance and the group action we can assume $z = ic_1 = (i, i, 0)$, and thus it suffices to show

$$\lim_{\substack{\varepsilon\to 0\\ (\varepsilon\in\Omega)}} \int_{\partial\Omega}\int_{\mathbb{R}^n} \frac{\Delta(2(\varepsilon+c_1))^{n-1} f(u+iv)}{|\Delta(u+i(\varepsilon+c_1+v))|^{2(n-1)}}\,du\,d\mu(v) = d_{\alpha_0}\int_0^\infty\int_{\mathbb{R}} \frac{f((r+is)c_1)}{[r^2+(1+s)^2]^{\frac{n}{2}}}\,dr\,s^{\frac{n}{2}-1}\frac{ds}{s} . \tag{4.1}$$

Also, the action of the group $\{g \in G(\Omega) \mid gc_1 = c_1\}$ lets us restrict the limit to $\varepsilon = (\varepsilon_1, \varepsilon_2, 0) \in \Omega$.

In order to prove (4.1) we shall use the coordinates of the *Peirce decomposition* in $\mathbb{R}^n$:

$$u = rc_1 + \rho c_2 + z, \qquad r, \rho \in \mathbb{R},\ z \in \mathbb{R}^{n-2} , \tag{4.2}$$

where $c_2 = (1, -1, 0)$ (see [6, Chapter IV]), and the coordinates defined in (3.8) for $v \in \partial\Omega$. Our purpose is to obtain a suitable expression for the quadratic form Δ in this coordinates, so that $\mathcal{P}_{\alpha_0}(ic_1 + i\varepsilon, u+iv)$ can be seen as an «approximate identity» in the variables ρ, z, σ. We shall often use the equality

$$\Delta(x+y) = \Delta(x) + \Delta(y) + 2\Delta(x, y) , \qquad x, y \in \mathbb{R}^n ,$$

where $\Delta(x, y) = x_1y_1 - (x_2y_2 + \ldots + x_ny_n)$.

LEMMA 4.3. *In the conditions above,*

$$\Delta(u + i(v + c_1 + \varepsilon)) = \Delta(\varepsilon, c_1)\, q_{\varepsilon, r, s}\left(\frac{\rho}{\Delta(\varepsilon, c_1)}, \frac{z}{\sqrt{\Delta(\varepsilon, c_1)}}, \frac{\sigma}{\sqrt{\Delta(\varepsilon, c_1)}}\right),$$

where

$$q_{\varepsilon, r, s}(\rho, z, \sigma) = -(|z|^2 - 4r\rho + 2(1 + s^2) + 4|\sigma|^2 + \Delta(\varepsilon, c_2)(1 + 2|\sigma|^2)) + \\ + 2i(r + 2r|\sigma|^2 + 2\rho(1 + s^2) + 2sz \cdot \sigma + \rho\Delta(\varepsilon, c_2)).$$

PROOF. This follows by explicit computation of $\Delta(u + i(v + c_1 + \varepsilon))$ in the coordinates (4.2), (3.8). The verification is routinary and left to the reader. □

With the previous expression we can write

$$\mathcal{P}_{\alpha_0}(ic_1 + i\varepsilon, u + iv) = \frac{c_{\alpha_0}\, [\Delta(\varepsilon, c_1)(2 + \Delta(\varepsilon, c_2))]^{n-1}}{\left|\Delta(\varepsilon, c_1) q_{\varepsilon, r, s}\left(\frac{\rho}{\Delta(\varepsilon, c_1)}, \frac{z}{\sqrt{\Delta(\varepsilon, c_1)}}, \frac{\sigma}{\sqrt{\Delta(\varepsilon, c_1)}}\right)\right|^{2(n-1)}}.$$

We shall define

$$p_{\varepsilon, r, s}(\rho, z, \sigma) = \frac{[2 + \Delta(\varepsilon, c_2)]^{n-1}}{|q_{\varepsilon, r, s}(\rho, z, \sigma)|^{2(n-1)}}.$$

LEMMA 4.4. *In the above conditions*

$$|p_{\varepsilon, r, s}(\rho, z, \sigma)| \le \frac{[2 + \Delta(\varepsilon, c_2)]^{n-1}}{|q_{0, r, s}(\rho, z, \sigma)|^{2(n-1)}} \in L^1(\mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2}).$$

Moreover,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}^{n-2}} \frac{d\rho\, dz\, d\sigma}{|q_{0, r, s}(\rho, z, \sigma)|^{2(n-1)}} = \frac{c_n}{(r^2 + (1 + s^2)^2)^{\frac{n}{2}}}. \tag{4.5}$$

PROOF. The first inequality is the same as saying

$$|q_{\varepsilon, r, s}(\rho, z, \sigma)|^2 \ge |q_{0, r, s}(\rho, z, \sigma)|^2,$$

which in turn follows from the expression of Δ in Lemma 4.3, after separating the terms involving ε and performing the appropriate cancellations. In order to compute the integral, one uses the following formula:

$$|q_{0, r, s}(\rho, z, \sigma)|^2 = 16a(\rho + A)^2 + b^2,$$

where $a = r^2 + (1 + s^2)^2$, $A = A(z, \sigma, r, s)$, and

$$b = 2\sqrt{a} + \frac{1 + s^2}{\sqrt{a}} \left|z + \frac{2rs\sigma}{1 + s^2}\right|^2 + 4\sqrt{a}\,\frac{|\sigma|^2}{1 + s^2}.$$

This can also be obtained explicitly from the expression in Lemma 4.3, after a routinary computation making first squares in ρ, and then in z. At this point, it is easy to calculate the integral

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}^{n-2}} \frac{d\rho\, dz\, d\sigma}{16a(\rho + A)^2 + b^2} = \cdots = \frac{c_n}{a^{\frac{n}{2}}} ,$$

which is the desired result in (2.2). □

The previous lemmas suggest looking at the family

$$\varphi_\eta^{r,s}(\rho, z, \sigma) = \eta^{-(n-1)} p_{\eta, r, s}\left(\frac{\rho}{\eta}, \frac{z}{\sqrt{\eta}}, \frac{\sigma}{\sqrt{\eta}}\right), \quad \eta > 0 ,$$

as an *approximation of the identity* in $\mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2}$, meaning that

$$\lim_{\eta \to 0} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}^{n-2}} \varphi_\eta^{r,s}(\rho, z, \sigma) g(\rho, z, \sigma) d\rho\, dz\, d\sigma = \frac{c_n g(0)}{(r^2 + (1 + s^2)^2)^{\frac{n}{2}}} ,$$

whenever $g \in C_c(\mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2})$. We can now transport this property to our original integral, obtaining,

$$\lim_{\varepsilon \to 0} \int_{\partial\Omega} \int_{\mathbb{R}^n} \mathcal{P}_{\alpha_0}(ic_1 + i\varepsilon, u + iv) f(u + iv) du\, d\mu(v) =$$

$$= c_n \lim_{\varepsilon \to 0} \int_0^\infty s^{n-3} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}^{n-2}} \varphi_{\Delta(\varepsilon, c_1)}^{r,s}(\rho, z, \sigma) f^{r,s}(\rho, z, \sigma) d\rho\, dz\, d\sigma \right] dr\, ds =$$

$$= c_n' \int_0^\infty s^{n-3} \int_{\mathbb{R}} \frac{f(rc_1 + is^2 c_1)}{[r^2 + (1 + s^2)^2]^{\frac{n}{2}}} dr\, ds ,$$

where in the last equality the limit inside the integrals is justified by Lemma 4.4. Thus, changing variables in s we obtain exactly the expression in (4.1), completing the proof of Theorem 1.8. □

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References

[1] D. Békollé - A. Bonami, *Estimates for the Bergman and Szegö projections in two symmetric domains of* $\mathbb{C}^n$. Colloq. Math., 68, 1995, 81-100.

[2] D. Bekollé - A. Bonami - G. Garrigós - F. Ricci, *Littlewood-Paley decompositions and Besov spaces related to symmetric cones*. Univ. Orléans, preprint 2001.

[3] D. Békollé - A. Bonami - M. Peloso - F. Ricci, *Boundedness of weighted Bergman projections on tube domains over light cones*. Math. Z., 237, 2001, 31-59.

[4] A. Bonami, *Three related problems on Bergman spaces of tube domains over symmetric cones*. Rend. Mat. Acc. Lincei, s. 9, v. 13, 2002, 183-197.

[5] J. Faraut - A. Korányi, *Analysis on symmetric cones*. Clarendon Press, Oxford 1994.

[6] C. Fefferman, *The multiplier problem for the ball*. Ann. of Math., 94, 1971, 330-336.

[7] G. Garrigós, *Generalized Hardy spaces in tube domains over cones*. Colloq. Math., 90 (2), 2001, 213-251.

[8] I. Gelfand - G. Shilov, *Generalized functions I*. Academic Press, New York 1964.

[9] H. Rossi - M. Vergne, *Équations de Cauchy-Riemann tangentielles associées a un domaine de Siegel*. Ann. scient. Éc. Norm. Sup., 9, 1976, 31-80.
[10] E. Stein, *Some problems in harmonic analysis suggested by symmetric spaces and semi-simple Lie groups*. Actes, Congrès intern. math. 1, 1970, 173-189.
[11] E. Stein, *Harmonic Analysis*. Princeton University Press, 1993.
[12] E. Stein - G. Weiss, *Fourier Analysis on Euclidean Spaces*. Princeton University Press, 1971.
[13] E. Stein - G. Weiss - M. Weiss, H^p *classes of holomorphic functions in tube domains*. Proc. Nat. Acad. Sci. USA, 52, 1964, 1035-1039.
[14] M. Vergne - H. Rossi, *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group*. Acta Math., 136, 1976, 1-59.

Universidad Autónoma de Madrid
Facultad de Ciencias C-XV
28049 Madrid (Spagna)
gustavo.garrigos@uam.es