Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **13** (2002), n.3-4, p. 219–232.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2002_9_13_3-4_219_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2002.

Rend. Mat. Acc. Lincei
s. 9, v. 13:219-232 (2002)

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CONTACT AND CONFORMAL MAPS ON IWASAWA N GROUPS

Abstract. — The action of the conformal group $O(1, n+1)$ on $\mathbb{R}^n \cup \{\infty\}$ may be characterized in differential geometric terms, even locally: a theorem of Liouville states that a C^4 map between domains U and V in $\mathbb{R}^n$ whose differential is a (variable) multiple of a (variable) isometry at each point of U is the restriction to U of a transformation $x \mapsto g \cdot x$, for some g in $O(1, n+1)$. In this paper, we consider the problem of characterizing the action of a more general semisimple Lie group G on the space G/P, where P is a parabolic subgroup. We solve this problem for the cases where G is $SL(3, \mathbb{R})$ or $Sp(2, \mathbb{R})$ and P is a minimal parabolic subgroup.

Key words: Semisimple Lie group; Contact map; Conformal map.

1. Introduction

In 1850, Liouville proved that any C^4 conformal map between domains in $\mathbb{R}^3$ is necessarily a composition of translations, dilations and inversions in spheres. This can be expressed in more modern language by first observing that the group $O(1, 4)$ acts naturally on the sphere S^3 by conformal transformations (and hence locally on $\mathbb{R}^3$, by stereographic projection), and then saying that any conformal map between two domains arises as the restriction of the action of some element of $O(1, 4)$. The same result also holds in $\mathbb{R}^n$ when $n > 3$ (see, for instance, [7]), and with metric rather than smoothness assumptions (see [3]).

In [6], a similar result was proved with the Heisenberg group in place of Euclidean space and the sphere in $\mathbb{C}^n$ with its CR structure in place of the real sphere. The notion of conformality used there is with respect to the so-called Levi metric; for smooth maps this means that the map is a contact map, and the restriction of its differential to the contact plane is a multiple of a unitary map. The conclusion is that all «conformal» maps belong to the group $SU(1, n)$.

In [8], P. Pansu showed that in the quaternionic and octonionic analogues of this set-up, there is a natural generalised contact structure of codimension greater than one, and the analogue of Liouville's theorem holds under the sole assumption that the map in question preserves the contact structure.

Similar phenomena have been studied in more general situations: see, for example, [1, 2, 4, 5].

The authors of this paper looked at these problems in what seems to be the widest context in which they can be reasonably formulated: the case of a semisimple Lie group G acting on the quotient space G/P, where P is a parabolic subgroup of G. In this context, G/P is the generalization of the sphere in $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$; it always contains a dense open cell, which may be identified with the Iwasawa subgroup N, which is the general analogue of $\mathbb{R}^n$ and the Heisenberg group. The action of G on G/P and on N is contact and conformal in a geometrically reasonable sense (see Lemma 1). At this point one can ask whether a Liouville type theorem can still hold.

In this article, we discuss two fairly typical examples, which will give a good idea of our general theory. We also derive some explicit formulae which may be of independent interest.

We recently discovered that many of our results were found previously by N. Yamaguchi [10]. His work is rarely cited in the literature, and seems to have been largely overlooked, and another aim of our work is to draw attention to his achievements. Yamaguchi's work relies on the theory of G structures, as developed by N. Tanaka [9]. Our methods are different – they are Lie theoretic but otherwise elementary. For this reason, we believe that it may still be of interest to present our approach in some detail.

1.1. *Contact and conformal maps.*

Let G be a semisimple Lie group, with Cartan involution Θ, and Iwasawa decomposition KAN. Consider the compact manifold G/P, where P is a parabolic subgroup with Langlands decomposition $M_P A_P \overline{N}_P$, containing the minimal parabolic subgroup $MA\overline{N}$, where $\overline{N} = \Theta N$. By considering the Bruhat decomposition, we see that N_P, defined to be $\Theta\overline{N}_P$, may be identified with an open dense subset of G/P. The Lie algebra $\mathfrak{n}_P$ has a «multistratification» and a stratification, the former being a decomposition of $\mathfrak{n}_P$ of the form $\bigoplus_{\gamma\in\Sigma_P^+}\mathfrak{g}_\gamma$, where Σ_P^+ denotes the set of restricted positive roots (a subset of the dual of the Lie algebra of A_P), and in particular $[\mathfrak{g}_\gamma\,,\mathfrak{g}_\delta]\subseteq\mathfrak{g}_{\gamma+\delta}$, and the latter being a decomposition of the form $\mathfrak{n}_1\oplus\mathfrak{n}_2\oplus\cdots\oplus\mathfrak{n}_h$, where $\mathfrak{n}_i$ is the sum of the root spaces $\mathfrak{g}_\gamma$ where γ is a sum of i simple $\mathfrak{a}_P$ roots, and in particular $[\mathfrak{n}_i\,,\mathfrak{n}_j]\subseteq\mathfrak{n}_{i+j}$. The group N_P admits a multistratification and a stratification arising from the corresponding multistratification and stratification of its Lie algebra.

1.2. *Examples.*

We consider two examples. For both, the Lie algebra $\mathfrak{m}$ is trivial, but we write it regularly, so that it is easy to see how to deal with the general situation.

The case where $G=\mathrm{SL}(3\,,\mathbb{R})$. Suppose that P is the minimal parabolic subgroup of G of lower triangular matrices. For x, y and z in $\mathbb{R}$, denote by $\nu(x\,,y\,,z)$ the matrix

$$\begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}.$$

Take α and β to be the simple roots relative to the standard Cartan subalgebra of $\mathfrak{sl}(3\,,\mathbb{R})$ of diagonal matrices: $\alpha(\operatorname{diag}(a\,,b\,,c))=(a-b)$ and $\beta(\operatorname{diag}(a\,,b\,,c))=(b-c)$.

Then

$$\begin{aligned}\mathfrak{g}_\alpha &= \{\nu(x\,,0\,,0) : x \in \mathbb{R}\}\ ,\\ \mathfrak{g}_\beta &= \{\nu(0\,,y,0) : y \in \mathbb{R}\}\ ,\\ \mathfrak{g}_{\alpha+\beta} &= \{\nu(0\,,0\,,z) : z \in \mathbb{R}\}\,.\end{aligned}$$

Further, $\mathfrak{n}_P = \mathfrak{n} = \{\nu(x, y, z) : x, y, z \in \mathbb{R}\}$; the algebra $\mathfrak{n}$ has the multistratification $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta}$ and the stratification $\mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_1 = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$ and $\mathfrak{n}_2 = \mathfrak{g}_{\alpha+\beta}$.

The case where $G = \mathrm{Sp}(2\,,\mathbb{R})$. Suppose that P is a minimal parabolic subgroup of G. The standard decomposition into restricted root spaces of the corresponding Iwasawa Lie algebra $\mathfrak{n}$ yields the multistratification $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta}$ and the stratification $\mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$, where $\mathfrak{n}_1 = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$, $\mathfrak{n}_2 = \mathfrak{g}_{\alpha+\beta}$ and $\mathfrak{n}_3 = \mathfrak{g}_{2\alpha+\beta}$. The details will be discussed below.

In this paper, we shall consider only the cases where G is $\mathrm{SL}(3\,,\mathbb{R})$ or $\mathrm{Sp}(2\,,\mathbb{R})$ and P is a minimal parabolic subgroup. The general case will be treated in subsequent papers. Thus, from now on, G/P is either the manifold of complete flags in $\mathbb{R}^3$ or the manifold of complete Lagrangian flags in $\mathbb{R}^4$.

The structure of $\mathfrak{n}$ allows us to give several generalised versions of conformal and contact mappings. To formulate these, we first identify $\mathfrak{n}$ with the tangent space to G/P at P, and then prove a lemma. Define $\phi\colon N \to G/P$ by the formula $\phi(n) = nP$. By the Bruhat decomposition, ϕ is injective, and its image is a dense open subset of G/P containing P. The differential ϕ_* then maps $\mathfrak{n}$, the tangent space to N at the identity e, onto T_P, the tangent space to G/P at P. When γ is a simple positive restricted root (or equivalence class thereof), we denote by $S_{\gamma\,,P}$ the subspace $\phi_*(\mathfrak{g}_\gamma)$ of T_P, and by $S_{1\,,P}$ the sum of the subspaces $\phi_*(\mathfrak{g}_\gamma)$ as γ ranges over all the simple positive restricted roots (or equivalence classes thereof).

Lemma 1. *The action of any element p of P on G/P induces an action p_* on the tangent space T_P which in turn induces an action $\phi_*^{-1} p_* \phi_*$ on $\mathfrak{n}$. This last action preserves each of the spaces $\mathfrak{g}_\gamma$ when γ is a simple root, and its restriction to $\mathfrak{n}_1$ lies in* $\mathrm{Ad}\,(MA)|_{\mathfrak{n}_1}$.

Proof. Since $P = MA\overline{N}$, it is enough to prove the statement for elements of MA and of $\overline{N}$ separately.

First, suppose that $z \in MA$. As $z(nMA\overline{N}) = znz^{-1}MA\overline{N}$, it follows immediately that $\phi_*^{-1} z_* \phi_* = \mathrm{Ad}\,(z)$. Thus, $\phi_*^{-1} z_* \phi_*$ is in $\mathrm{Ad}\,(MA)$, which preserves all the root spaces $\mathfrak{g}_\gamma$, and their direct sums.

Next, suppose that $\overline{n} \in \overline{N}$. For Z in $\mathfrak{n}$, consider the curve $t \mapsto \exp(tZ)MA\overline{N}$ based at P. Write $\overline{n}\exp(tZ)MA\overline{N}$ as $[\exp(t\,\mathrm{Ad}\,(\overline{n})Z)]_N MA\overline{N}$, where $[g]_N$ stands for the N-component of g in the Bruhat decomposition. We claim that

$$\frac{d}{dt}[\exp t(\mathrm{Ad}\,(\overline{n})Z)]_N\Big|_{t=0} = \pi(\mathrm{Ad}\,(\overline{n})Z)\ , \tag{1}$$

where $\pi\colon \mathfrak{n}\oplus(\mathfrak{m}\oplus\mathfrak{a}\oplus\bar{\mathfrak{n}}) \to \mathfrak{n}$ is the canonical projection. Indeed, decomposing $\mathrm{Ad}\,(\bar{n})Z$ as $U+V$, where $U \in \mathfrak{n}$ and $V \in \mathfrak{m}\oplus\mathfrak{a}\oplus\bar{\mathfrak{n}}$, then by the Baker-Campbell-Hausdorff formula,

$$\exp tU = [\exp tU \exp tV]_N = \left[\exp\left(t(U+V)+O(t^2)\right)\right]_N ,$$

so that

$$\frac{d}{dt}[\exp(t\,\mathrm{Ad}\,(\bar{n})Z)]_N\bigg|_{t=0} = \frac{d}{dt}[\exp t(U+V)]_N\bigg|_{t=0} = U ,$$

and (1) holds. Therefore, $\phi_*^{-1}\bar{n}_*\phi_* = \pi\circ\mathrm{Ad}\,(\bar{n})$, as claimed.

Now for X in $\bar{\mathfrak{n}}$ and Y in $\mathfrak{g}_\gamma$, where γ is a simple root, we have

$$\mathrm{Ad}\,(\exp X)\,Y = e^{\mathrm{ad}\,X}\,Y = Y + \sum_{n=1}^{+\infty}\frac{(\mathrm{ad}\,X)^n}{n!}\,Y\,.$$

By writing X as the sum $\sum_{\delta\in\Sigma_P^+}X_{-\delta}$, where $X_{-\delta}\in\mathfrak{g}_{-\delta}$, we see that $\mathrm{ad}\,(X)\,Y\in\mathfrak{m}\oplus\mathfrak{a}\oplus\bar{\mathfrak{n}}$, since $\gamma-\delta$ is either 0, not a root, or a negative root. Then $\mathrm{ad}\,(X)^n\,Y\in\mathfrak{m}\oplus\mathfrak{a}\oplus\bar{\mathfrak{n}}$ for all positive integers n. Consequently $\pi(\mathrm{Ad}\,(\exp X)\,Y) = Y$, as required to conclude the proof. □

This lemma allows us to identify $\mathfrak{n}$ with the tangent space T_x at any point x in G/P, and to identify the subspaces $\mathfrak{g}_\gamma$ of $\mathfrak{n}$, where γ is a simple root, with subspaces $S_{\gamma,x}$ of T_x. These subspaces have conformal structures. Indeed, $x = gP$, the images $g_*\phi_*\mathfrak{g}_\gamma$ are well defined, independently of the representative g of the coset. Further, $\mathfrak{n}$ has a canonical inner product $(X, Y)\mapsto -B(\theta X, Y)$, where B denotes the Killing form, and this induces a conformal structure on all the subspaces $S_{\gamma,x}$.

Consider a diffeomorphism f of G/P or, more generally, $f\colon\mathcal{U}\to\mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are open subsets of G/P. We identify the tangent space T_x at any point x in G/P with $\mathfrak{n}$ as above; for simplicity of notation, we now write $\mathfrak{n}_1$ instead of $g_*\phi_*\mathfrak{n}_1$, and $\mathfrak{g}_\gamma$ instead of $S_{\gamma,gP}$. We say that f is

(*i*) a *contact* map if f_* maps $\mathfrak{n}_1$ into itself;
(*ii*) a *multicontact* map if f_* maps $\mathfrak{g}_\gamma$ into itself for every simple root γ;
(*iii*) a *multiconformal* map if it is a multicontact map and the restriction $(f_*)_\gamma$ of f_* to $\mathfrak{g}_\gamma$ is conformal for every simple root γ.

It is clear that multiconformal maps are multicontact, and multicontact maps are contact.

Examples of multiconformal maps are obtained by considering the natural action of G on G/P, as clarified in the next theorem. It is clear that the theorem holds for any semisimple G and any minimal parabolic P, and even for arbitrary parabolic subgroups after some appropriate definitions have been made.

Theorem 2. *The action of G on G/P is multiconformal.*

It is obvious that the restriction of the action of a given element of G to an open subset of G/P is still multiconformal.

If we restrict the map $x \mapsto gx$ on G/P to appropriate subsets of G/P, we may consider this to be a map between subsets of N: we write $g \cdot n$ for $\phi^{-1} g\phi(n)$. For maps on N, we may introduce other notions of generalised conformality. We identify the tangent space T_x at any point x in N with $\mathfrak{n}$ by left translations. Suppose that R is a group of transformations of $\mathfrak{n}_1$. We say that a diffeomorphism $f: \mathcal{U} \to \mathcal{V}$ is

(*iv*) an R-contact map if f_* preserves $\mathfrak{n}_1$, and the restriction $f_*|_{\mathfrak{n}_1}$ lies in R (1);

(*v*) a conformal map if f_* preserves $\mathfrak{n}_1$, and the restriction $f_*|_{\mathfrak{n}_1}$ is a multiple of an isometry.

The previous definitions, of multicontact and so on, also apply in this context.

THEOREM 3. *If $\mathcal{U}$ and $\mathcal{V}$ are open subsets on N, and $g \cdot \mathcal{U} \subseteq \mathcal{V}$, then the map $n \mapsto g \cdot n$ is* $\mathrm{Ad}\,(MA)|_{\mathfrak{n}_1}$*-contact.*

PROOF. By composing with translations with elements of N, we may reduce the proof to showing that, if $g \cdot e = e$ (*i.e.*, if $g \in P$), then the differential of the map $n \mapsto g \cdot n$, acting on the subspace $\mathfrak{n}_1$ of the tangent space to N at e, lies in $\mathrm{Ad}\,(MA)|_{\mathfrak{n}_1}$. This follows from Lemma 1. $\square$

We will discuss conformal maps later in this paper.

There is another family of algebraic maps of G/P which might have nice geometric properties. Suppose that Ξ is an automorphism of G and that $\Xi(P) \subseteq P$. Then the mapping $\dot{\Xi}: gP \mapsto \Xi(g)P$ is well-defined. If this automorphism is inner, then necessarily $\Xi(g) = pgp^{-1}$ for some p in P (since P is its own normaliser in G), whence $\dot{\Xi}(gP) = pgP$, and this is one of the maps we have studied. However, both $\mathrm{SL}(3\,,\mathbb{R})$ and $\mathrm{Sp}(2\,,\mathbb{R})$ have some additional automorphisms which fix P, and we now discuss these briefly.

The «flip» automorphism Ψ of $\mathrm{SL}(3\,,\mathbb{R})$ is defined to be the map sending the matrix (a_{ij}) to the matrix $(a_{4-j,4-i})^{-1}$; its differential ψ acts on $\mathfrak{g}$ sending (a_{ij}) to $(-a_{4-j,4-i})$. Clearly $\Psi(P) = P$. It is easy to see that every automorphism of $\mathrm{SL}(3\,,\mathbb{R})$ is either inner or is the product of an inner automorphism and the flip. At the Lie algebra level, $\psi(\mathfrak{g}_\alpha) = \mathfrak{g}_\beta$ and $\psi(\mathfrak{g}_\beta) = \mathfrak{g}_\alpha$. It follows that $\dot{\Psi}$ is a contact map but not a multicontact map.

The grading automorphism Z of $\mathrm{Sp}(2\,,\mathbb{R})$ is the map sending the matrix $(a_{i,j})$ to $((-1)^{i-j} a_{i,j})$. Its differential $\zeta: \mathfrak{g} \to \mathfrak{g}$ may be described as follows: we write the Lie algebra $\mathfrak{g}$ as $\mathfrak{g}_{\mathrm{odd}} \oplus \mathfrak{g}_{\mathrm{even}}$, where

$$\mathfrak{g}_{\mathrm{odd}} = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + \mathfrak{g}_\beta + \mathfrak{g}_{-\beta} + \mathfrak{g}_{2\alpha+\beta} + \mathfrak{g}_{-2\alpha-\beta}$$

$$\mathfrak{g}_{\mathrm{even}} = \mathfrak{m} + \mathfrak{a} + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{-\alpha-\beta};$$

the subscripts refer to the parity of the number of simple roots involved in the root spaces making up $\mathfrak{g}_{\mathrm{odd}}$ and $\mathfrak{g}_{\mathrm{even}}$. This is a grading of $\mathfrak{g}$, and the map ζ, given by

(1) It is possible to define R-contact maps on G/P provided that $R \supseteq \mathrm{Ad}\,(MA)$.

$$\zeta(X) = \begin{cases} +X & \text{if } X \in \mathfrak{g}_{\text{even}} \\ -X & \text{if } X \in \mathfrak{g}_{\text{odd}} \,, \end{cases}$$

is the differential of Z. The complex linear extension of Z to $\mathrm{Sp}(2\,,\mathbb{C})$ is inner, but Z itself is not. It is easy to check that every automorphism of $\mathrm{Sp}(2\,,\mathbb{R})$ which preserves P is either inner or a product of an inner automorphism and the grading automorphism Z. The induced map $\dot{\mathrm{Z}}$ of G/P is a multiconformal map, but (restricted to N) is not $\mathrm{Ad}\,(MA)|_{\mathfrak{n}_1}$-contact.

2. Multicontact mappings

The main result in this paper is the following partial converse of Theorem 2.

Theorem 4. *If $\mathcal{U}$ and $\mathcal{V}$ are connected open subsets of G/P and $f{:}\mathcal{U} \to \mathcal{V}$ is a multicontact map, then there is a unique element g of G such that f is the restriction to $\mathcal{U}$ of the map $xP \mapsto g\,xP$, or (in the case where $G = \mathrm{Sp}(2\,,\mathbb{R})$) the restriction to $\mathcal{U}$ of the map $xP \mapsto \dot{\mathrm{Z}}g\,xP$.*

Proof. The proof consists of several steps. Since NP is open and dense in G, there is no loss of generality in assuming that $\mathcal{U}$ and $\mathcal{V}$ are subsets of N.

2.1. *Step 1.*

We consider multicontact vector fields, that is, vector fields V on $\mathcal{U}$ whose local flow $\{\phi_t^V\}$ consists of multicontact maps. If $X_\gamma \in \mathfrak{g}_\gamma$ and γ is a simple root, then

$$\frac{d}{dt}(\phi_t^V)_*(X_\gamma)\Big|_{t=0} = -\mathcal{L}_V(X_\gamma) = [X_\gamma\,,\,V]$$

(where $\mathcal{L}$ denotes the Lie derivative). Thus a smooth vector field V on $\mathcal{U}$ is a multicontact vector field if and only if

$$[V, \mathfrak{g}_\gamma] \subseteq \mathfrak{g}_\gamma \qquad \text{for every simple root } \gamma\,. \tag{2}$$

We define a representation τ of the Lie algebra $\mathfrak{g}$ of G as a set of vector fields on N as follows:

$$(\tau(X)f)(n) = \frac{d}{dt}f(\exp(-tX)\cdot n)\Big|_{t=0}\,.$$

By Theorem 2, $\tau(X)$ is a multiconformal vector field on N and *a fortiori* a multicontact vector field. In Steps 2 and 3, we show that all multicontact vector fields arise in this way.

2.2. *Step 2.*

This is the crucial part of the proof. We show that a multicontact vector field has polynomial components in the chosen coordinate system. We discuss the cases where $G = \mathrm{SL}(3\,,\mathbb{R})$ and $G = \mathrm{Sp}(2\,,\mathbb{R})$ separately.

The case where $G = \mathrm{SL}(3, \mathbb{R})$. We write a vector field V on $\mathcal{U}$ as $fX + gY + hZ$, where f, g and h are smooth functions on $\mathcal{U}$ in the coordinates x, y, z and $\{X, Y, Z\}$ is the canonical basis of $\mathfrak{n}$ (viewed as left-invariant vector fields), *i.e.*,

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad Z = \frac{\partial}{\partial z}.$$

Clearly

$$[X, Y] = Z, \qquad [X, Z] = [Y, Z] = 0.$$

The multicontact vector field equations (2) state that $[V, X] = \lambda X$ and $[V, Y] = \mu Y$, for some smooth functions λ and μ on $\mathcal{U}$. These equations imply immediately that

$$\lambda X = -gZ - (Xf)X - (Xg)Y - (Xh)Z$$

$$\mu Y = fZ - (Yf)X - (Yg)Y - (Yh)Z ,$$

which in turn imply that

$$\begin{aligned} Xf &= -\lambda & \qquad Yg &= -\mu \\ Xg &= 0 & Yf &= 0 \\ Xh &= -g & Yh &= f . \end{aligned}$$

We see at once that f and g are determined by h and h itself satisfies the differential equations

$$X^2 h = Y^2 h = 0 . \tag{3}$$

The equation $X^2 h = \partial^2 h/\partial x^2 = 0$ has the general solution

$$h(x, y, z) = h_0(y, z) + xh_1(y, z) ,$$

for some functions h_0 and h_1. The equation $Y^2 h = 0$ then becomes

$$0 = \left(\frac{\partial^2}{\partial y^2} + 2x\frac{\partial^2}{\partial y \partial z} + x^2\frac{\partial^2}{\partial z^2}\right)(h_0 + xh_1) =$$

$$= x^3\left(\frac{\partial^2 h_1}{\partial z^2}\right) + x^2\left(\frac{\partial^2 h_0}{\partial z^2} + 2\frac{\partial^2 h_1}{\partial y \partial z}\right) + x\left(\frac{\partial^2 h_1}{\partial y^2} + 2\frac{\partial^2 h_0}{\partial y \partial z}\right) + \left(\frac{\partial^2 h_0}{\partial y^2}\right).$$

Since the right hand side vanishes identically in some open set, the coefficients of the various powers of x must vanish. Considering the x^3 term, we see that $\partial^2 h_1/\partial z^2 = 0$. Differentiating the coefficient of the x^2 term once with respect to z, we deduce that $\partial^3 h_0/\partial z^3 = 0$. Next, considering the constant term yields $\partial^2 h_0/\partial y^2 = 0$, and then differentiating the coefficient of the x term once with respect to y, we deduce that $\partial^3 h_1/\partial y^3 = 0$. Summarizing, we have shown that

$$\frac{\partial^3}{\partial z^3}h_0 = 0, \qquad \frac{\partial^2}{\partial z^2}h_1 = 0, \qquad \frac{\partial^2}{\partial y^2}h_0 = 0, \qquad \frac{\partial^3}{\partial y^3}h_1 = 0.$$

The first two equations imply that

$$h_0(y, z) = z^2 a(y) + zb(y) + c(y), \qquad h_1(y, z) = zd(y) + e(y),$$

and the second two equations then imply that $a'' = b'' = c'' = d''' = e''' = 0$, so that both h_0 and h_1 are polynomials, whence h is too.

The case where $G = \mathrm{Sp}(2, \mathbb{R})$. We view G as the group of four-by-four real matrices which preserve the symplectic form $(\vec{u}, \vec{v}) \mapsto u_1 v_4 + u_2 v_3 - u_3 v_2 - u_4 v_1$, with the Cartan subalgebra of diagonal matrices $\{H_{s,t} : s, t \in \mathbb{R}\}$, where $H_{s,t} = \mathrm{diag}(s, t, -t, -s)$. Let $\nu(u, x, y, z)$ denote the matrix

$$\begin{bmatrix} 0 & \frac{u}{\sqrt{2}} & \frac{y}{\sqrt{2}} & z \\ 0 & 0 & x & \frac{y}{\sqrt{2}} \\ 0 & 0 & 0 & -\frac{u}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and write U for $\nu(1, 0, 0, 0)$, X for $\nu(0, 1, 0, 0)$, and so on. Then the vectors U, X, Y and Z are orthonormal relative to the matrix inner product ($A \cdot B = \sum_{i,j} a_{i,j} b_{i,j}$) and hence are multiples of orthonormal vectors relative to the inner product derived from the Killing form ($A \cdot B = \mathrm{tr}(A\theta B)$). Further, we have the following commutation relations:

$$\begin{aligned}
&[H_{s,t}, U] = (s-t)U &\quad& [U, X] = Y &\quad& [X, Y] = 0 \\
&[H_{s,t}, X] = 2tX && [U, Y] = Z && [X, Z] = 0 \\
&[H_{s,t}, Y] = (s+t)Y && [U, Z] = 0 && [Y, Z] = 0 \\
&[H_{s,t}, Z] = 2sZ.
\end{aligned}$$

Thus we may write $\mathfrak{g}_\alpha$ for span $\{U\}$, $\mathfrak{g}_\beta$ for span $\{X\}$, $\mathfrak{g}_{\alpha+\beta}$ for span $\{Y\}$, and $\mathfrak{g}_{2\alpha+\beta}$ for span $\{Z\}$, and $\mathfrak{n}$ for their direct sum, and the standard root commutation relations hold. If we co-ordinatize N by writing (u, x, y, z) for $\exp(xX + yY + zZ)\exp(uU)$, then the left-invariant vector fields associated to U, X, Y and Z are

$$\frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial x} + u\frac{\partial}{\partial y} + \frac{u^2}{2}\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial y} + u\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial z}.$$

Take a multicontact vector field V on an open subset $\mathcal{U}$ of N, write V as $fU + gX + hY + kZ$, and consider the equations (2), which state that $[V, U] = \lambda U$ and $[V, X] = \mu X$, for some smooth functions λ and μ on $\mathcal{U}$. These equations yield

$$\begin{aligned}
\lambda U &= -gY - hZ - (Uf)U - (Ug)X - (Uh)Y - (Uk)Z \\
\mu X &= fY - (Xf)U - (Xg)X - (Xh)Y - (Xk)Z,
\end{aligned}$$

which in turn imply that

$$
\begin{aligned}
Uf &= \lambda & \qquad Xg &= -\mu \\
Ug &= 0 & Xf &= 0 \\
Uh &= -g & Xh &= f \\
Uk &= -h & Xk &= 0\,.
\end{aligned}
$$

We see at once that the components f, g and h are determined by k and that k itself satisfies the differential equations

$$Xk = U^3 k = 0\,. \tag{4}$$

The equation $U^3 k = \partial^3 k/\partial u^3 = 0$ has the general solution

$$k(u,x,y,z) = k_0(x,y,z) + uk_1(x,y,z) + u^2 k_2(x,y,z)\,,$$

for some functions k_0, k_1 and k_2. Hence the equation $Xk = 0$ becomes

$$
\begin{aligned}
0 &= \left(\frac{\partial}{\partial x} + u\frac{\partial}{\partial y} + \frac{u^2}{2}\frac{\partial}{\partial z}\right)(k_0 + uk_1 + u^2 k_2) = \\
&= \left(\frac{\partial k_0}{\partial x}\right) + u\left(\frac{\partial k_1}{\partial x} + \frac{\partial k_0}{\partial y}\right) + u^2\left(\frac{\partial k_2}{\partial x} + \frac{\partial k_1}{\partial y} + \frac{1}{2}\frac{\partial k_0}{\partial z}\right) + \\
&\qquad\qquad + u^3\left(\frac{\partial k_2}{\partial y} + \frac{1}{2}\frac{\partial k_1}{\partial z}\right) + u^4\left(\frac{1}{2}\frac{\partial k_2}{\partial z}\right)\,.
\end{aligned}
$$

By considering the term which is independent of u, and differentiating the u term once with respect to x and the u^2 term twice with respect to x, we see that k_i is a polynomial of degree i in x with coefficients that are functions of y and z, for all i. Similarly, by considering the u^4 term and differentiating the u^3 term once with respect to z and the u^2 term twice with respect to z, we see that k_i is a polynomial of degree $(2 - i)$ in z. Thus

$$
\begin{aligned}
k(u,x,y,z) &= k_{000}(y) + zk_{001}(y) + z^2 k_{002}(y) + \\
&\qquad + uk_{100}(y) + uxk_{110}(y) + uzk_{101}(y) + uxzk_{111}(y) + \\
&\qquad\qquad + u^2 k_{200}(y) + u^2 xk_{210}(y) + u^2 x^2 k_{220}(y)\,.
\end{aligned}
$$

Now X commutes with $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$. In particular $X\partial^2 k/\partial x^2 = 0$, $X\partial^2 k/\partial x\partial z = 0$ and $X\partial^2 k/\partial z^2 = 0$. These imply that $k'_{220} = 0$, $k'_{111} = 0$ and $k'_{002} = 0$. Thus

$$
\begin{aligned}
\frac{\partial k}{\partial y}(u,x,y,z) &= k'_{000}(y) + zk'_{001}(y) + \\
&\qquad + uk'_{100}(y) + uxk'_{110}(y) + uzk'_{101}(y) + u^2 k'_{200}(y) + u^2 xk'_{210}(y)\,.
\end{aligned}
$$

Further, $X\partial^2 k/\partial x\partial y = 0$ and $X\partial^2 k/\partial z\partial y = 0$. These imply that $k''_{110} = k''_{210} = 0$ and $k''_{001} = k''_{101} = 0$, whence

$$\frac{\partial^2 k}{\partial y^2}(u,x,y,z) = k''_{000}(y) + uk''_{100}(y) + u^2 k''_{200}(y)\,.$$

Finally, $X\partial^2 k/\partial y^2 = 0$. This implies that $k'''_{000} = k'''_{100} = k'''_{200} = 0$. In conclusion, each k_i is a polynomial of total degree at most 2.

2.3. *Step 3.*

The differential systems (3) and (4) can be integrated quite explicitly. The solutions of (3) are

$$h(x, y, z) = c_0 + c_1 x + c_2 y + c_3 z + c_4 xy + c_5 x(z - xy) + c_6 zy + c_7 z(z - xy) \ , \tag{5}$$

where $c_0, \ldots, c_7 \in \mathbb{R}$, and the solution of (4) are

$$\begin{aligned} k(u, x, y, z) = c_0 + c_1 u + c_2 u^2 + c_3(y - ux) + c_4(uy - 2z) + c_5 u(y - ux) + \\ + c_6 u(uy - 2z) + c_7(y - ux)^2 + c_8(y - ux)(uy - 2z) + c_9(uy - 2z)^2 \ , \end{aligned}$$

where $c_0, \ldots, c_9 \in \mathbb{R}$. Since the Lie algebras of multicontact vector fields contain $\tau(\mathfrak{g})$, as argued in Step 1, we conclude that in fact they coincide with these copies of $\mathfrak{g}$ for reasons of dimension.

The same conclusion, and more, may be inferred from Step 2 by general homogeneity arguments, using the polynomial nature of the solutions without integrating the differential systems. This is done by selecting a characteristic element H_0 in the Cartan subalgebra $\mathfrak{a}$, that is, an element satisfying $\gamma(H_0) = 1$ for all simple positive roots γ. A function f on N is said to be homogeneous of degree r if it does not vanish identically and it satisfies $\tau(H_0)f = rf$, and a vector field V is said to be homogeneous of degree s if it does not vanish identically and it satisfies $[\tau(H_0), V] = -sV$. Hence

$$\begin{aligned} \deg(fV) &= \deg(V) - \deg(f) \,, \\ \deg(V(f)) &= \deg(f) - \deg(V) \qquad (\text{except when } V(f) = 0) \,, \\ \deg([V, W]) &= \deg(V) + \deg(W) \quad (\text{except when } V \text{ and } W \text{ commute}) \,. \end{aligned}$$

In particular, all vector fields in the stratum $\mathfrak{n}_j$ are of degree j, and no polynomial vector field can have degree greater than h, the height of $\mathfrak{n}$, *i.e.*, the length of its stratification.

Next, we take a general multicontact vector field V, which is a polynomial solution of (3) or (4). We may write V as a sum of homogeneous components, each of which is also a solution of the equations.

Fix Y in $\mathfrak{n}_1$ and assume that $\deg(V) > 0$. Then

$$\deg([Y, V]) = \deg(Y) + \deg(V) = 1 + \deg(V) > 1 \ ,$$

so it cannot be true that $[Y, V] \in \mathfrak{n}_1$ unless $[Y, V] = 0$. Therefore $[Y, V] = 0$ for all Y in $\mathfrak{n}_1$, and since $\mathfrak{n}_1$ generates $\mathfrak{n}$, it follows that $[Y, V] = 0$ for all Y in $\mathfrak{n}$. If a vector field commutes with infinitesimal right translations, then it is an infinitesimal left translation. Consequently, $V = \tau(X)$ for some X in $\mathfrak{n}$, and multicontact vector fields of positive degree correspond to elements in $\mathfrak{n}$.

To treat the case where $\deg(V) < 0$, we consider the inversion map on N, induced by the action of s on G/P, where s is a representative of the longest Weyl group element. The induced map s_* has the property that $\deg(s_* V) = -\deg(V)$ for homogeneous vector fields V. Consequently, if V is a homogeneous multicontact vector field of negative degree, than $s_* V$ is of positive degree, so $s_* V = \tau(X)$ for some X in $\mathfrak{n}$, whence $V = \tau(s_*^{-1}X)$, and V corresponds to an element of $\bar{\mathfrak{n}}$.

Finally, if $\deg(V) = 0$, then ad V preserves both $\tau(\mathfrak{n})$ and $\tau(\bar{\mathfrak{n}})$, and hence also preserves $\tau(\mathfrak{g})$, from the Jacobi identity, because the algebra generated by $\tau(\mathfrak{n})$ and $\tau(\bar{\mathfrak{n}})$ is $\tau(\mathfrak{g})$. As ad V is a derivation of the semisimple Lie algebra $\tau(\mathfrak{g})$, there exists Y in $\mathfrak{g}$ such that $V - \tau(Y)$ commutes with $\tau(\mathfrak{g})$. If a vector field commutes with $\tau(\mathfrak{g})$, then in particular it commutes with infinitesimal left translations, so it is an infinitesimal right translation, and since it also commutes with dilations, it is zero. Hence $V = \tau(Y)$, and since $\deg(V) = 0$, it follows that $Y \in \mathfrak{m} + \mathfrak{a}$. We conclude our proof that the multicontact vector fields correspond to $\mathfrak{g}$ by recalling that $\mathfrak{g} = \mathfrak{n} + \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{a}$.

In the case where $G = \mathrm{SL}(3, \mathbb{R})$, it is easy to check the following correspondence (up to constants) between polynomials and Lie algebra generators, in the sense that the polynomial p corresponds to the unique multicontact vector field whose Z component is pZ:

$$\begin{array}{ll} Z \leftrightarrow 1 & \theta Z \leftrightarrow z(z - xy) \\ Y \leftrightarrow x & \theta Y \leftrightarrow yz \\ X \leftrightarrow y & \theta X \leftrightarrow x(z - xy) \\ H_\alpha \leftrightarrow z - 2xy & H_\beta \leftrightarrow z + xy\ , \end{array}$$

where H_α and H_β are the elements in $\mathfrak{a}$ that represent the simple roots by the Killing form. The above table should be compared with (5). One can draw up a similar table for $\mathrm{Sp}(2, \mathbb{R})$.

2.4. *Step 4.*

The final step involves integration. We wish to show that any multicontact map from $\mathcal{U}$ to $\mathcal{V}$ is the restriction to $\mathcal{U}$ of a translation by a group element. By composing with group translations, we may assume that $e \in \mathcal{U} \cap \mathcal{V}$, and it is enough to show that any multicontact map f from $\mathcal{U}$ to $\mathcal{V}$ which preserves the identity e is such a restriction.

We claim that the mapping f induces an automorphism of $\mathfrak{g}$. Indeed, if $V \in \tau(\mathfrak{g})$, then V is completely determined by $V|_{\mathcal{U}}$ and we may consider the vector field $f_* V$ defined only on $\mathcal{V}$. Since it too generates a local 1-parameter group of multicontact transformations, it determines a unique element in $\tau(\mathfrak{g})$. The induced automorphism of $\mathfrak{g}$ is the map $\tau^{-1} f_* \tau$.

Consider the action of G on G/P. The subgroup P fixes the coset P, and no other point. Consequently, the vector fields in $\tau(\mathfrak{m} + \mathfrak{a} + \bar{\mathfrak{n}})$ all vanish at e, and at no other point. Since f is a diffeomorphism and f fixes e, all the vector fields in the set $f_* \tau(\mathfrak{m} + \mathfrak{a} + \bar{\mathfrak{n}})$ also vanish at e. Thus the automorphism $\tau^{-1} f_* \tau$ of $\mathfrak{g}$ maps $\mathfrak{m} + \mathfrak{a} + \bar{\mathfrak{n}}$ into itself. Finally, since f is a multicontact map, then consideration of f_* at the identity shows that $\tau^{-1} f_* \tau$ also maps the positive simple root spaces $\mathfrak{g}_\gamma$ into themselves.

By composing the map f with the grading automorphism Z if necessary (in the case where $G = \mathrm{Sp}(2\,,\mathbb{R})$), we may suppose that $\tau^{-1} f_* \tau = \mathrm{Ad}\,(p)$, for some p in P. Then $F\colon n \mapsto p^{-1} \cdot f(n)$ is a diffeomorphism, $F(e) = e$, and the induced automorphism $\tau^{-1} F_* \tau$ is trivial. Since $F_* \tau(X) = \tau(X)$ for all X in $\mathfrak{n}$, F commutes with left translations, and so is a right translation. Since $F(e) = e$, it follows that F is trivial, and so $f(n) = p \cdot n$ for all n in $\mathcal{U}$, as required. □

3. Contact mappings

In contrast with the case of multicontact mappings, the space of contact mappings, that is, mappings preserving the contact plane $\mathfrak{n}_1 = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$, is infinite-dimensional. This may be proved by considering the space of contact vector fields, that is, vector fields which give rise to flows of contact mappings, and showing that this space is infinite-dimensional. We consider the analogue of equations (2), namely

$$[V, \mathfrak{g}_\alpha + \mathfrak{g}_\beta] \subseteq \mathfrak{n}_1 \,, \tag{6}$$

and analyze this, using the notation introduced in Step 2 above.

The case where $G = \mathrm{SL}(3\,,\mathbb{R})$. The equations (6) are equivalent to $f = -Yh$ and $g = Xh$. Thus h determines both f and g but is not itself subject to any condition. Then the space of contact vector fields on N corresponds to the space of smooth functions on N.

The case where $G = \mathrm{Sp}(2\,,\mathbb{R})$. The equations (6) are equivalent to $h = -Uk$, $g = -Uh = U^2 k$ and $f = Xh = -XUk$, where k satisfies the single equation $Xk = 0$. The space of solutions to $Xk = 0$ on $\mathcal{U}$ therefore contains the space

$$\left\{ \left(-u^2 x + \frac{1}{2} uy + z \right) g(u) : g \in C^\infty(\mathcal{U}) \right\} \,,$$

and is infinite-dimensional.

4. Conformal mappings

The situation with conformal mappings (in the sense defined above) is perhaps the most curious. In both cases, the set of conformal mappings is finite-dimensional, but in one case it is (nearly) a subgroup of G and in the other case it is not. Again, this is proved by considering the corresponding class of vector fields. It is not hard to show that the equations describing conformality of vector fields are the contact equations (*i.e.*, (6) above), together with the additional condition that ad V, restricted to the contact plane, must lie in $\mathbb{R}\,\mathrm{id} \oplus \mathfrak{so}(2)$.

The case where $G = \mathrm{SL}(3\,,\mathbb{R})$. If $\mathcal{U}$ and $\mathcal{V}$ are open subsets of N and $f\colon\mathcal{U} \mapsto \mathcal{V}$ is conformal, then by results of Korányi and Reimann [6], f is actually the restriction to $\mathcal{U}$ of a map of the Heisenberg group N coming from the group $\mathrm{SU}(2\,,1)$, which also has a Heisenberg group as its Iwasawa N subgroup. Most conformal maps do not extend naturally to all of G/P.

The case where $G = \mathrm{Sp}(2, \mathbb{R})$. If $V = fU + gX + hY + kZ$ and V is a conformal vector field, then, as before, $h = -Uk$, $g = -Uh = U^2 k$ and $f = Xh = -XUk$, where k satisfies the single equation $Xk = 0$.

Further, the matrix

$$\begin{bmatrix} Uf & Xf \\ Ug & Xg \end{bmatrix}$$

must satisfy the Cauchy-Riemann equations: $Uf = Xg$ and $Xf = -Ug$. Thus k determines everything, and k satisfies the three equations

$$Xk = 0\,, \qquad XU^2 k = -UXUk\,, \qquad U^3 k = -X^2 Uk\,.$$

Since $Xk = 0$, the last equation implies that

$$U^3 k = -X^2 Uk = -X(XU - UX)k = XYk = YXk = 0\,.$$

Thus k certainly satisfies the equations $Xk = U^3 k = 0$, and f, g, h are determined as in the multicontact case. The additional information is the equation $XU^2 k = -UXUk$.

Solving these equations, much as before, leads to the conclusion that there are constants $c_0, c_1, \ldots, c_4$ such that

$$k(u, x, y, z) = c_0 + c_1 u + c_2 u^2 + c_3(y - ux) + c_4(4uy - u^2 x - 6z)\,.$$

The corresponding elements of $\mathfrak{g}$ are those which, integrated, give rise to left translations on N by elements of N, and the dilations $(u, x, y, z) \mapsto (su, sx, s^2 y, s^3 z)$, where $s \in \mathbb{R}^+$. The group of automorphisms of the Lie algebra of conformal vector fields which induce conformal maps is generated by inner automorphisms from MAN and the grading automorphism. Similar arguments to those for the multicontact case allow us to deduce that the conformal maps of N are generated by left translations, dilations and the grading automorphism.

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