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A TRIPLE RATIO ON THE SILOV BOUNDARY OF A BOUNDED SYMMETRIC DOMAIN

Abstract. — Let D be a Hermitian symmetric space of tube type, S its Silov boundary and G the neutral component of the group of bi-holomorphic diffeomorphisms of D. Our main interest is in studying the action of G on $S^3 = S \times S \times S$. Sections 1 and 2 are part of a joint work with B. Ørsted (see [4]). In Section 1, as a pedagogical introduction, we study the case where D is the unit disc and S is the circle. This is a fairly elementary and explicit case, where one can easily get a flavour of the more general results. In Section 2, we study the case of tube type domains, for which we show that there is a finite number of open G-orbits in S^3, and to each orbit we associate an integer, called the Maslov index. In the special case where D is the Siegel disc, then G is (isomorphic to) the symplectic group and S is the manifold of Lagrangian subspaces. The result on the orbits and the number which we construct coincides with the classical theory of the *Maslov index* (see *e.g.* [7]), hence the name. We describe a formula for computing the Maslov index, using the automorphy kernel of the domain D. In the special case of the Lagrangian manifold, this formula was obtained by Magneron [8] in a different approach. In Section 3, we study the case where D is the unit ball in a (rectangular) matrix space. There is now an infinite family of orbits, and we construct characteristic invariants for the action of G on S^3. For the special case where D is the unit ball in $\mathbb{C}^2$, this coincides with an invariant constructed by E. Cartan for the «hypersphere» (see [2]). In all cases, we follow the following method: from an appropriate automorphy kernel for D we construct a kernel on $D \times D \times D$, satisfying a simple transformation property under the action of G. We then define a dense open set of S^3 (the set of mutually transversal points in S), on which the kernel (or some function of it) can be extended continuously, and the resulting kernel is invariant or at least transforms nicely under the action of G.

Key words: Bounded symmetric domains; Silov boundary; Maslov index.

1. The circle

Let S be the circle, and denote by

$$S^3_\top = \{(\sigma_1 , \sigma_2 , \sigma_3) \in S^3 , \sigma_i \neq \sigma_j \text{ for } i \neq j\} .$$

Thinking of S as the projective line $P_1(\mathbb{R})$, it is a basic result in projective geometry that the group $PGL(2 , \mathbb{R})$ acts transitively on $S^3_\top$. If we think of S as the boundary $\partial\mathcal{D}$ of the open unit disc in the complex plane

$$\mathcal{D} = \left\{ z = x + iy \in \mathbb{C} , |z| < 1 \right\} ,$$

then it is natural to look for the action of $G = PSU(1 , 1)$ ($\simeq$ the connected component of $PGL(2 , \mathbb{R})$) on $S^3_\top$. There are now **two** orbits, due to the fact that any bi-holomorphic map of $\mathcal{D}$ extends to a neighbourhood of $\overline{\mathcal{D}}$ and induces a transformation of S which preserves the orientation of S. An equivalent way of looking at this result is to introduce the Maslov index. Let $\sigma_1 , \sigma_2 , \sigma_3$ be three distinct points on S. Then starting from σ_1 and travelling counter-clockwise, one meets first either σ_2 or σ_3. The *Maslov index* $\iota(\sigma_1 , \sigma_2 , \sigma_3)$ is defined to be $+\,1$ in the first case and -1

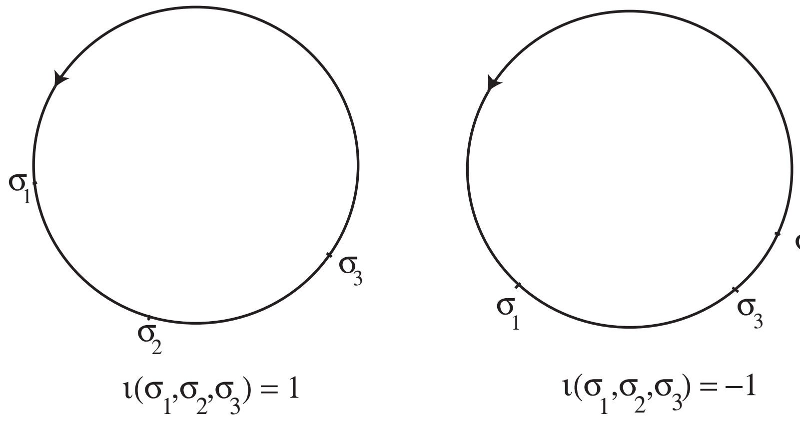

Fig. 1.

in the second case, as shown on figure 1. The Maslov index is clearly invariant under $PSU(1\,,1)$ and characterizes the two orbits of G in $S^3_\top$.

Another way of computing the Maslov index is to use the *ideal triangle* having $\sigma_1\,,\sigma_2\,,\sigma_3$ as summits. Recall that the sides of the ideal triangle are (infinite) geodesics for the Poincaré metrics on $\mathcal{D}$, hence (arcs of) circles orthogonal to S.

Now a very classical result shows that the area of such an ideal triangle $\mathcal{T}$ is finite and equal to π, with the standard normalization of the Poincaré metrics. More precisely, it is possible to define the *oriented* area $\mathcal{A}(T)$, which is $+\,\pi$ if travelling along the sides of the ideal triangle following the order $\sigma_1\,,\sigma_2\,,\sigma_3$ corresponds to the counter-clockwise orientation, and $-\pi$ in the opposite case. Then we have the formula

$$\iota(\sigma_1\,,\sigma_2\,,\sigma_3) = \frac{1}{\pi}\mathcal{A}(\mathcal{T})\,.$$

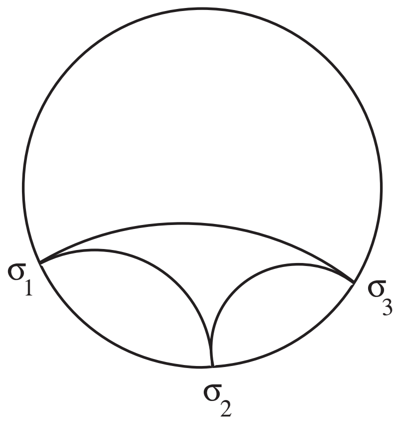

Fig. 2.

An ideal triangle may be seen as a limit of a geodesic triangle $(z_1\ z_2\ z_3)$, where $z_i \in \mathcal{D}$ for $1 \leq i \leq 3$, whose summits tend respectively to the boundary points $\sigma_1, \sigma_2, \sigma_3$. But formulae for the area of a geodesic triangle are known, and for instance, it is not difficult to prove that for the geodesic (oriented) triangle $\mathcal{T} = (z_1\ z_2\ z_3)$ its area $\mathcal{A}(T)$ is given by

$$\mathcal{A}(\mathcal{T}) = \arg\left(\frac{1 - z_1\overline{z}_2}{1 - \overline{z}_1 z_2}\right) + \arg\left(\frac{1 - z_2\overline{z}_3}{1 - \overline{z}_2 z_3}\right) + \arg\left(\frac{1 - z_3\overline{z}_1}{1 - \overline{z}_3 z_1}\right).$$

In the formula, an analysis-minded person will certainly recognize the role of the *automorphy kernel* of $\mathcal{D}$. For $z, w \in \mathcal{D}$, define

$$k(z, w) = 1 - z\overline{w}.$$

This kernel is well defined and does not vanish on $\mathcal{D} \times \mathcal{D}$, is holomorphic in z, antiholomorphic in w and obeys a simple transformation rule under the action of $PSU(1, 1)$, namely, for $g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in PSU(1, 1)$

$$k(gz, gw) = (\overline{\beta}z + \overline{\alpha})^{-1} k(z, w)(\beta\overline{w} + \alpha)^{-1}.$$

Then define for $z_1, z_2, z_3 \in \mathcal{D}$

$$c(z_1, z_2, z_3) = k(z_1, z_2)k(z_2, z_1)^{-1} k(z_2, z_3)k(z_3, z_2)^{-1} k(z_3, z_1)k(z_1, z_3)^{-1}.$$

As $\mathcal{D}$ is simply connected, it is possible to define an argument for the function c over $\mathcal{D} \times \mathcal{D} \times \mathcal{D}$, so that

$$\mathcal{A}(\mathcal{T}) = \arg c(z_1, z_2, z_3).$$

Then, for $(\sigma_1, \sigma_2, \sigma_3) \in S^3_\top$

$$\iota(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{\pi} \lim \arg c(z_1, z_2, z_3)$$

for $z_j \longrightarrow \sigma_j, j = 1, 2, 3$.

This formula will be the key to generalize the Maslov index in the higher rank case.

2. Tube type domains and the generalized Maslov index

Let $D = G/K$ be a Hermitian symmetric space of the non-compact type. The group G is a real semi-simple Lie group, whose Lie algebra $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let J be the complex structure on $\mathfrak{p}$, and accordingly, decompose the complexification $\mathfrak{p}^{\mathbb{C}}$ as $\mathfrak{p}^+ \oplus \mathfrak{p}^-$, where $J_{|\mathfrak{p}^+} = i, J_{|\mathfrak{p}^-} = -i$.

The Harish Chandra's embedding realizes G/K as a bounded domain D in $\mathfrak{p}^+$.

Let S be its Silov boundary. Then D is of tube-type if

$$\dim_{\mathbb{R}} S = \dim_{\mathbb{C}} D.$$

There is a nice approach to tube-type domains through the theory of *Euclidean Jordan algebras*. By a Euclidean Jordan algebra we mean a (finite dimensional) vector space

V over $\mathbb{R}$, with a bilinear map $V \times V \longrightarrow V$, a unit element e and an inner product $\langle\ ,\ \rangle$, such that

$$(J1) \qquad xy = yx\,, \quad ex = x$$

$$(J2) \qquad x^2(xy) = x(x^2y)$$

$$(J3) \qquad \langle x\,, yz\rangle = \langle xy\,, z\rangle$$

for all $x, y, z \in V$.

The basic example is $V = \mathrm{Sym}\,(r, \mathbb{R})$ with the Jordan multiplication $x.y = \frac{1}{2}(xy + yx)$, the unit element $e = I_r$ and the inner product $\langle x, y\rangle = \mathrm{Tr}\,(xy)$. If $r = 1$, then the corresponding Jordan algebra is just $\mathbb{R}$ with its usual sum and multiplication.

We refer to [5] for details on the structure of the Euclidean Jordan algebras. There is a spectral analysis for elements of V, and in particular there is a specific linear form, called the *trace* and denoted by tr, and a certain polynomial on V called the *determinant* and denoted by det. The degree r of this polynomial is called the rank of the Jordan algebra.

Denote by $L(x)$ the multiplication operator $y \mapsto xy$, and let

$$x\Box y = L(xy) + [L(x, L(y)]\,.$$

Let $\mathbb{V}$ the complexification of V and extend all previously defined operators by $\mathbb{C}$-linearity. Then

$$D = \{z \in \mathbb{V}\,,\ \ I - z\Box\overline{z} \gg 0\}$$

is a bounded symmetric domain of tube-type, and all such domains are obtained by this process. Let G be the neutral component of the group of all bi-holomorphic transforms of D.

The Silov boundary of D is easily described as

$$S = \{z \in \mathbb{V}\,, \overline{z} = z^{-1}\}\,.$$

The reason to call these domains «of tube type» comes from another realization, which we now describe. Let $\Omega = \{x \in V \mid L(x) \gg 0\}$. It is an open proper convex self-adjoint cone in V, which is homogeneous under the action of $G(\Omega)$, the group of linear transformations preserving Ω. Now form the *tube*

$$T_\Omega = V + i\Omega \subset \mathbb{V}\,.$$

Define the *Cayley transform* c by the formula

$$c(w) = i(e + w)(e - w)^{-1}\,.$$

Then c is well-defined on D and it is a bi-holomorphic map from D onto T_Ω.

The corresponding group $G(T_\Omega) = c \circ G \circ c^{-1}$ is given a simple description: it the group generated by

i) the translations $t_v : z \mapsto z + v$ with $v \in V$
ii) the complexified action of the linear group $G(\Omega)$
iii) the inversion $s : z \mapsto -z^{-1}$.

The Cayley transform extends to (almost all of) the Silov boundary, and maps it onto $V \subset \overline{T}_\Omega$. The (rational) action of group $G(T_\Omega)$ on V can be seen as a realization of the *conformal group* of the Jordan algebra V (see [1]).

Example. For $V = \mathrm{Sym}\,(r, \mathbb{R})$, we get

$$T_\Omega = \text{Siegel half-plane}$$
$$G(T_\Omega) \simeq Sp_{2r}(\mathbb{R})$$
$$S \simeq \text{manifold of Lagrangian subspaces in } \mathbb{R}^{2r}.$$

We are interested in the action of G on triplets of points in S. We need the important notion of transversality.

Two points $\sigma, \zeta \in S$ are said to be *transverse* (we then denote this property by $\sigma \top \zeta$) iff $\det(\sigma - \zeta) \neq 0$. A few facts are easily established for this relation. It is stable under the action of G. The elements $\sigma \in S$ transverse to e are exactly the points of S for which the Cayley transform c is well defined, and so the set of such elements is in $1-1$ correspondance with V under the Cayley transform.

Let

$$S^3_\top = \left\{ (\sigma_1, \sigma_2, \sigma_3) \in S^3, \sigma_i \top \sigma_j, \text{ for } i \neq j \right\}.$$

Theorem 1. *G has a finite number of orbits in $S^3_\top$.*

To give a precise description of the orbits, we need a *Peirce decomposition* $e = c_1 + c_2 + \cdots + c_r$, where the $(c_i), 1 \leq i \leq r$ form a system of primitive orthogonal idempotents. For $0 \leq j \leq r$, let

$$\epsilon_j = \sum_{i=1}^{j} c_i - \sum_{i=j+1}^{r} c_i .$$

Then $\{(e, -e, -i\epsilon_j), 0 \leq j \leq r\}$ is an exhaustive family of representatives of the orbits in $S^3_\top$.

Define the generalized *Maslov index* by

$$\iota(\sigma_1, \sigma_2, \sigma_3) = j - (r - j) = 2j - r$$

if $(\sigma_1, \sigma_2, \sigma_3)$ belongs to the orbit of $(e, -e, -i\epsilon_j)$.

By construction, the Maslov index is invariant under the action of G, but the definition is not really useful for investigating its properties. Inspired by the results presented for the circle in the first section, we similarly introduce the (scalar) *canonical automorphy kernel* $k(z, w)$ for the domain D. Its definition is somewhat involved (see [9] for a precise statement). The Bergman kernel (familiar to classical analysts) is a certain power of this scalar automorphy kernel. It is well defined and does not vanish on $D \times D$, is holomorphic in z, satisfies the symmetry property $k(z, w) = \overline{k(w, z)}$ (hence is antiholomorphic in w) and transforms under the action of G by the following rule:

$$k(g(z), g(w)) = j(g, z) k(z, w) \overline{j(g, w)}$$

where $j(g, z) = \frac{d(g.z)}{dz}$ is the so called *automorphy factor*.

Now as before, let for z_1, z_2, z_3

$$c(z_1, z_2, z_3) = k(z_1, z_2)k(z_2, z_1)^{-1}k(z_2, z_3)k(z_3, z_2)^{-1}k(z_3, z_1)k(z_1, z_3)^{-1}.$$

There is a continuous determination of the argument of c on $D \times D \times D$ which can be continued to $S^3_\top$.

THEOREM 2. *For* $(\sigma_1, \sigma_2, \sigma_3) \in S^3_\top$

$$i(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{2\pi} \arg c(\sigma_1, \sigma_2, \sigma_3).$$

The proof consists in proving the formula for the canonical representatives of the different orbits as described after Theorem 1.

The main (and immediate) consequence of this theorem is the *cocycle relation* for the generalized Maslov index,

$$\iota(\sigma_1, \sigma_2, \sigma_3) = \iota(\sigma_1, \sigma_2, \sigma_4) + \iota(\sigma_2, \sigma_3, \sigma_4) + \iota(\sigma_3, \sigma_1, \sigma_4)$$

if $\sigma_i \top \sigma_j$, for $i \neq j$. In fact, from its definition it is easily seen that c satisfies a similar (multiplicative) cocycle relation on $D \times D \times D$. Taking the argument and passing to the limit gives the relation for the Maslov index.

APPENDIX. Classification of tube type domains and their Silov boundaries.

V	$\mathbb{V}$	$\mathcal{D} \simeq G/U$	S
$\mathrm{Sym}(r, \mathbb{R})$	$\mathrm{Sym}(r, \mathbb{C})$	$\mathrm{Sp}(2r, \mathbb{R})/U(r)$	$U(r)/O(r)$
$\mathrm{Herm}(r, \mathbb{C})$	$\mathrm{Mat}(r, \mathbb{C})$	$SU(r, r)/S(U(r) \times U(r))$	$U(r)$
$\mathrm{Herm}(r, \mathbb{H})$	$\mathrm{Skew}(2r, \mathbb{C})$	$SO^*(4m)/U(2r)$	$U(2r)/SU(r, \mathbb{H})$
$\mathbb{R} \times \mathbb{R}^{n-1}$	$\mathbb{C} \times \mathbb{C}^{n-1}$	$SO_0(2, n)/SO(2) \times SO(n)$	$(U(1) \times S^{q-1})/\mathbb{Z}_2$
$\mathrm{Herm}(3, \mathbb{O})$	$\mathrm{Mat}(3, \mathbb{O})$	$E_{7(-25)}/U(1).E_6$	$U(1).E_6/F_4$

REMARK. In Section 1, we use the relation of the kernel $c(z_1, z_2, z_3)$ with the area of the geodesic triangle having z_1, z_2, z_3 as summits. In the case of a general tube type domain, a similar interpretation is possible. Instead of the area, one has to integrate the canonical Kähler 2-form of the domain D on any surface bounded by the geodesic triangle through the three points. As the Kähler form is closed, the integral does not depend on the surface. In turn, this is related to the study of the bounded cohomology of Hermitian symmetric spaces (see *e.g.* [10]).

3. THE UNITARY STIEFEL MANIFOLD AND E. CARTAN'S INVARIANT

Similar ideas can be used for Hermitian symmetric spaces not of tube type. It is no longer true that there are a finbite number of open orbits in $S^3_\top$, but it is still possible to construct characteristic invariants. We treat here the case of the unit ball in a matrix space.

Let $p \geq q$ and

$$D = \left\{ z \in \mathrm{Mat}\,(p \times q\,, \mathbb{C}) \mid \mathbb{I}_q - z^* z \gg 0 \right\}$$

It is a bounded symmetric domain, with group of holomorphic transforms $G = SU(p\,, q)$ acting by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} g(z) = (az + b)(cz + d)^{-1}\,.$$

The corresponding (matrix valued) automorphy factor is

$$j(g\,, z) = cz + d\,.$$

If $z \in D$ and $g \in G$, then $j(g\,, z)$ is invertible.

The Silov boundary of D is the *unitary Stiefel manifold*

$$S = \left\{ \sigma \in \mathrm{Mat}\,(p \times q\,, \mathbb{C}) \mid \sigma^* \sigma = \mathbf{1}_q \right\}.$$

Let $\sigma\,, \xi$ be two elements of S. The transversality condition now reads

$$\sigma \top \xi \Longleftrightarrow \det\,(\mathbf{1}_q - \xi^* \sigma) \neq 0$$

and define $S^3_\top$ as before.

If $p \neq q$, then there are an infinite number of orbits in $S^3_\top$. It is possible to describe representatives of each G-orbit, but it requires some more work (see [3] for details). There is also a Cayley transform in this case. The corresponding domain $c(D)$ is no longer of tube type, but is a Siegel domain of type II, which makes the description more complicated. Here we concentrate on the construction of characteristic invariants by a process similar to what we did for tube-type domains.

For $z\,, w \in D$, let

$$k(z\,, w) = (\mathbb{I}_q - w^* z)^{-1}\,.$$

It takes values in $GL(q\,, \mathbb{C})$ and is a (kind of) matrix-valued automorphy kernel for D, satisfying holomorphy properties and a simple transformation rule under the action of G. Form

$$T(z_1\,, z_2\,, z_3) = k(z_1\,, z_2) k(z_3\,, z_2)^{-1} k(z_3\,, z_1)\,.$$

The transformation property of the automorphy kernel under G implies the following transformation property for T:

$$T(g(z_1)\,, g(z_2)\,, g(z_3)) = j(g\,, z_1)\, T(z_1\,, z_2\,, z_3) j(g\,, z_1)^*\,.$$

One can prove that T can be extended by continuity to $S^3_\top$. The limit, denoted by $T(\sigma_1\,, \sigma_2\,, \sigma_3)$ is still invertible.

THEOREM 3. *Let $(\sigma_1\,, \sigma_2\,, \sigma_3)$ and $(\tau_1\,, \tau_2\,, \tau_3)$ belong to $S^3_\top$. They belong to the same G-orbit if and only if $T(\sigma_1\,, \sigma_2\,, \sigma_3)$ and $T(\tau_1\,, \tau_2\,, \tau_3)$ belong to the same $GL(q\,, \mathbb{C})$-orbit for the action*

$$(\gamma\,, X) \longmapsto \gamma X \gamma^*$$

of $GL(q\,, \mathbb{C})$ on $\mathrm{Mat}\,(q\,, \mathbb{C})$.

The determination of the orbits of this action is possible, but there is another version of this theorem, which refers now to the (more natural) action of $GL(q, \mathbb{C})$ by conjugacy. Introduce the associated *angular matrix*

$$A(z_1, z_2, z_3) = T(z_1, z_2, z_3)^{*^{-1}} T(z_1, z_2, z_3).$$

Now, under the action of G, the matrix-valued function A transforms by conjugacy (we use the notation $A \sim B$ to denote conjugacy under $GL(q, \mathbb{C})$). The kernel A still extends by continuity to $S^3_\top$. We also need to construct a continuous determination of $\arg\det\, T(z_1, z_2, z_3)$ on $D \times D \times D$ and its extension to $S^3_\top$.

THEOREM 4. *Let* $(\sigma_1, \sigma_2, \sigma_3)$ *and* (τ_1, τ_2, τ_3) *belong to* $S^3_\top$. *They belong to the same* G*-orbit if and only if*

$$A(\sigma_1, \sigma_2, \sigma_3) \sim A(\tau_1, \tau_2, \tau_3)$$

and

$$\arg\det\, T(\sigma_1, \sigma_2, \sigma_3) = \arg\det\, T(\tau_1, \tau_2, \tau_3).$$

In the special case where $p = 2$ and $q = 1$, the first condition is void and the second corresponds to an invariant already constructed by E. Cartan in 1935 (see [2]). If (slightly) more genrally, we consider the unit ball in $\mathbb{C}^p$, then the Silov boundary coincides with the topological boundary and is just the unit sphere. Using a Cayley transform, the boundary is realized as the $(2p-1)$-dimensional Heisenberg group under the action of its group of «conformal transformations» $PSU(p, 1)$. M. Reimann and A. Korányi in 1987, working on this model of the Silov boundary essentially rediscovered this invariant (see [6]).

REFERENCES

[1] W. BERTRAM, *Un théorème de Liouville pour les algèbres de Jordan*. Bull. Math. Soc. France, 124, 1996, 299-327.

[2] E. CARTAN, *Sur le groupe de la géométrie hypersphérique*. Comm. Math. Helv., 4, 1932, 158-171.

[3] J.-L. CLERC, *A triple ratio on the unitary Stiefel manifold*. L'Enseignement Mathématique, to appear.

[4] J.-L. CLERC - B. ØRSTED, *The Maslov index revisited*. Transformation Groups, to appear.

[5] J. FARAUT - A. KORÁNYI, *Analysis on symmetric cones*. Oxford Mathematical Monographs, Clarendon Press, Oxford 1994.

[6] A. KORÁNYI - H.M. REIMANN, *The complex cross ratio on the Heisenberg group*. L'Ens. Math., 33, 1987, 291-300.

[7] G. LION - M. VERGNE, *The Weil representation, Maslov index and Theta series*. Progress in Mathematics, 6, Birkhäuser, Boston 1980.

[8] B. MAGNERON, *Spineurs symplectiques purs et indice de Maslov de plans lagrangiens positifs*. J. Funct. Anal., 59, 1984, 90-122.

[9] I. Satake, *Algebraic structures of symmetric domains*. Kanô Memorial Lectures, 4, Iwanami Shoten and Princeton University Press, Princeton 1980.
[10] D. Toledo, *Representations of surface groups in complex hyperbolic space*. J. of Diff. Geom., 29, 1989, 125-133.

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