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THREE RELATED PROBLEMS OF BERGMAN SPACES OF TUBE DOMAINS OVER SYMMETRIC CONES

Abstract. — It has been known for a long time that the Szegö projection of tube domains over irreducible symmetric cones is unbounded in L^p for $p \neq 2$. Indeed, this is a consequence of the fact that the characteristic function of a disc is not a Fourier multiplier, a fundamental theorem proved by C. Fefferman in the 70's. The same problem, related to the Bergman projection, deserves a different approach. In this survey, based on joint work of the author with D. Békollé, G. Garrigós, M. Peloso and F. Ricci, we give partial results on the range of p for which it is bounded. We also show that there are two equivalent problems, of independent interest. One is a generalization of Hardy inequality for holomorphic functions. The other one is the characterization of the boundary values of functions in the Bergman spaces in terms of an adapted Littlewood-Paley theory. This last point of view leads naturally to extend the study to spaces with mixed norm as well.

Key words: Whitney decomposition; Symmetric cone; Bergman projector; Littlewood-Paley; Hardy inequality.

1. Introduction

Let Ω be an irreducible symmetric cone in the Euclidean space V, and $T_\Omega = V + i\Omega$ the corresponding tube domain in the complexified space $V^{\mathbb{C}}$. We shall note n the dimension of V and r the rank of Ω. Moreover, we shall denote by $(x|y)$ the scalar product in V, and by Δ the determinant function. For the description of such cones Ω in terms of Jordan algebras, one may use the book of Faraut and Korányi [8]. One may also have in mind the typical example that one obtains when V is the space of real symmetric $r \times r$ matrices, and Ω is the cone of positive definite matrices. In this example, the scalar product on V is induced by the Hilbert-Schmidt norm of the matrices, and the determinant function is given by the determinant of the matrices. The rank is r, while the dimension is $\frac{r(r+1)}{2}$.

We shall also make use of the *generalized wave operator* on V, given by $\Box = \Delta(\frac{1}{i}\frac{\partial}{\partial x})$. This is a differential operator of degree r, defined by the equality

$$\Delta\left(\frac{1}{i}\frac{\partial}{\partial x}\right)[e^{i(x|\xi)}] = \Delta(\xi)e^{i(x|\xi)}\,, \quad \xi \in V\,.$$

It is the usual derivative (up to a constant) when Ω is the half-line $(0\,,\infty)$. Its name is due to another fundamental example, given by the forward light cone in $\mathbb{R}^n$,

$$\left\{x \in \mathbb{R}^n;\, x_1 > \sqrt{x_2^2 + \cdots + x_n^2}\right\}\,,$$

which is of rank 2. In this case, the determinant function is equal to

$$\Delta(x) = x_1^2 - x_2^2 - \cdots - x_n^2\,,$$

and the operator $\Box$ is the wave operator. One may look at [5, 2], which deal with this particular case.

For $1 \le p, q < \infty$, let $A^{p,q}_\nu = A^{p,q}_\nu(T_\Omega)$ denote the weighted Bergman spaces in the tube domain T_Ω, that is the space of of holomorphic functions $F \in \mathcal{H}(T_\Omega)$, satisfying the integrability condition

$$\|F\|_{A^{p,q}_\nu} = \|F\|_{L^{p,q}_\nu} = \left[\int_\Omega \left[\int_V |F(x+iy)|^p\,dx\right]^{\frac{q}{p}} \Delta^{\nu-\frac{n}{r}}(y)\,dy\right]^{\frac{1}{q}} < \infty\,. \tag{1.1}$$

We shall impose $\nu > \frac{n}{r} - 1$ to avoid trivial cases where $A^{p,q}_\nu = \{0\}$.

The mixed Lebesgue spaces $L^{p,q}_\nu$ are defined in an obvious way. When $p = q$, we write L^p_ν and A^p_ν to simplify. The space $A^{p,q}_\nu$ is a closed subspace of $L^{p,q}_\nu$.

The case $p = q = 2$ is of special interest. It is well-known that every $F \in A^2_\nu$ can be written as

$$F(z) = \mathcal{L}f(z) = \int_\Omega e^{i(z|\xi)}\hat{f}(\xi)\,d\xi\,, \quad z \in T_\Omega\,, \tag{1.2}$$

for some function $\hat{f} \in L^2(\Omega; \Delta^{-\nu}(\xi)d\xi)$ (see [8, Chapter XIII]). The operator $\mathcal{L}$ will be called the *Fourier-Laplace transform* of f (using the usual terminology, it is the Laplace transform of its Fourier transform). The function f may be seen as the (Shilov) boundary value of the holomorphic function F. The orthogonal projection from L^2_ν onto A^2_ν, which is called the (weighted) Bergman projection, is denoted by P_ν and explicitly given by

$$P_\nu F(z) = \int_{T_\Omega} B_\nu(z - \overline{w})F(w)\,\Delta(\mathrm{Im}\ w)^{\nu-\frac{n}{r}}\,dw\,,$$

where $B_\nu(z-\overline{w}) = d(\nu)\Delta^{-(\nu+\frac{n}{r})}((z-\overline{w})/i)$ is the reproducing kernel of A^2_ν (see [8]). For simplification, we have written $dw = du\,dv$, for $w = u + iv$ an element of T_Ω.

We can now state the three problems under consideration in this survey.

Problem 1. *Boundedness of the Bergman projection.* The question, here, is to know the exact range of p, q for which the projection P_ν extends as a bounded operator on $L^{p,q}_\nu$. For obvious reasons (self-adjointness and interpolation), the set of couples $(\frac{1}{p}, \frac{1}{q})$ for which it is bounded is a convex set in $(0,1)\times(0,1)$, which is symmetric around $(\frac{1}{2}, \frac{1}{2})$.

Let us recall that, for the upper half-plane, this convex set is the whole square $(0,1)\times(0,1)$. For higher rank, the situation is different. From the convexity and symmetry given above, we may restrict our interest to values of q which are larger than 2. We shall first see that there is a *small critical index* $q_\nu > 2$ such that P_ν defines a bounded operator on $L^{p,q}_\nu$ for $2 \le q < q_\nu$ for all values of p. Moreover, in this range, one has still a bounded operator when the kernel B_ν is replaced by its absolute value, that is when one considers the positive operator given by

$$P^+_\nu F(z) = \int_{T_\Omega} |B_\nu(z-\overline{w})|F(w)\,\Delta(\mathrm{Im}\ w)^{\nu-\frac{n}{r}}\,dw\,. \tag{1.3}$$

We shall see that the index q_ν is sharp for this continuity property.

In the other direction, there is a *large critical index*, depending on p, that we shall call $\widetilde{q}_{\nu,p}$, such that, for $q \geq \widetilde{q}_{\nu,p}$, the projection P_ν fails to be bounded for obvious reasons.

Let us recall that the situation is completely different for the Szegö projection, which is unbounded in $L^p(V)$ for $p \neq 2$ (see [11, 9]).

PROBLEM 2. *Hardy inequality in Bergman spaces.* The question, here, is to know the range of (p, q) for which one has a Hardy type inequality for holomorphic functions on the tube domain T_Ω

$$\|F\|_{L^{p,q}_\nu} \leq C_{pq} \|\Delta(\mathrm{Im}\, w)\Box F\|_{L^{p,q}_\nu}. \tag{1.4}$$

Again, for the upper half-plane, one knows the exact range, and in fact it is valid for all p and q in the interval $[1, \infty)$. It is an easy consequence of the usual Hardy inequality, which gives an integral inequality between a function and its derivative. Let us remark that, since we deal with holomorphic functions, the differential operator $\Box$ may be defined as a polynomial in $\partial/\partial x$, as we did, or in $\partial/\partial z$, or in $\partial/\partial y$.

The converse inequality, where left and right hand side of (1.4) are exchanged, is always valid, as a consequence of the mean value property.

PROBLEM 3. *Characterizations of boundary values of Bergman spaces.* For the upper half-plane, Bergman spaces are characterized by the fact that their boundary values belong to some Besov space. So, the functions of the Bergman spaces may be obtained as Fourier-Laplace transforms of these boundary values, a property which generalizes the situation of A^2.

One would like to have an equivalent characterization in higher rank. We will show that it is indeed the case for some values of p, q. We will need a precise description of the geometry of the cone to be able to describe these objects, which come from an adapted Littlewood-Paley decomposition. So, we will not be able to state properly Problem 3 before Section 4.

It turns out that the three problems are in some sense equivalent. The same critical indices occur in the three problems. In particular, all three possess a negative answer for $q > \widetilde{q}_{\nu,p}$ for obvious reasons. So, the equivalence between the three problems is only interesting for $2 \leq q \leq \widetilde{q}_{\nu,p}$. We may see Problems 2 and 3 as equivalent formulations of Problem 1 which help to take care of the oscillations of the kernel.

We will give precise statements in the other sections, and give a complete answer for $p \leq 2$, with the exact range of values $q > 2$ for which the projection P_ν is bounded on $L^{p,q}_\nu$. For $p > 2$ (and, in particular for $p = q$), there is a gap in the results. We will see in the last section how the question opened by this gap may be related to Littlewood-Paley theory for functions on V with spectrum in Ω. We will then state a last problem, Problem 4.

The present survey is based on joint work of the author with David Békollé, Gustavo Garrigós, Marco Peloso and Fulvio Ricci [1, 3-5]. While the first papers dealt only with the forward light cone, the two last ones deal with the general case. Once the geometric aspects of the proofs have been developed, using the formalism of Jordan

algebras as in [8], there is no difficulty to write our results in the general case, which we do here. Sections 3 and 4 contain some new statements. We tried to give some easy proofs, when they help for the general understanding of the subject. We refer to the different papers for the difficult ones.

Let us mention that part of the results of [1], which are related to the small critical index q_ν, have also been generalized by Békollé and Temgoua Kagou [6], using the formalism of Gindikin for the description of the cones. Let us also mention that one source of inspiration has been the work of Coifman and Rochberg on atomic decomposition of Bergman spaces [7].

Finally, I would like to thank Gustavo Garrigós, whose comments were very helpful. All this survey has been enriched by discussions with him.

2. Geometry and analysis on the cone

In order to describe precisely the results, and specially to define Besov spaces, we start with the description of the geometry of the cone. We refer to [8] for the context, and to [3] and [4] for the geometric lemmas.

Considering V as a Jordan algebra, we denote its unit element by $\mathbf{e}$ (think of the identity matrix for the fundamental example of real symmetric matrices). Let G be the identity component of the group of invertible linear transformations which leave the cone Ω invariant. It is well known that G acts transitively on Ω, which may be identified with the Riemannian symmetric space G/K, where K is the compact subgroup of elements of G which leave $\mathbf{e}$ invariant. The G-invariant Riemannian metric can be defined by

$$\langle \xi , \eta \rangle_y := (t^{-1}\xi \mid t^{-1}\eta)$$

if $y = t\mathbf{e}$ and ξ , η are tangent vectors at $y \in \Omega$. We shall denote by d the corresponding distance, and by $B_\delta(\xi)$ the invariant ball centered at ξ of radius δ. The invariance implies that, for $g \in G$, $B_\delta(g\xi) = gB_\delta(\xi)$.

The determinant function is also preserved by g, in such a way that

$$\Delta(gy) = \Delta(g\mathbf{e})\Delta(y) = \mathrm{Det}(\mathrm{g})^{\frac{r}{n}}\Delta(y)\,. \tag{2.1}$$

It follows from this formula that an invariant measure in Ω is given by $\Delta(y)^{-\frac{n}{r}}\,dy$.

The invariance properties allow also to prove that the determinant function is almost constant on the balls of a given radius, as well as scalar products.

Lemma 2.1. *There is a constant $\gamma > 0$ such that, for $y \in \overline{\Omega}$, if $\xi , \xi' \in \Omega$ with $d(\xi , \xi') \le 2$, then*

$$\frac{1}{\gamma} \le \frac{\Delta(\xi)}{\Delta(\xi')} \le \gamma\,; \tag{2.2}$$

$$\frac{1}{\gamma} \le \frac{(\xi \mid y)}{(\xi' \mid y)} \le \gamma\,; \tag{2.3}$$

$$\frac{1}{\gamma} \le \frac{|\xi|}{|\xi'|} \le \gamma\,. \tag{2.4}$$

From the previous lemma, it follows that, for all $y \in \Omega$ and $0 < \delta \leq 2$,

$$\text{meas}\,(B_\delta(y)) = \text{meas}\,(B_\delta(\mathbf{e})) \sim \text{Vol}\,(B_\delta(\mathbf{e})) \sim \delta^n\ ,$$

where $\text{Vol}\,(B)$ stands for the Euclidean volume of B, while $\text{meas}\,(B)$ stands for its measure for the invariant measure.

Next, we need the analog for a general cone of the decomposition of the real half-line $(0\,,+\infty)$ into an union of dyadic intervals $[2^j\,,2^{j+1})$, which may be seen as invariant balls of constant size. This is given by the next lemma.

LEMMA 2.2. *There exists a sequence of points $\{\xi_j\}_j$ in Ω, and an associated family of disjoint sets $\{E_j\}_j$ covering Ω, such that*

$$B_{1/2}(\xi_j) \subset E_j \subset B_1(\xi_j)\,.$$

A sequence of points $\{\xi_j\}_j$ with the above properties is called a *lattice* of the cone, and the associated partition $\{E_j\}_j$ a *Whitney decomposition of the cone* Ω .

From considerations on the volume of balls we get easily that, for a fixed radius $R \geq 1$, the balls $B_R(\xi_j)$ have *the finite intersection property*. That is, there is an integer $N = N(\Omega\,, R)$ so that each point in Ω belongs to at most N of these balls.

We will use an appropriate partition of unity related to a lattice in order to define Besov spaces. Its existence is given in the next proposition.

PROPOSITION 2.3. *There exists a sequence of smooth functions ψ_j such that*

1. $\widehat{\psi_j} \in C_c^\infty(B(\xi_j\,,1))$;
2. $0 \leq \widehat{\psi_j} \leq 1$, *and* $\sum_j \widehat{\psi_j}(\xi) = 1\,,\ \forall\, \xi \in \Omega$;
3. *The functions ψ_j are uniformly bounded in $L^1(\mathbb{R}^n)$.*

This implies, in particular, the existence of some constant $c > 0$ such that

$$\|\psi_j * f\|_p \leq c\,\|f\|_p\,, \quad \forall f \in L^p(\mathbb{R}^n)\,,\ \forall j,\ 1 \leq p < \infty. \tag{2.5}$$

Roughly speaking, the ψ_j's are obtained from a fixed function by the action of an element of G which sends $\mathbf{e}$ to ξ_j. This allows to compute easily their L^p norms.

Associated with the operator $\Box$ and the Whitney decomposition, we can now introduce the family of Besov-type spaces, $B_\nu^{p,q}$, naturally adapted to the geometry of the cone. They are defined as equivalent classes of tempered distributions on V, by means of the seminorm:

$$\|f\|_{B_\nu^{p,q}} = \left[\sum_j \Delta^{-\nu}(\xi_j)\,\|f * \psi_j\|_p^q\right]^{\frac{1}{q}}\,, \quad f \in \mathcal{S}'(V)\,. \tag{2.6}$$

The Whitney decomposition of the cone has other applications. It allows to discretize integrals which involve almost constant quantities on each piece. Let us give an example of such a situation. The proof is a direct consequence of the lemmas.

PROPOSITION 2.4. *Let $0 < \delta \leq 1$ be fixed, and $\{\xi_j\}_j$ be a lattice with associated Whitney decomposition $\{E_j\}_j$. Then, for every $s \in \mathbb{R}$, $y \in \overline{\Omega}$, and for every non-negative function f on*

the cone, we have

$$\frac{1}{C}\sum_j e^{-\gamma(y|\xi_j)}\Delta^s(\xi_j)\int_{E_j} f(\xi)\,\frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}} \le \int_\Omega f(\xi)e^{-(y|\xi)}\Delta^s(\xi)\,\frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}} \le$$

$$\le C\sum_j e^{-\frac{1}{\gamma}(y|\xi_j)}\Delta^s(\xi_j)\int_{E_j} f(\xi)\,\frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}}\,,$$

where γ is the constant in (2.3) *and C depends only s.*

One may think, at first view, that such estimates will be difficult to use because of the constant γ. But a further integration in the y variable transforms exponentials into powers, as given in the next lemma.

LEMMA 2.5. *For $y \in \Omega$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > \frac{n}{r} - 1$, then*

$$\int_\Omega e^{-(\xi|y)}\,\Delta^s(\xi)\,\frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}} = \Gamma_\Omega(s)\,\Delta^{-s}(y)\,.$$

Moreover, the integral does not converge for other values of s.

$\Gamma_\Omega(s)$ is the Gamma function in Ω, which may be computed in terms of the usual Gamma function. We will need the following lemma, which is an easy consequence of the previous one. Here $L^q_\nu(\Omega)$ denotes the space of functions on Ω whose q-th power is integrable for the measure $\Delta(y)^{\nu-\frac{n}{r}}\,dy$.

LEMMA 2.6. *The function $\Delta(y+\mathbf{e})^{-s}$ is in $L^q_\nu(\Omega)$ if and only if $s > \frac{1}{q}(\nu + \frac{n}{r} - 1)$.*

In fact, we also need in the proofs the generalized powers of Δ. We give here their definitions for completeness, but refer to Chapter VI of [8] and to [4] for their use in estimates. Let $\{c_1, \dots, c_r\}$ be a fixed Jordan frame in V (think of diagonal matrices for which the diagonal entries are all zero except for one equal to 1). Let $\Delta_1(x), \dots, \Delta_r(x)$ the *principal minors* of $x \in V$, with respect to the fixed Jordan frame $\{c_1, \dots, c_r\}$. The *generalized power function* in Ω is defined as

$$\Delta_{\mathbf{s}}(x) = \Delta_1^{s_1-s_2}(x)\,\Delta_2^{s_2-s_3}(x)\cdots\Delta_r^{s_r}(x)\,,\quad \mathbf{s} = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r\,,\quad x \in \Omega\,.$$

When all s_j are equal to s, we see that $\Delta_{\mathbf{s}} = \Delta^s$.

3. The small critical index

The small critical index is given by

$$q_\nu = 1 + \frac{\nu}{\frac{n}{r} - 1}\,. \tag{3.1}$$

Let us mention, even if we will not use it, that for the forward light cone and the usual Bergman space ($r = 2$, $\nu = \frac{n}{2}$), it is the critical index for Bochner-Riesz means in $\mathbb{R}^{n-1}$.

We see first that the small critical index is related to Problem 2. More precisely, it occurs when generalizing the Hardy inequality on the real half-line, given by

$$\int_0^\infty \left(\int_v^\infty f(y)dy \right)^q v^{\nu-1} dv \leq C \int_0^\infty f(y)^q y^{\nu-1} dy$$

for positive f. To replace the integration, the first idea that one has in mind is to use an explicit solution of the equation $\Box^m g = f$ inside the cone, with m large enough so that its elementary solution, given by $c\Delta^{m-\frac{n}{r}}\chi_\Omega$, is locally integrable (which is the case if $m > \frac{n}{r} - 1$). Then

$$T_m f(v) = \int_\Omega f(v+y)\Delta(y)^{m-\frac{n}{r}} dy$$

satisfies the equation $\Box^m g = f$. We will prove the following proposition, which can be called the *Hardy inequality of order m* on Ω.

PROPOSITION 3.1. *There exists a constant C such that, for all positive functions f,*

$$\int_\Omega [T_m f(v)]^q \Delta(v)^{\nu-\frac{n}{r}} dv \leq C \int_\Omega [\Delta(y)^m f(y)]^q \Delta(y)^{\nu-\frac{n}{r}} dy \tag{3.2}$$

if and only if $1 \leq q < q_\nu$.

PROOF. It is equivalent to prove that the operator T, with kernel given by

$$\Delta(y-v)^{m-\frac{n}{r}} \chi_\Omega(y-v)\Delta(y)^{-m+\frac{n}{r}-\nu} ,$$

is bounded in $L^q_\nu(\Omega)$. A necessary condition is that $T^* f$ belongs to $L^{q'}_\nu(\Omega)$, with T^* the formal adjoint of T, when f is the characteristic function of the ball $B_1(\mathbf{e})$. One can show that, for $y \in N\mathbf{e} + \Omega$, with N a fixed integer which is large enough, one has the bound below

$$T^* f(y) > c\Delta(y)^{-\nu} .$$

The necessary condition follows from Lemma 2.6.

The sufficiency follows from a routine argument, using Schur's lemma and generalized powers of Δ: we find that there exists some generalized power $\mathbf{s}$ such that

$$T(\Delta_{\mathbf{s}}^{1/p}) \leq C\Delta_{\mathbf{s}}^{1/p} \qquad T^*(\Delta_{\mathbf{s}}^{1/p'}) \leq C\Delta_{\mathbf{s}}^{1/p'} .$$

This is based on integrability conditions. We shall not go into details, and refer to the bibliography for this kind of computations. □

This proposition leads to a Hardy inequality of order m for holomorphic functions on the tube domain T_Ω,

$$\|F\|_{L^{p,q}_\nu} \leq C_{pq} \|\Delta(\operatorname{Im} w)^m \Box^m F\|_{L^{p,q}_\nu} \tag{3.3}$$

for all $p, q \in [1, \infty)$, with $q < q_\nu$.

We will see that this range of values can always be extended. So this first method, inspired by the method on the upper half-plane, does not give optimal results.

Let us now show the role of the small critical index in Problem 1. We will prove the following theorem.

THEOREM 3.2. *The operator* P_ν^+, *given by* (1.3), *is bounded on* $L_\nu^{p,q}$ *if and only if* $q_\nu' < q < q_\nu$.

PROOF. Let us first prove the sufficient condition. Clearly, P_ν^+ acts as a convolution operator in the x, u variable (if we note $z = x + iy$, $w = u + iv$). Moreover, the norm of this convolution operator acting in $L^q(V)$ is bounded by the L^1 norm of the kernel $|\Delta(\cdot + i(y+v))|^{-(\nu+\frac{n}{r})}$. This norm is easily computed, using Lemma 2.5 and Plancherel formula. Indeed, we have that

LEMMA 3.3. *For* $\alpha \in \mathbb{R}$, *the integral*

$$\int_V |\Delta(x+iy)|^{-\alpha}\,dx, \quad y \in \Omega$$

is finite if and only if $\alpha > \frac{2n}{r} - 1$. *In this case, it is equal to* $c(\alpha)\,\Delta(y)^{-\alpha+\frac{n}{r}}$.

Using also Minkowski inequality, we see that

$$\left(\int_V |P_\nu^+ f(x+iy)|^p dx\right)^{\frac{1}{p}} \le c\int_\Omega \Delta(y+v)^{-\nu} F(v) Q(v)^{\nu-\frac{n}{r}}\,dv\,,$$

with

$$F(v) = \left(\int_V |f(u+iv)|^p du\right)^{\frac{1}{p}}.$$

By assumption, F belongs to $L_\nu^q(\Omega)$, and has norm equal to the norm of f in $L_\nu^{p,q}$. To conclude, we use the next proposition.

PROPOSITION 3.4. *The operator with kernel* $\Delta(y+v)^{-\nu}$ *is bounded on* $L_\nu^q(\Omega)$ *if and only if* $q_\nu' < q < q_\nu$.

We do not give the proof of this proposition. It follows the same lines as the proof of Proposition 3.1.

Let us now prove the necessary condition of the theorem. We test the operator P_ν^+ on functions $f(x+iy) = \chi_{|x|<2}(x)g(y)$,with g a positive function supported in the intersection of the cone with the Euclidean ball of radius $\frac{1}{2\gamma}$ centered at 0. The constant γ, here, is the constant of Lemma 2.1. Let us take for granted that there exists a constant c such that, for $y \in \Omega$ with $|y| < 1/\gamma$, one has

$$\int_{|x|<1} |\Delta(x+iy)|^{-a} dx \ge c\Delta(y)^{-a+\frac{n}{r}}\,. \tag{3.4}$$

We postpone the proof of this inequality, and go on with the proof of the theorem. For x such that $|x| < 1/2$, and $y \in \Omega$ such that $|y| < \frac{1}{2\gamma}$, one has the inequality

$$P_\nu^+ f(x+iy) \ge c\int_\Omega \Delta(y+v)^{-\nu} g(v) Q(v)^{\nu-\frac{n}{r}}\,dv\,.$$

By assumption, there exists a constant C independent of g, such that

$$\int_{y\in\Omega\,,\,|y|<\frac{1}{2\gamma}} \left(\int_\Omega \Delta(y+v)^{-\nu} g(v)\Delta(v)^{\nu-\frac{n}{r}}\,dv\right)^p \Delta(y)^\nu\,dy \le C\int_\Omega g(v)^p\,\Delta(v)^{\nu-\frac{n}{r}}\,dv\,.$$

By homogeneity of the kernel, we can replace the constant $\frac{1}{2\gamma}$ by any positive constant N: for every positive function g on Ω, we have the inequality

$$\int_{y\in\Omega\,,|y|<N}\left(\int_{\Omega}\Delta(y+v)^{-\nu}g(v)\Delta(v)^{\nu-\frac{n}{r}}dv\right)^{q}\Delta(y)^{\nu-\frac{n}{r}}dy\leq C\int_{v\in\Omega\,,|v|<N}g(v)^{p}\Delta(v)^{\nu-\frac{n}{r}}dv\,.$$

Using the density of compactly supported functions, we get the same inequality without any bound on integrals. The necessary condition of the theorem is then a consequence of the necessary condition in Proposition 3.4.

It remains to prove (3.4). It is sufficient to prove the inequality

$$\int_{B_1(y)}|\Delta(x+iy)|^{-a}dx\geq c\Delta(y)^{-a+\frac{n}{r}}\,.$$

Indeed, we deduce from Lemma 2.1 that the invariant ball $B_1(y)$ is contained in the Euclidean ball $\{|x|<1\}$. Now, we can use the fact that Δ is almost constant on the invariant ball, which allows to write that the left hand side is equivalent to

$$\Delta(y)^{\frac{n}{r}}\int_{B_1(y)}|\Delta(x+iy)|^{-a}\frac{dx}{\Delta(x)^{\frac{n}{r}}}\,.$$

Using the action of G and the formula of change of variable for Δ, we see that this last quantity is equal to $\Delta(y)^{-a+\frac{n}{r}}$, multiplied by the same integral when computed for $y=\mathbf{e}$. This last factor is clearly a positive constant. □

For both Problems 1 and 2, we see that the study below the small critical index can be deduced from the boundedness of positive operators on the cone Ω. One needs different methods to take into account the oscillations of the Bergman kernel.

4. The large critical index

The large critical index is equal to

$$\widetilde{q}_{\nu\,,p}=\frac{\frac{n}{r}-1}{\left(\frac{n}{rp'}-1\right)_{+}}\,q_{\nu}=\frac{\nu+\frac{n}{r}-1}{\left(\frac{n}{rp'}-1\right)_{+}}\,,$$

with the convention that $\widetilde{q}_{\nu\,,p}=\infty$ if $p'\geq\frac{n}{r}$, that is $p\leq 1+(\frac{n}{r}-1)^{-1}$.

Let us first consider its relation with Problem 1. It follows from Lemma 2.6 that the Bergman kernel $B_{\nu}(\cdot+i\mathbf{e})=d(\nu)\Delta^{-(\nu+\frac{n}{r})}(\mathbf{e}-i\cdot)$ is in $L^{p'\,,q'}_{\nu}$ if and only when $q<\widetilde{q}_{\nu\,,p}$. But, if the Bergman projection P_{ν} is bounded in $L^{p,q}_{\nu}$, then it is also bounded in $L^{p'\,,q'}_{\nu}$. Moreover, the Bergman projection of the characteristic function of an Euclidean ball, which is centered at $i\mathbf{e}$ and contained inside T_{Ω}, belongs to $L^{p'\,,q'}_{\nu}$. By the mean value equality, this is the function $B_{\nu}(\cdot+i\mathbf{e})$, up to a constant. So the condition $q<\widetilde{q}_{\nu\,,p}$ is necessary for the boundedness of the Bergman projection P_{ν}.

We do not know whether this condition is sufficient for the boundedness of the Bergman projection. Nevertheless, it is sufficient to have a reproducing formula for $A^{p,q}_{\nu}$

functions in terms of the Bergman kernel: for $f \in A^{p,q}_\nu$ and $z \in T_\Omega$, we may write

$$f(z) = \int_{T_\Omega} B_\nu(z - \bar{w}) f(w) \Delta(\operatorname{Im} w)^{\nu - \frac{n}{r}} dw \,. \tag{4.1}$$

Indeed, this identity is valid for $f \in A^2_\nu$. Such functions are dense in $A^{p,q}_\nu$, and we can pass to the limit since $B_\nu(z - \cdot)$ is in $L^{p',q'}_\nu$.

Let us now consider Problem 2. We will only consider the values of p for which $p > 2 + (\frac{n}{r} - 1)^{-1}$, and refer to [4] for other values. We prove first that there is no Hardy inequality for $q > \widetilde{q}_{\nu,p}$. Indeed, there exists a function which is in $A^{p,q}_\nu$, and which is annihilated by the $\Box$ operator. It is sufficient to consider the function $\Delta(z + i\mathbf{e})^{-\frac{n}{r}+1}$, and to use again Lemma 2.6. When $q = \widetilde{q}_{\nu,p}$ (we only consider here the case when $\widetilde{q}_{\nu,p} < \infty$), the proof is more technical: one considers the function

$$F(z) = \Delta\big((z + i\mathbf{e})/i\big)^{-\frac{n}{r}+1} \big(1 + \log \Delta((z + i\mathbf{e})/i)\big)^{-\frac{1}{q}} \,, \quad z \in T_\Omega \,.$$

It is possible to compute explicitly $\Box F$, and to see that, in its expression, the Logarithm appears with a square (see [8, p. 142]). It is also possible to compute the $L^{p,q}_\nu$ norms of both functions, and see that F has an infinite norm, while $\Box F$ has a finite one. Similar computations are done in [4] for another counterexample which is used later. We conclude that, as for Problem 1, one can find easily that Problem 2 has a negative answer above the large critical index.

Let us mention a related problem, *the injectivity of the* $\Box$ *operator*. We have seen that, for $q > \widetilde{q}_{\nu,p}$ and $p > 2 + (\frac{n}{r} - 1)^{-1}$, there exists a function $F \in A^{p,q}_\nu$ such that $\Box F = 0$. Can one prove that it is not the case for the other values such that $q \le \widetilde{q}_{\nu,p}$? For $q < \widetilde{q}_{\nu,p}$, using the representative formula (4.1), we can write that

$$\Box^m F(z) = c \int_{T_\Omega} B_{\nu+m}(z - \bar{w}) F(w) \Delta(\operatorname{Im} w)^{\nu - \frac{n}{r}} dw \,.$$

We used the fact that $\Box^m \Delta^{-\alpha} = c\Delta^{-\alpha-m}$, which implies that $\Box^m B_\nu(\cdot - w) = c B_{\nu+m}(\cdot - w)$. To prove that there does not exist such a function with $\Box^m F = 0$, it is sufficient to prove the density of the functions $B_{\nu+m}(\bar{z} - \cdot)$ in $A^{p'q'}_\nu$. Indeed, if it is the case and if $\Box^m F(z) = 0$ for all z, then the scalar product of F with $B_\nu(\bar{z} - \cdot)$ is also 0, which implies, by the representative formula, that F is identically 0. For m large, the density follows from the fact that the projection $P_{\nu+m}$ is bounded in $L^{p,q}_\nu$, see [5]. We have proved that the $\Box$ operator is injective for $q < \widetilde{q}_{\nu,p}$. It remains to consider the other cases, for which we have no conjecture.

Let us finally consider Problem 3, and the Besov spaces that we have introduced in Section 2. For F a closed set in V, let us denote by $S'_F = S'_F(V)$ the space of tempered distributions with Fourier transform supported in F. It is clear that the natural definition for $B^{p,q}_\nu$ is the following.

Definition 4.1. Given $\nu \in \mathbb{R}$, $1 \le p, q < \infty$, we define $B^{p,q}_\nu$ as the space of equivalence classes of tempered distributions:

$$B^{p,q}_\nu = \big\{ f \in \mathcal{S}'_{\bar{\Omega}} \mid \|f\|_{B^{p,q}_\nu} < \infty \big\} \,/\, \mathcal{S}'_{\partial\Omega} \,.$$

One would like to identify an element of $B_\nu^{p,q}$ with a representative of the equivalence class, and also to define its Fourier-Laplace transform. Again, we claim that this possibility is related with the condition $q < \widetilde{q}_{\nu,p}$. This is based on the next proposition.

PROPOSITION 4.2. *Let (f_j) a sequence of functions on V such that f_j has spectrum in $B_1(\xi_j)$, and*

$$\sum \Delta(\xi_j)^{-\nu} \|f_j\|_p^q < \infty .$$

Then, if $q < \widetilde{q}_{\nu,p}$, the series $\sum_j f_j$ converges in $\mathcal{S}'$. Moreover, this property holds for all such sequences only if $q < \widetilde{q}_{\nu,p}$.

PROOF. Let us prove the necessary condition. Assume that φ is a smooth function whose Fourier transform has compact support, and is 1 in a neighborhood of 0. Since the series $\sum_j \langle f_j , \varphi \rangle$ converges for any order which is chosen on the ξ_j's, it means that $\sum_j |\langle f_j , \varphi \rangle|$ converges, with the sum taken for ξ_j in a neighborhood of 0. If f_j has spectrum in $B_{1/2}(\xi_j)$, then this means that $\sum_j |\langle f_j , \psi_j \rangle|$ converges. Using the action of G, we may assume that ψ_j is equal to $\Delta(\xi_j)^{-\frac{n}{r}} \psi \circ g$, and take also $f_j = a_j f \circ g$, where f is a fixed function whose spectrum is contained in $B_{1/2}(\mathbf{e})$ and such that $\langle f , \psi \rangle$ is not zero. We take for g an element of G such that $\xi_j = g\mathbf{e}$. Then, we have that $\sum_j |a_j| < \infty$ whenever $\sum \Delta(\xi_j)^{-\nu - \frac{nq}{rp}} |a_j|^q < \infty$. This implies the inequality

$$\sum \Delta(\xi_j)^{\nu q'/q + \frac{nq'}{rp}} < \infty ,$$

where the sum is taken for ξ_j in a neighborhood of 0. Using Proposition 2.4, the fact that this sum is finite is equivalent to the fact that

$$\int_{y \in V ; |y| < \varepsilon} \Delta(y)^{\nu \frac{q'}{q} + \frac{nq'}{rp} - \frac{n}{r}} dy < \infty .$$

This last inequality is valid if and only if $q < \widetilde{q}_{\nu,p}$.

We refer to [4] for the proof of the fact that, whenever $q < \widetilde{q}_{\nu,p}$ and φ belongs to $\mathcal{S}'$, then the semi-norm of φ in $B^{p',q'}_{-\nu q'/q}$ is finite. So the series $\sum_j \langle f_j , \varphi \rangle$ is absolutely convergent. □

Assume that $q < \widetilde{q}_{\nu,p}$. If we use the previous proposition for $f_j = f * \psi_j$, with $f \in \mathcal{S}'_\Omega$ such that $\|f\|_{B_\nu^{p,q}} < \infty$, we see that $\sum_j f * \psi_j$ converges in $\mathcal{S}'$ to an element $f^\#$ which depends only on the equivalence class of f. The mapping $f \mapsto f^\#$ defines a mapping from $B_\nu^{p,q}$ to $\mathcal{S}'$. Moreover, it is an injective mapping. To prove this, it is sufficient to prove that $f * \psi_j = 0$ for all j whenever $f^\# = 0$. But, then

$$f * \psi_j(x) = \langle f, \overline{\psi_j(x - \cdot)} \rangle = \left\langle \sum_k f * \psi_k , \overline{\psi_j(x - \cdot)} \right\rangle = \langle f^\# , \overline{\psi_j(x - \cdot)} \rangle = 0 .$$

In the previous identities, the infinite sum on k can be replaced by a finite one, because of the finite intersection property, and this allows to pass to the limit. This means that, below the large critical index, $B_\nu^{p,q}$ identifies with a space of tempered distributions.

One may also define the space of holomorphic functions $\mathcal{B}^{p,q}_\nu$ as the Fourier Laplace transform of $B^{p,q}_\nu$. Indeed, for $f \in B^{p,q}_\nu$ with $q < \widetilde{q}_{\nu,p}$, we can define $\mathcal{L}(f) = \mathcal{L}(f^{\#}) = \sum_j \mathcal{L}(f * \psi_j)$. As before, to prove that this last sum is well defined, it is sufficient to prove that, for $y \in \Omega$, the function whose Fourier transform is $\exp(-(y|\cdot)$ has a finite semi-norm in $B^{p',q'}_{-\nu q'/q}$. The computation is nearly the same as the previous one, and it is the case when $q < \widetilde{q}_{\nu,p}$.

When $q < \widetilde{q}_{\nu,p}$, it makes sense to ask whether $\mathcal{B}^{p,q}_\nu$ is equal to $A^{p,q}_\nu$. This is Problem 3.

5. In between; results and open questions

We now state the results, and refer to [4] for the proofs. Let us first consider Problems 2 and 3 for $q \le 2$. The next theorem gives a complete answer.

Theorem 5.1. *For $q \le 2$ and for all $p \in [1, \infty)$, the spaces $A^{p,q}_\nu$ and $\mathcal{B}^{p,q}_\nu$ coincide. Moreover, for m a positive integer, the $\Box^m$ operator is an isomorphism between $A^{p,q}_\nu$ and $A^{p,q}_{\nu+qm}$.*

We will only sketch the proof for Problem 2, to show again the importance of representation formulas. Remember that $\Box^m B_\nu(\cdot - w) = cB_{\nu+m}(\cdot - w)$. For F a function in $A^{p,q}_{\nu+qm}$, which may be written as

$$F(z) = \int_{T_\Omega} B_{\nu+m}(z - \overline{w}) F(w) \Delta(\operatorname{Im} w)^{\nu+m-\frac{n}{r}} dw \,,$$

a natural solution of Equation $\Box^m G = F$ is given by

$$G(z) = \int_{T_\Omega} B_\nu(z - \overline{w}) F(w) \Delta(\operatorname{Im} w)^{\nu+m-\frac{n}{r}} dw \,.$$

It remains to see that this makes sense, and gives the only solution (remember that we have proved the uniqueness).

In the next theorem, we state the equivalence between Problems 1, 2 and 3.

Theorem 5.2. *Let $p \in [1, \infty)$ and $2 < q < \widetilde{q}_{\nu,p}$. Then there exists an integer m_0 such that the three properties are equivalent*:

1. *The projection P_ν extends into a continuous operator in $L^{p,q}_\nu$.*
2. *A holomorphic function F belongs to $A^{p,q}_\nu$ if and only if it may be written as $\mathcal{L}(f)$, with $f \in B^{p,q}_\nu$.*
3. *For some m larger than m_0, then there exists a constant C such that the Hardy inequality of order m,*

$$\|F\|_{L^{p,q}_\nu} \le C_{pq} \|\Delta(\operatorname{Im} w)^m \Box^m F\|_{L^{p,q}_\nu} \,,$$

holds for all $F \in A^{p,q}_\nu$.

Moreover, if one of the properties is satisfied, then inequalities of Hardy are valid at all orders.

We do not know whether m_0 can be taken equal to 1, or whether there exists a range for which the Hardy inequality of order 1 holds, while Hardy inequalities of higher order do not.

It follows from this theorem and Theorem 1.3 that Problems 1, 2 and 3 have a positive answer in the range $[2, q_\nu)$. It remains to explore the range $[q_\nu, \widetilde{q}_{\nu,p})$. The next theorem will give a partial answer. Let us first consider another related problem, which has its own interest.

PROBLEM 4. *Besov spaces for $\nu = 0$, and purely Fourier analysis approach.* Up to now, we only considered values of ν for which $\nu > \frac{n}{r} - 1$. Such values are related to the weighted Bergman spaces. The case $\nu = 0$ is related to Hardy spaces: by this, we mean that Laplace transforms of functions which are in $L^2(V)$ are functions of the Hardy space H^2, and conversely.

Once one has a Whitney decomposition of the cone Ω, one may ask whether there is an associated Littlewood-Paley inequality for functions in $L^p(V)$, that is, whether there exists a constant C such that, for $f \in L^p(V)$,

$$\left\| \left(\sum_j |f * \psi_j|^2 \right)^{\frac{1}{2}} \right\|_p \leq C \|f\|_p . \tag{5.1}$$

By duality, it implies that, for f_j with spectra in $B_1(\xi_j)$, one has the inequality

$$\left\| \sum_j f_j \right\|_{p'} \leq C \left\| \left(\sum_j |f_j|^2 \right)^{\frac{1}{2}} \right\|_{p'} .$$

For $p \neq 2$, both Littlewood-Paley inequalities for p and p' cannot be valid in the same time, since the characteristic function of the cone Ω is not a Fourier multiplier.

We shall in fact consider a different property, which is weaker when $p \leq 2$ and $s \geq 2$: the existence of some constant C such that

$$\left(\sum_j \|f * \psi_j\|_p^s \right)^{\frac{1}{s}} \leq C \|f\|_p . \tag{5.2}$$

Such an inequality can only be valid for $s \geq 2$. Indeed, take f_j with disjoint spectra in $B_{\frac{1}{2}}(\xi_j)$ and $f = \sum \varepsilon_j f_j$, where the ε_j's are independent ± 1 given by Rademacher functions. Then, using Khintchine inequalities and assuming that (5.2) holds, we find that

$$\left(\sum_j \|f_j * \psi_j\|_p^s \right)^{\frac{1}{s}} \leq C \left\| \left(\sum_j |f_j|^2 \right)^{\frac{1}{2}} \right\|_p .$$

We test this last inequality on N functions f_j with same modulus (taking translations of the same function on the Fourier side) to find a contradiction if $s < 2$.

In the other direction, (5.2) is certainly valid for $s = \max(p, p')$, by interpolation between the cases $p = 1$ or $p = \infty$, for which it is a consequence of the fact that the

norms of ψ_j in L^1 are uniformly bounded, and $p = 2$, for which it follows from the finite intersection property.

By duality, it is equivalent to the fact that, for a finite sequence of functions f_j whose Fourier transforms are supported in $B_1(\xi_j)$, one has the inequality

$$\left\| \sum_j f_j \right\|_{p'} \leq C \left(\sum_j \|f_j\|_{p'}^{s'} \right)^{\frac{1}{s'}} .$$

We call $(\mathcal{C}_{p'}(s'))$ this last property. This means in particular that, when $(\mathcal{C}_p(s))$ holds, one has the following inclusion related to the Besov space for $\nu = 0$:

$$B_0^{p,s} \subset L^p(V) .$$

Problem 4 consists in finding the critical index for $(\mathcal{C}_p(s))$, between $\min(p, p')$ and 2.

Let us remark that $(\mathcal{C}_p(s))$ implies, in particular, that infinite sums $\sum_j f_j$ for which $\sum_j \|f_j\|_p^s$ is finite, converge in $\mathcal{S}'$. This indicates, by Proposition 4.2, that $s < \widetilde{q}_{0,p} = \frac{\frac{n}{r}-1}{(\frac{n}{rp'}-1)_+}$. This constraint is only interesting when this number is smaller than 2, that is $p \geq \frac{2n}{n-r}$.

We will answer to Problem 4 when $p \leq 2$: then p is the best possible index. For $p > 2$, it seems to be a difficult problem, which is related to the other ones, as it can be seen in the next theorem.

THEOREM 5.3. *If the condition* $(\mathcal{C}_p(s))$ *holds, then Problems* 1, 2 *and* 3 *have a positive answer for* q *in the range* $[2, sq_\nu)$. *It is the case, in particular, when* $s = \min(p, p')$. *Moreover, this last result is optimal when* $p \leq 2$.

Again, we refer to [4] for the proof. The counterexamples are given, as before, by functions of the determinant function which involve powers and logarithms. We also prove there that a necessary condition for a positive answer to Problems 1, 2 and 3 is the existence of a constant C such that, for all finite sequences of functions f_j whose Fourier transforms are supported in $B_1(\xi_j)$, one has the inequality

$$\left\| \sum f_j \right\|_p^q \leq C \sum \Delta(\xi_j)^{-\nu} \|f_j\|_p^q , \tag{5.3}$$

where the sum is restricted to those ξ_j's which are of Euclidean norm less than 1. An easy consequence of this, using Khintchine inequalities as before, is the necessary condition $q < 2q_\nu$ for all p: the larger range is obtained for $p = 2$.

These results leave a gap, for Problems 1 to 3, as well as for Problem 4, for $p > 2$. It is possible that solving the problems in the gap is of considerable difficulty. Moreover, the sufficient conditions given by Problem 4 and the necessary conditions (5.3) seem very close, and give a purely Fourier analysis formulation of the different problems. Indeed, work in progress allows to fill part of the gap when using it for the forward light cone in dimension 3.

Among other open problems, let us mention the boundedness of the projection P_ν for the limit case $\nu = \frac{n}{r} - 1$ (see [10]). One does not know whether there is an interval of values of p for which it is bounded in L^p.

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