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Isabeau Birindelli, Ermanno Laconelli

## A Note on one dimensional symmetry in Carnot groups

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**Analisi matematica.** — *A* Note *on one dimensional symmetry in Carnot groups.* Nota (*) di Isabeau Birindelli e Ermanno Lanconelli, presentata dal Socio G. Da Prato.

Abstract. — In this *Note* we extend Gibbons conjecture to Carnot groups using the sliding method and the maximum principle in unbounded domains.

Key words: Monotony properties; Semi-linear equations; Carnot groups.

Riassunto. — *Proprietà di simmetria unidimensionale nei gruppi di Carnot.* In questa *Nota* dimostriamo che la congettura di Gibbons può essere estesa ai gruppi di Carnot nella direzione della base del gruppo usando la tecnica di scivolamento dei domini.

## 1. Introduction

Recently the so called Gibbons conjecture has been completely proved simultaneously by Barlow, Bass, Gui [2], Berestycki, Hamel and Monneau [4] and Farina [8]. Birindelli and Prajapat in [6] have extended the result to the Kohn Laplacian in the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$. Precisely they obtain:

Theorem 1.1. *Let u be a solution of*

$$-\Delta_{\mathbb{H}^n} u = u - u^3 \ \textit{in} \ \mathbb{H}^n , \tag{1.1}$$

*that satisfies* $|u| \leq 1$ *together with the asymptotic conditions*:

$$u(x_1, \cdots, x_n, y_1, \cdots, y_n, t) \to \pm 1 \quad \textit{as} \ \ x_1 \to \pm\infty$$

*uniformly with respect to the other variables. Then* $\frac{\partial u}{\partial x_1} > 0$ *and u depends only on* $x_1$.

In the Euclidean case, when the limit is not uniform, this is a conjecture of De Giorgi, which has been proved in the two dimensional space by Ghoussoub and Gui in [9] and in the three dimensional space by Ambrosio and Cabré in [1].

The proof of Theorem 1.1 relies on the techniques developed in [4] that use the sliding method (see [5]) and a maximum principle in unbounded domains.

In particular in [6] such a maximum principle is proved in sub domains of $\mathbb{H}^n$ contained in half spaces by constructing suitable barrier functions in cones.

Using a completely different approach, Bonfiglioli and Lanconelli in [7] have generalized the maximum principle on unbounded domains to sub-Laplacian in Carnot groups $\mathbb{G} = (\mathbb{R}^N, \circ)$.

Namely they obtain the following result:

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set whose complementary* $\mathbb{R}^N \setminus \Omega$ *contains a*

(*) Pervenuta in forma definitiva all'Accademia il 10 settembre 2001.

*$\mathbb{G}$-cone. Let $c : \Omega \to \mathbb{R}$, $c \leq 0$ and let $u$ be a bounded above $C^2(\Omega)$ function satisfying*

$$\begin{cases} \Delta_{\mathbb{G}} u + cu \geq 0 \ in\, \Omega\,, \\ \limsup_{y\to x} u(y) \leq 0\,,\ \forall x \in \partial\Omega \end{cases} \tag{1.2}$$

*then $u \leq 0$ in $\Omega$.*

We have denoted by $\Delta_{\mathbb{G}} := \sum_{k=1}^{p} X_k^2$ the sub Laplacian in $\mathbb{G}$.

When $\mathbb{G} = \mathbb{H}^n$ is the Heisenberg group, this Theorem corresponds to Proposition 2.1 in [6]. For the classical case see [3, Lemma 2.1].

We recall that a Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ)$ is the Lie group associated to a Lie Algebra $\mathcal{G}$ nilpotent, stratified and with constant rank $N$. $\mathcal{G}$ admits a direct sum decomposition

$$\mathcal{G} = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2} \oplus \cdots \oplus \mathbb{R}^{N_m}$$

with $N_1 = p$ and $N_1 + N_2 + \cdots + N_m = N$.

For more details on Carnot groups and definition of $\mathbb{G}$-cone we refer to the next section, nevertheless let us just make the following

Remark 1. The group action $\circ$ restricted to $\mathbb{R}^{N_1}$ coincides with the Euclidean action. Precisely denoting the elements of $\mathbb{G}$ by $x = (x^{(1)}, x^{(2)})$ with $x^{(1)} \in \mathbb{R}^{N_1}$ and $x^{(2)} \in \mathbb{R}^{N-N_1}$, we have that

$$x \circ y = (x^{(1)} + y^{(1)}, \sigma)$$

where $\sigma$ is an element of $\mathbb{R}^{N-N_1}$ that depends on $x$ and $y$.

This property and Theorem 1.2 allow us to use the sliding method to prove our main result: Theorem 1.3 stated below.

In the next theorem we shall use the following notation for $x \in \mathbb{G}$: $x = (x_1, \hat{x})$, where $x_1$ indicates any direction in $\mathbb{R}^{N_1}$ and $\hat{x}$ are the remaining variables *i.e.* $\hat{x} \in \mathbb{R}^{N-1}$.

Theorem 1.3. *Let $u$ be a solution of*

$$\Delta_{\mathbb{G}} u + f(u) = 0 \ in\ \mathbb{G} \tag{1.3}$$

*which satisfies $|u| \leq 1$ together with the asymptotic conditions*

$$u(x_1, \hat{x}) \to \pm 1 \ \ when\ \ x_1 \to \pm\infty \tag{1.4}$$

*uniformly in $\hat{x} \in \mathbb{R}^{N-1}$. We assume that $f$ is Lipschitz continuous in $[-1, 1]$, $f(\pm 1) = 0$ and that there exists $\delta > 0$ such that*

$$f \text{ is nonincreasing on } [-1, -1+\delta] \text{ and on } [1-\delta, 1]. \tag{1.5}$$

*Then $\frac{\partial u}{\partial x_1} > 0$ and*

$$u(x_1, \hat{x}) = U(x_1).$$

The sliding method and the Maximum Principle on unbounded domains also allow us to prove monotonicity results in all other coordinate directions, starting from the following property of the group law:

$$(0, \dots, \overset{i}{s}, \dots, 0) \circ (x_1, \dots, x_N) = (x_1, \dots, x_{i-1}, x_i + s, \sigma) \tag{1.6}$$

for any $i > N_1$. Here we have denoted by $\sigma$ an element of $\mathbb{R}^{N-i}$.

In the next theorem we shall use the following notation for elements $x \in \mathbb{G}$: $x = (\check{x}, x_i)$ where $x_i$ is any direction not in $\mathbb{R}^{N_1}$, and $\check{x} \in \mathbb{R}^{N-1}$ are all the remaining directions.

Theorem 1.4. *Let $u$ be a solution of* (1.3) *which satisfies $|u| \leq 1$ together with the asymptotic conditions*

$$u(\check{x}, x_i) \to \pm 1 \quad \textit{when} \quad x_i \to \pm\infty \tag{1.7}$$

*uniformly with respect to $\check{x}$. If $f$ satisfies the hypotheses of Theorem* 1.3 *then $\frac{\partial u}{\partial x_i} \geq 0$ and the inequality is strict if $\frac{\partial}{\partial x_i}$ commutes with the $X_k$ fields.*

## 2. Proofs and basic facts about Carnots groups

We recall that a Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ)$ is a Lie group whose associated Lie Algebra $\mathcal{G}$ admits a direct sum decomposition of vector fields

$$\mathcal{G} = V_1 \oplus \ldots \oplus V_m$$

with $dim V_j = N_j$, $[V_1, V_j] = V_{j+1}$ for $1 \leq j < m$ and $[V_1, V_m] = 0$. Thus $V_1$ generates $\mathcal{G}$ as a Lie algebra.

More precisely, given a Lie algebra $(\mathcal{G}, [\ .\ ])$ satisfying the above conditions, consider $\mathbb{R}^N$, where $N = \sum_{j=1}^{l} N_j$ with the group operation $\circ$ determined by the Campbell-Hausdorff formula

$$\eta \circ \xi = \eta + \xi + \frac{1}{2}[\eta, \xi] + \frac{1}{12}[\eta, [\eta, \xi]] + \frac{1}{12}[\xi, [\xi, \eta]] + \ldots \tag{2.8}$$

Note that since $\mathcal{G}$ is nilpotent there are only a finite number of nonzero terms in the above sum; precisely those involving commutators of $\xi$ and $\eta$ of length less than $m$. Then $(\mathbb{G}, \circ) = (\mathbb{R}^N, \circ)$ is the Lie group whose Lie algebra of left-invariant vector fields coincides with $(\mathcal{G}, [\ ,\ ])$.

Remark 1 and (1.6) follow easily from the Campbell-Hausdorff formula (2.8).

For $\lambda > 0$ let us denote by $\delta_\lambda$ the dilations of $\mathbb{G}$ defined by:

$$\delta_\lambda(x) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \cdots, \lambda^m x^{(m)})$$

where $x^{(j)}$ are the elements of $\mathbb{R}^{N_j}$.

Definition 2.1. *A subset $K$ of $\mathbb{G}$ is said to be a $\mathbb{G}$-cone with vertex at the origin if*

$$\delta_\lambda x \in K, \quad \forall x \in K, \quad \forall \lambda > 0.$$

*If $K$ is such a cone then $x_o \circ K$ is a $\mathbb{G}$-cone with vertex at $x_o$.*

For what follows it is important to note that any half space of $\mathbb{R}^N$ contains a $\mathbb{G}$-cone, hence the Maximum Principle of Theorem 1.2 holds for any open set $\Omega$ contained in a half space.

An immediate consequence of this remark is the following comparison result:

COROLLARY 2.1. *Let $f$ be a Lipschitz continuous function, non-increasing on $[-1, -1+\delta]$ and on $[1-\delta, 1]$ for some $\delta > 0$. Assume that $u_1$, $u_2$ are solutions of*

$$\Delta_{\mathbb{G}} u_i + f(u_i) = 0 \ \text{in} \ \Omega$$

*and are such that $|u_i| \leq 1$, $i = 1, 2$. Furthermore, assume that*

$$u_2 \geq u_1 \ \text{on} \ \partial\Omega$$

*and that either*

$$u_2 \geq 1 - \delta \ \text{in} \ \Omega$$

*or*

$$u_1 \leq 1 + \delta \ \text{in} \ \Omega.$$

*If $\Omega \subset \mathbb{G}$ is an open set contained in a half space then $u_2 \geq u_1$ in $\Omega$.*

PROOF OF THEOREM 1.3. The proof of Theorem 1.3 is along the lines of Theorem 1 in [4] and Theorem 3.1 of [6]. Of course, here we use the group action $\circ$ of $\mathbb{G}$, we rely on the fact that the sub-Laplacian is invariant with respect to $\circ$ and on Remark 1. Using the notations of Theorem 1.3, we begin by proving

CLAIM 1. For any $y = (y_1, \hat{y}) \in \mathbb{G}$ with $y_1 > 0$ and any $s > 0$, we have

$$u_s(x) := u(sy \circ x) \geq u(x) \ \text{for all} \ x \in \mathbb{G}. \tag{2.9}$$

PROOF. Using the condition (1.4), for $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$$u(x_1, \hat{x}) > 1 - \delta \quad \text{for } x_1 \geq N \tag{2.10}$$

$$u(x_1, \hat{x}) < -1 + \delta \quad \text{for } x_1 \leq -N\,. \tag{2.11}$$

Hence, using Remark 1 of the Introduction, for $s > 2N/y_1$, we have

$$u_s(x) = u(sy_1 + x_1, \widehat{sy \circ x}) > 1 - \delta \quad \text{for } x_1 \geq -N\,. \tag{2.12}$$

Furthermore, the function $u_s$ satisfies the equation (1.3) and

$$u_s(-N, \hat{x}) > u(-N, \hat{x})\,.$$

We now apply Corollary 2.1 to the functions $u_s$ and $u$ in the half spaces $\{x \in \mathbb{G} : x_1 \geq -N\}$ and $\{x \in \mathbb{G} : x_1 \leq -N\}$ to conclude that

$$u_s(x) \geq u(x) \ \text{for all} \ x \in \mathbb{G}\,.$$

Let $\tau = \inf\{s : u_s(x) \geq u(x) \ \textit{for all} \ x \in \mathbb{G}\}$. We claim that $\tau = 0$. Since the proof of this claim is identical to the same claim in Theorem 3.1 of [6] we skip it here and Claim 1 is proved.

Let $R_x : \mathbb{G} \to \mathbb{G}$ be the right multiplication by $x$: $R_x\, y = y \circ x$, then from Claim 1 we conclude that

$$du_{|x} \cdot (dR_x)_0(y) = \lim_{s\to 0} \frac{u_s(x) - u(x)}{s} \geq 0 \tag{2.13}$$

for any $y$ such that $y_1 > 0$. By continuity (2.13) still holds if we choose $y$ such that $y_1 = 0$. Let $W$ denote the subspace of $\mathbb{G}$ of $y := (0\,, \hat{y})$.

Since $(dR_x)_0(-y) = -(dR_x)_0(y)$, then we have obtained that

$$du_{|x} \cdot (dR_x)_0(y) = 0 \tag{2.14}$$

for any $y \in W$.

Now we can conclude that $u$ is a function of $x_1$ alone. Indeed, since $R_x$ is a diffeomorphism then $dR_x$ is an isomorphism from the tangent space of $\mathbb{G}$ in 0: $T_0\mathbb{G} = \mathcal{G}$ to $T_x\mathbb{G} = \mathcal{G}$. Furthermore the stratified properties imply that $dR_x : W \to W$ and therefore the restriction of $dR_x$ to $W$ is surjective, equation (2.14) becomes:

$$\nabla u \cdot v = 0 \quad \forall\, v \in W$$

and $u$ is constant in all directions except $x_1$.

To prove that $\frac{\partial u}{\partial x_1} > 0$, it is enough to choose in (2.13) $y$ such that $y_1 = 1$ and all other variables are 0. Then one gets $\frac{\partial u}{\partial x_1} \geq 0$.

On the other hand, $v := \partial_{x_1} u$ satisfies $\Delta_{\mathbb{G}} v + f'(u)v = 0$. As a consequence, by the strong maximum principle $\partial_{x_1} u$ is strictly positive.

This concludes the proof of Theorem 1.3.

For the proof of Theorem 1.4 just proceed as in the proof of Theorem 1.3: Choose $y = (0\,, \ldots\,, \overset{i}{s}\,, \ldots\,, 0)$ where $i > N_1$ is the direction of the asimptotic condition and define $u_s(x) = u(sy \circ x)$. Then it is easy to see, using (1.6), that $u_s(x) \geq u(x)$ for any $s \geq 0$. This implies that $\partial_{x_i} u \geq 0$. Furthermore if $\partial_{x_i}$ commutes with $\Delta_{\mathbb{G}}$ then $\partial_{x_i} u$ satisfies a linear equation and by the maximum principle $\partial_{x_i} u > 0$.

This concludes the proof of Theorem 1.4.

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I. Birindelli:
Dipartimento di Matematica
Università degli Studi di Roma «La Sapienza»
Piazzale A. Moro, 2 - 00185 Roma
isabeau@mat.uniroma1.it

E. Lanconelli:
Dipartimento di Matematica
Università degli Studi di Bologna
Piazza di Porta San Donato, 5 - 40127 Bologna
lanconel@dm.unibo.it