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Geometria algebrica. — *Limit Weierstrass schemes on stable curves with* 2 *irreducible components.* Nota (*) di Marc Coppens e Letterio Gatto, presentata dal Socio E. Vesentini.

Abstract. — We are concerned with limits of Weierstrass points under degeneration of smooth curves to stable curves of non compact type, union of two irreducible smooth components meeting transversely at $m \geq 1$ points. The case $m = 1$ having already been treated by Eisenbud and Harris in [8], we analyze the situation for $m > 1$.

Key words: Stable curves of non compact type; Limits of Weierstrass points on reducible curves; Limit Weierstraßschemes.

Riassunto. — *Limiti di schemi di Weierstrass su curve stabili con* 2 *componenti irriducibili.* Si studiano limiti di punti di Weierstrass per degenerazioni di curve lisce a curve stabili di tipo non compatto, unione di due componenti lisce irriducibili che si intersecano trasversalmente in $m \geq 1$ punti. Il caso $m = 1$, essendo di tipo compatto, è già stato trattato da Eisenbud e Harris in [8], sicché nella presente *Nota* verrà analizzata la situazione per $m > 1$.

1. Introduction

1.1. In the celebrated paper [8], using techniques introduced in [7], Eisenbud and Harris study how Weierstrass points on general smooth curves degenerate when the smooth curve degenerates, in a flat proper family, onto a stable curve of *compact type*. The purpose of this paper is to show how relatively easy technical tools provide a considerable amount of new informations when the special fiber of the family is a stable curve of non-compact type which is the union of two irreducible smooth components intersecting in $m > 1$ points (the case $m = 1$ being covered by the theory of Eisenbud and Harris). We recall here that a stable curve C_0 is said to be of compact type if its (generalized) jacobian is compact or, equivalently, if all of the irreducible components of C_0 are smooth curves and if its *dual graph*, whose vertices correspond to the irreducible components and an edge joins two vertices if the two components have a point in common, is a tree. We review below some basic – and eventually very classical – definitions, in order to set the framework and the general motivations of this research.

1.2. On any projective smooth complex curve of genus g (a compact Riemann Surface) some distinguished points live, called *Weierstrass points*. A Weierstrass point is a ramification point of the canonical linear system, K_C, and it lies in the zero scheme

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of a section $\mathbb{W}$, said to be the *wronskian*, of the line bundle $K_C^{\otimes \frac{g(g+1)}{2}}$. More precisely, the zero scheme $Z(\mathbb{W})$ associated to the wronskian, in the following said to be the *Weierstrass scheme*, induces a well defined cycle on the curve C, namely:

$$[Z(\mathbb{W})] = \sum_{P \in C} \mathrm{wt}(P) P \ ,$$

where $\mathrm{wt}(P)$ is the *Weierstrass weight* of the point. A point P is a Weierstrass point if and only if $\mathrm{wt}(P) > 0$ and, of course, $\mathrm{wt}(P) = 0$ for all $P \in C$ but finitely many (see *e.g.* [2, 13, 14, 18, 19]).

1.3. In [24], C. Widland and R. Lax extend this general theory to the case of *complete irreducible Gorenstein curves* C. Things work almost the same if one replaces the canonical bundle with the dualizing sheaf ω_C of C, that is invertible by the Gorenstein assumption. Again, one may define a wronskian $\mathbb{W}$, and this gives rise to a Weierstrass subscheme of C of lenght $(g-1)g(g+1)$. The theory of Widland and Lax for irreducible Gorenstein curves behaves nicely under degeneration, as shown in [22]. Even if one is interested only in the classical theory (C smooth), studying Weierstrass points on singular curves can be useful: smooth curves degenerate to special types of singular curves.

1.4. What about reducible curves? It is very natural to start thinking about the easiest situation, by considering reducible curves with only nodes as singularities. This is reasonable and interesting because, first of all, noded curves are Gorenstein (hence carry an invertible dualizing sheaf). Secondly, the nodal reducible curves that are *stable* (*i.e.* projective, connected, reduced and such that each smooth rational component intersects the rest of the curve in at least 3 points) are the building blocks for the so-called Deligne-Mumford compactification of the moduli space $\overline{M}_g$, M_g, of smooth curves of genus $g \geq 2$ [5]. However, trying to straightforwardly extend the definition of Lax and Widland on a stable reducible curve, one gets in trouble very soon. This is because, if the stable curve has a non-rational irreducible component, then any reasonable substitute of the *wronskian* would vanish identically on that component of the curve, turning the notion of Weierstrass point meaningless. We are left to deal with some more delicate problem: suppose that we are given a smoothing of a stable reducible curve C_0, *i.e.* of a family of smooth curves degenerating to C_0. How do the Weierstrass points on the general (smooth) fiber degenerate onto the special fiber? Or, put otherwise, and using the current terminology of the literature, *what are the limits of Weierstrass points of smooth curves degenerating to a stable reducible curve*?

1.5. In [8], using the celebrated theory of *limit linear series* as formulated in [7], Eisenbud and Harris investigate, amongst other things, how the Weierstrass points degenerate under specialization to a (reducible) stable curve C_0 of *compact type*. In their investigation of 1-parameter degenerations, Eisenbud and Harris consider proper flat families of semistable curves of genus $g \geq 3$, $\pi : \mathcal{X} \longrightarrow S$ with $S = \mathrm{Spec}(\mathbb{C}[[T]])$ such that:

1. the geometric generic fiber $\mathcal{X}_{\overline{\eta}}$ is smooth;
2. the total space $\mathcal{X}$ is smooth;
3. the special fiber $\mathcal{X}_0$ has C_0 as a stable model.

These data induce, in a unique way, a linear system $g^{g-1}_{2g-2}(X)$ on each irreducible component X of $\mathcal{X}_0$, whose ramification points in the smooth locus of $\mathcal{X}_0$ are precisely the limits of Weierstrass points of $\mathcal{X}_{\overline{\eta}}$ degenerating to the smooth part of the component X. Such a collection of linear systems $g^{g-1}_{2g-2}(X)$ is called a canonical limit linear system on $\mathcal{X}_0$; the linear system $g^{g-1}_{2g-2}(X)$ is called the X-aspect of this *canonical limit linear system*. Those aspects satisfy suitable ramification conditions at each node of $\mathcal{X}_0$. Now, fix the dual graph (being a tree) and the genera related to the vertices and take C_0 having that graph. For general C_0, the aspects on the irreducible components of C_0 are independent from the family, hence the limits of the Weierstrass points are completely determined by C_0. Moreover the nodes are not limits of Weierstrass points on nearby smooth curves.

1.6. At p. 499 of their paper [8], the Authors ask the question: «*What are the limits of Weierstrass points in families of curves degenerating to stable curves not of compact type*?». As already mentioned, the purpose of this paper is trying to deal with such a question for the case of a stable curve C_0 that is the union of 2 smooth irreducible components X and Y. The main difference with respect to the situation studied by Eisenbud and Harris is that, even in the most general case, as soon as the curve C_0 is not of compact type, the limits of Weierstrass points *depend* on the smoothing family. For instance, if C_0 is the union of two smooth curves, X and Y, each of genus ≥ 1, intersecting transversally at two points A and B, there is a 1-dimensional family of limits. More precisely, if P_0 is any smooth point of C_0, there exists a family $\pi : \mathcal{X} \longrightarrow S$ such that $S = \mathrm{Spec}(\mathbb{C}[[T]]) = \{0\,,\eta\}$, with a Weierstrass point $P_\eta \in \mathcal{X}_\eta$ such that $P_0 \in \overline{\{P_\eta\}}$. This fact, very well known from ten years at least, is worked out in all the details in the paper [3, p. 333], as a standard application of the theory of Harris and Mumford [21, p. 56 and ff.] of the compactification of *Hurwitz schemes* by means of *admissible covering*. Because we shall use such tools, we advise the reader to look at the papers [3, 21], especially at the quoted pages, for foundational material. So, from the paper [3] one knows that a general stable curve with two irreducible components with two nodes admits a 1-dimensional family of limits of Weierstrass points. What about if the two curves intersect in $m > 2$ points? In this situation the theory of admissible covering does not say very much: we only may conclude, arguing as in [3] for curves with two nodes, that there exists a family of limits of Weierstrass points which is at least 1-dimensional. Admissible coverings are not suited to refine such an information, because they only allow to follow the degeneration of a Weierstrass point at a time. So, the main result of this paper is:

THEOREM 3.2. *Fix m, $g_X \geq 1$, $g_Y \geq 1$ and let C_0 be a general union of 2 irreducible components X and Y of genus respectively g_X and g_Y, intersecting transversally at m points. Then the space parametrizing limits of Weierstrass schemes on C_0 has dimension $m-1$.*

Relying on this theorem we are finally able to find the limits of Weierstrass points on one component of a stable curve with two irreducible components such that one of them has genus 1 (Theorem 5.3).

1.7. The quoted paper by Cukierman [3] should be enough to convince the reader that the limits of Weierstrass points move in positive dimensional families, so clarifying the relevance of Theorem 3.2. However, it may be worth to work out the first case unsettled by the theory of Eisenbud and Harris, namely the union C_0 of two elliptic curves meeting transversally in two points A and B, at least to give the feeling of the occurring phenomena. We warn the reader, by the way, that the result we are going to informally explain, using here an ad hoc procedure, is nothing but a particular case of the stuff presented in the rest of the paper, which, in a sense, generalizes the following ideas.

Let $C_0 = X \cup Y$ and choose arbitrarily a smooth point P_0, say $P_0 \in Y$, to fix the ideas. We claim that P_0 is limit of a Weierstrass point on nearby smooth curves. In fact there exists (a unique!) $C \in Y$ such that $3P_0 \sim A + B + Q$ ($\sim$ meaning *linear equivalence*). Moreover $h^0(X, O_X(A + B)) = 2$ so that there is a $2:1$ covering of the projective line such that $A + B$ is a fiber of such a covering. By attaching a smooth rational component at C, we get the following *admissible covering*:

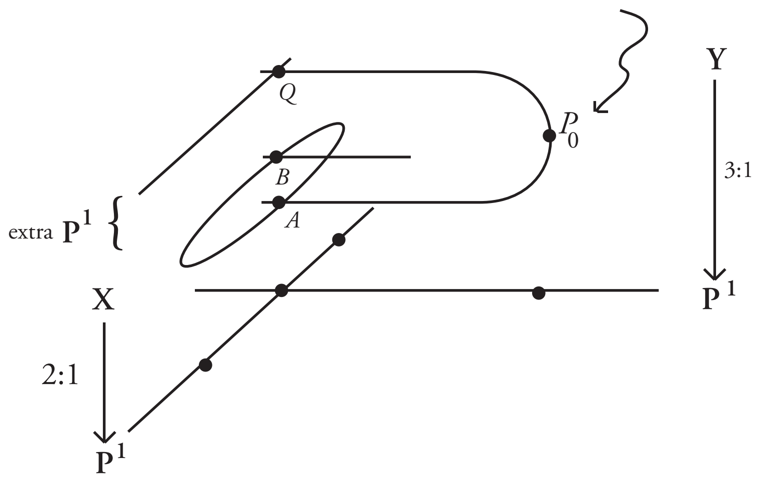

Fig. 1.

The picture above, by the theory of Harris and Mumford, is the degeneration of a family of ramified coverings of degree 3, $C_t \longrightarrow \mathbb{P}^1$, where C_t are smooth curves of genus 3, with a total ramification point P_t (hence a Weierstrass point) degenerating to $P_0 \in C_0$. In other words $P_0 \in C_0$ is a *limit* of a Weierstrass point. Such a result is of course a particular case of our Theorem 3.2, but this example gives the reader the feeling of what happens for curves of non compact type.

At this point is not very much work to study where are located the other limits of the Weierstrass points on Y, corresponding to the smoothing family gotten from the choice of the point P_0. Similar results hare been obtained also in [10]. Here we should use a local analysis performed by [4, pp. 48-49], which ensures us that we may find a germ of stable curve $\mathcal{X} \longrightarrow \mathrm{Spec}(\mathbb{C}[[T]])$ such that $\mathcal{X}_0 = C_0$, $\mathcal{X}_{\overline{\eta}}$ smooth and P_η a Weierstrass point on $\mathcal{X}_{\overline{\eta}}$ degenerating onto P_0, such that the *total space $\mathcal{X}$ of the family is a smooth surface*. This allows us to use elementary intersection theory on a smooth surface. One easily check that the natural restriction map:

$$\pi_* \omega_\pi \otimes \mathbf{k}(0) \cong H^0(C_0, \omega_\pi(X)_{|C_0}) \longrightarrow H^0(Y, O_Y(2A + 2B)) \ ,$$

is an injection. Hence the pair $(\pi_* \omega_\pi(D) \otimes \mathbf{k}(0), O_Y(2A + 2B))$ defines a g_4^2 on Y, which ramifies at P_0 (see Example 4.6). Moreover such a g_4^2 contains the linear system $|A + B| + A + B$, a g_4^1 on Y with two base points (namely A and B). To show that the 12 ramification points of the g_4^2 above are the limits of the Weierstrass points of $\mathcal{X}_\eta$ degenerating onto Y, fact that is a particular case of our Theorem 5.3, we shall argue by translating the question in a problem of plane quartic curves. In fact, $\pi_* \omega_\pi(X) \otimes \mathbf{k}(\eta) = H^0(\mathcal{X}_\eta, \omega_{\mathcal{X}_\eta})$, so that we get a morphism:

$$\phi : \mathcal{X} \longrightarrow \mathbb{P}(\pi_*(\omega_\pi(X))) \ ,$$

such that $\phi_\eta(\mathcal{X}_\eta)$ is a smooth quartic curve in $\mathbb{P}^2_\eta$, degenerating to a quartic $\phi_0(C_0)$ which is the image of C_0 in $\mathbb{P}^2$ via the linear system $\pi_* \omega_\pi \otimes \mathbf{k}(0)$ (the g_4^2 previously described). The image turns out to be a quartic curve with a tacnode at a point $\phi_0(A) = \phi_0(B)$, whose normalization is isomorphic to Y and where the component X of C_0 has been contracted into the singularity.

According to the theory of Lax and Widland, the weight of a tacnode on a curve of arithmetic genus 3 is 12. We conclude that 12 Weierstrass points of $\mathcal{X}_\eta$ degenerate onto the component X (contracted into the tacnode) while the other 12 degenerate to the points which are made via ϕ_0 onto the flexes of $\phi_0(C_0)$, where the given g_4^2 is precisely cut out by the linear system of lines. This concludes the proof that the ramification point of the $g_4^2(Y)$ are contained in the limit Weierstrass scheme.

1.8. The paper is organized as follows. In Section 2 we set the notation, the basic definitions of limit Weierstrass schemes, and state a few results without proof but giving precise references to the literature. There, we also discuss the general problems inspiring our investigations.

In Section 3 we state and prove the main theorem of the paper (Theorem 3.2) about the dimension of the space of limits Weierstrass schemes.

In Section 4, after defining the notion of a *suited divisor* for the relative dualizing sheaf of a 1-parameter family $\pi : \mathcal{X} \longrightarrow S$, whose total space $\mathcal{X}$ is smooth, we study special cases where we start from a degeneration to C_0, having an elliptic component, and we try to find a linear system of dimension $g - 1$ on the irreducible components of C_0 such that the limits of Weierstrass points are related to the ramification points of that linear system. In other words, we try to mimic the construction of canonical limit linear series of [8]. For instance, in our previous example of the union of two elliptic curves with two nodes, X was a suited divisor for the component Y. This example is generalized in our Example 4.6. We also define the *aspects* of a limit Weierstrass scheme and attack the general problem of determining them in the case of stable curves with 2 components. We define the notion of *suited divisor* for a component of the special fiber of a good family $\pi : \mathcal{X} \longrightarrow S$ of stable curves.

Section 5 is still devoted to a deeper study of examples which in our opinion are significant of how the construction of a general theory of limits of Weierstrass points on any stable curve is a very delicate subject. In one case (Example 5.9) we are able to detect simultaneously the aspects on both components of the union of an elliptic curve and a curve of genus 2 meeting transversally in two general points. Such example is fully compatible with our Theorem 3.2. Moreover, it shows that the nodes are limits of Weierstrass points coming from Weierstrass points on any general smoothing family. The estimation of the *weight* of the WP's degenerating onto the nodes is, in our opinion, a nice new application of the theory by Widland and Lax ([24]) about Weierstrass points on Gorenstein curves. We end the paper with some remarks about the irreducibility of the Weierstrass scheme (see Section 2 for definition)

1.9. This paper circulated for a couple of years in a preprint form. In the meantime E. Esteves and N. Medeiros, who knew our results, studied this problem in a sistematic and definitely deeper way, up to give a complete and satisfactory solution of it, announced in [11, 12]. Their work studies, more generally, limits of canonical series on curves with two components: it gets our results as particular cases, in a more general framework populated by many more new, and interesting, results.

2. Generalities and problems

2.1. We work over the field of complex numbers $\mathbb{C}$. Let $\overline{M}_{g,n}$ (resp. $M_{g,n}$) be the coarse moduli space of stable (resp. smooth) n-pointed curves of genus g. To fix our general set-up we need the following natural proposition, whose proof can be found *e.g.* in [3, p. 335].

2.2. Proposition. *Let* $B \in M_{g,n}$ *be a locally closed subset and* $[(C; P_1, \dots, P_n)] \in \overline{M}_{g,n}$ *a point. Then* $[(C; P_1, \dots, P_n)]$ *belongs to the closure of* B *in* $\overline{M}_{g,n}$ *if and only if there exists a family of curves* $\pi : \mathcal{X} \longrightarrow S = \mathrm{Spec}(\mathbb{C}[[T]])$ *together with* n *pairwise disjoint sections,* $\sigma_1, \dots, \sigma_n : S \longrightarrow \mathcal{X}$*, such that:*

1. $\mathcal{X}$ *is a smooth surface;*
2. *the geometric generic fiber is smooth;*
3. *if* $\mathcal{X}_0$ *is the central fiber of the family, then* $(\mathcal{X}_0, \sigma_1(0), \dots, \sigma_n(0))$ *is semistably equivalent to* $(C, P_1, \dots, P_n)$*;*
4. *the induced moduli map* $m : S \longrightarrow \overline{M}_{g,n}$ *sends the generic point of* S *to* B.

Proof [3, p. 335].

2.3. The main reference for this subsection is [16, Section 2] (see also [9]).

Let S be a connected scheme of finite type over $\mathbb{C}$. By a stable (resp. smooth) curve of genus g over S we shall intend a flat proper morphism $\pi : \mathcal{X} \longrightarrow S$ such that each scheme theoretical fiber is a complex stable (resp. smooth projective) curve. Let $\Omega^1_{\mathcal{X}/S}$ and ω_π be respectively the *sheaf of relative differentials* and the *relative dualizing sheaf* of the family. The latter is a line bundle over $\mathcal{X}$ [5, p. 76]. Let $d : O_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/S}$ and $\mathcal{R} : \Omega^1_{\mathcal{X}/S} \longrightarrow \omega_\pi$ be respectively the *exterior differential* and the *residue map*. The composition:

$$d_\pi := \mathcal{R} \circ d : O_{\mathcal{X}} \longrightarrow \omega_\pi \ ,$$

will be said the *derivative* along the fibers of π. Let $f \in O_{\mathcal{X}}(U)$ where U is an open set such that $\omega_\pi(U) = O_{\mathcal{X}}(U) \cdot \sigma$. Then the k-th derivative $f^{(k)}$ of f with respect to $\sigma \in \omega_\pi(U)$ is recursively defined by:

$$\begin{cases} f^{(0)} = f \\ d_\pi(f^{(k-1)}) = f^{(k)} \cdot \sigma \, . \end{cases} \tag{1}$$

Let $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X}/S)$ such that $h^0(\mathcal{X}\,,\mathcal{L}) > 0$. If $\lambda \in H^0(\mathcal{X}\,,\mathcal{L})$, and ψ generates $\mathcal{L}(U)$ over $O_{\mathcal{X}}(U)$, set:

$$\lambda_{|_U} = \ell \cdot \sigma \, , \quad \ell \in O_{\mathcal{X}}(U)\, .$$

Then, for each $k \geq 0$, the data:

$$(\ell\,, \ell'\,, \ldots\,, \ell^{(k)})$$

is the local representation of a section, denoted by $D^k\lambda$, of a vector bundle of rank $k+1$, denoted by $J^k\mathcal{L}$, whose transition functions are prescribed by the transformation rules (from one to another open set) of the $k+1$-tuple $(\ell\,, \ell'\,, \ldots\,, \ell^{(k)})$.

2.4. Let $\pi : \mathcal{X} \longrightarrow S$ be a stable curve of genus g and assume that the general fiber is irreducible. The *Hodge bundle* $\mathbb{E}_\pi$ of the family is the rank g vector bundle over S defined as:

$$\mathbb{E}_\pi = \pi_* \omega_\pi \, .$$

We shall often write simply $\mathbb{E}$, by skipping the subscript π when this is clear from the contest. Let us denote by $\mathcal{I}_\pi \subset \mathcal{X}$ the inflectionary locus of the relative dualizing sheaf, *i.e.* the locus of points P of $\mathcal{X}$ such that there exists a dualizing differential α on the fiber $\mathcal{X}_{\pi(P)}$ vanishing at P with multiplicity at least g, in the sense that $D^{g-1}(\alpha)(P) = 0$. Such an inflectionary locus can be given a scheme structure as follows: it is the locus of points $P \in \mathcal{X}$ where the natural map of vector bundles over $\mathcal{X}$:

$$\begin{array}{ccc} \pi^*\mathbb{E} & \xrightarrow{D^{g-1}} & J^{g-1}_\pi \omega_\pi \\ & \searrow \quad \swarrow & \\ & \mathcal{X} & \\ & \downarrow \pi & \\ & S & \end{array} \tag{2}$$

drops rank. In other words, taking the top exterior power of the map D^{g-1} one has:

$$\mathcal{I}_\pi = Z(\mathbb{W}_\pi)$$

where $\mathbb{W}_\pi$ is the relative wronskian, *i.e.*:

$$\mathbb{W}_\pi := \wedge^g D^{g-1} \in H^0(\mathcal{X}, \omega_\pi^{\otimes \frac{g(g+1)}{2}} \otimes \wedge^g \pi^* \mathbb{E}^\vee) , \tag{3}$$

where we observed that $\wedge^g J^{g-1}\omega_\pi \cong \omega_\pi^{\otimes \frac{g(g+1)}{2}}$. Let S_{irr} be the locus of points of S corresponding to irreducible fibers and $\mathcal{X}_{\text{irr}}$ the induced family. Then $\mathcal{I}_{\pi|\mathcal{X}_{\text{irr}}}$ cuts out the locus of $P \in \mathcal{X}$ which are Weierstrass points (in the extended sense of Widland and Lax, see [24]) on the corresponding fibers $\mathcal{X}_{\pi(P)}$. Moreover the inflectionary locus $\mathcal{I}_\pi$ contains entirely the components of the reducible fibers with some multiplicity (see *e.g.* [3, p. 325] for one parameter families having a special fiber a stable curve of compact type with two irreducible components). This has to do with the fact that, if the curve is reducible, for each component there are non zero dualizing differentials vanishing identically on that component.

2.5. Let C_0 be now a (complex) stable curve of genus $g \geq 2$. Let R be the ring of formal power series in $3g-3$ indeterminates with $\mathbb{C}$-coefficients, *i.e.* $R = \mathbb{C}[[T_1, \ldots, T_{3g-3}]]$. Let $\mathcal{M}_{C_0} = \text{Spec}(\mathbb{C}[[T_1, \ldots, T_{3g-3}]])$ be the formal moduli space around C_0 and let $\mathcal{X}_{C_0}$ be its universal family together with the structural morphism $\pi : \mathcal{X}_{C_0} \longrightarrow \text{Spec}(R)$ [5, p. 81]. Let $\mathbb{K} = \mathbb{C}((T_1, \ldots, T_{3g-3}))$, so that $\mathcal{X}_\eta \longrightarrow \text{Spec}(\mathbb{K}) = \eta$ is the generic fiber of the family. One has the cartesian diagram:

$$\begin{array}{ccc} \mathcal{X}_\eta & \xrightarrow{i_\eta} & \mathcal{X}_{C_0} \\ {\scriptstyle \pi_\eta}\downarrow & & \downarrow{\scriptstyle \pi} \\ \text{Spec}(\mathbb{K}) & \xrightarrow{j_\eta} & \mathcal{M}_{C_0} \end{array} \tag{4}$$

Because of the flat base change theorem (see *e.g.* [20, p. 255, Proposition 9.3]) we have:

$$\mathbb{E}_\eta := j_\eta^*(\mathbb{E}) = \pi_{\eta,*} \circ \iota_\eta^*(\omega_\pi) = H^0(\mathcal{X}_\eta, \omega_{\mathcal{X}_\eta}) .$$

Let $\mathbb{W}_\eta$ be the wronskian section of the line bundle $\omega_{\mathcal{X}_\eta}^{\otimes \frac{g(g+1)}{2}}$. As previously said, such a wronskian defines a closed subscheme $Z(\mathbb{W}_\eta)$ of $\mathcal{X}_\eta$ of lenght $(g-1)g(g+1)$, hence a point in the Hilbert scheme $\text{Hilb}^{(g-1)g(g+1)}(\mathcal{X}_\eta)$. Let

$$\pi_H : \mathcal{H} = \text{Hilb}_\pi^{(g-1)g(g+1)} \longrightarrow \mathcal{M}_{C_0} ,$$

be the relative Hilbert scheme of 0-dimensional subschemes of the fibers of π of lenght $(g-1)g(g+1)$ (see [17]). Hence $Z(\mathbb{W}_\eta)$ is a point of $\mathcal{H}_\eta$.

2.6. Definition The *Weierstrass scheme* associated to the family

$$\pi : \mathcal{X}_{C_0} \longrightarrow \mathcal{M}_{C_0}$$

is the closure $\mathcal{W}$ of $Z(\mathbb{W}_\eta)$ in $\mathcal{H}$.

Let $\pi_{\mathcal{W}} : \mathcal{W} \longrightarrow \mathcal{M}_{C_0}$ be the restriction of the map π_H to $\mathcal{W}$. Let $\mathcal{W}_0$ be the fiber of $\mathcal{W}$ over the closed point of $\mathcal{M}_{C_0}$.

2.7. Definition A *limit Weierstrass scheme* of C_0 is a zero dimensional subscheme of C_0 corresponding to a point of $\mathcal{W}_0$.

The examples studied in [3] (or the example of the union of two elliptic curves widely discussed in the introduction) prove that in general the map π_W is not a flat map. The fiber over the generic point is zero-dimensional, while the fiber over C_0 may be positive dimensional, as remarked.

2.8. The general problem we study is the following one. Fix a dual graph related to a stable curve of genus g; fix the genus of each vertex. Take a general stable curve C_0 of genus g belonging to the topological stratum of $\overline{M}_g$ defined by such a graph.

1. Give a description of $\mathcal{W}_0$. More concretely: what is the dimension of the irreducible components of $\mathcal{W}_0$; is $\mathcal{W}_0$ irreducible?
2. Let W_0 be an irreducible component of $\mathcal{W}_0$. Take $\mathcal{Z} \in \mathcal{W}_0$ general. If $P \in \mathcal{Z}$ is a smooth point of C_0, is the multiplicity of $\mathcal{Z}$ at P equal to 1? If P is a node of C_0, what is the multiplicity of $\mathcal{Z}$ at P? Does there exist a fixed closed subscheme S with support at P such that $\mathcal{Z} = S$ locally at P?
3. Under what assumption on C_0, does C_0 satisfy general behaviour? As already mentioned in the introduction, in this paper we study those questions when C_0 is the union of two irreducible components.

3. The dimension of the Weierstrass scheme

This section shall be entirely devoted to prove the main result of this paper, concerning the dimension of the limit Weierstrass scheme on a general stable curve which is the union of two irreducible smooth components. We first start with a lemma which is essentially nothing more than a (although useful) remark. Set:

$$\mathcal{W}_n = \{[(C, A_1, \dots, A_n)] \in M_{g,n} : A_1, \dots, A_n \text{ are Weierstrass points of } C\}$$

Then:

3.1. Lemma. *Let $(C_0, P_1, \dots, P_n)$ be a stable n pointed curve such that:*

1. *C_0 is irreducible (not necessarily smooth);*
2. *$P_1, \dots, P_n$ are Weierstrass points of C_0.*

Then $(C_0; P_1, \dots, P_n) \in \overline{\mathcal{W}}_n$, where the closure is taken in $\overline{M}_{g,n}$.

Proof. If C_0 is smooth the claim is obvious. If it is not smooth take any one parameter family (certainly existing) $\pi : \mathcal{X} \longrightarrow S$ such that $S = \mathrm{Spec}(\mathbb{C}[[T]])$, $\mathcal{X}_\eta$ smooth. Now the zero scheme $Z(\mathbb{W}_\pi)$ cuts exactly the locus of Weierstrass points on fibers of π. In particular there are Weierstrass points $P_{1,\eta}, \dots, P_{n,\eta}$ degenerating to $P_1, \dots, P_n$. The family comes so equipped with n sections $\sigma_1, \dots, \sigma_n : S \longrightarrow \mathcal{X}$ such that $\sigma_i(\eta)$ is a Weierstrass point on $\mathcal{X}_\eta$ and $\sigma_i(0) = P_i$ for each $1 \le i \le n$. The total space $\mathcal{X}$ of the family need not to be smooth, however all the singularities are concentrated into the nodes of C_0. Around each node (see [21]) the formal equation of $\mathcal{X}$ is $xy - t^n = 0$ (*i.e.* there is an A_{n-1} singularity). A minimal resolution of $\mathcal{X}$ gotten by repeatedly blowing up the singular points has the effect of inserting a chain of $(n-1)$ (-2)-rational curves. Hence we get a new family such that $\mathcal{X}_\eta$ is the same as before but with $(\mathcal{X}_0, \sigma_1(0), \dots, \sigma_n(0))$ semistably equivalent to $(C_0, P_1, \dots, P_n)$. This proves, by virtue of Proposition 2.2, that $(C_0; P_1, \dots, P_n) \in \overline{W}_n$. □

Then, according to the notation of the previous section, let C_0 be the union of two general smooth projective curves X and Y of genus $g_X \geq 1$ and $g_Y \geq 1$ with $X \cap Y = \{Q_1, \dots, Q_m\}$ generally chosen on X and Y, so that the arithmetic genus of C_0 is $g = g_X + g_Y + m - 1$.

3.2. Theorem. *Let* $C_0 = X \cup Y$ *be general as above. Then* $\dim \mathcal{W}_0 = m - 1$.

Proof. For the reader's convenience we shall split the proof in several steps.

Step 1. We first show that $\dim(\mathcal{W}_0) \leq m - 1$. Suppose not, and assume that $\dim(\mathcal{W}_0) \geq m$. Set:

$$\mathcal{M}^{\mathrm{irr}}_{C_0} = \{\eta' \in \mathcal{M}_{C_0} : \mathcal{X}_{\eta'} \text{ is irreducible}\} ,$$

and

$$\mathcal{M}'_{C_0} = \{\eta'' \in \mathcal{M}_{C_0} : \mathcal{X}_{\eta''} \text{ is of type } X_1 \cup Y_1 \text{with } g_{X_1} = g_X \text{ and } g_{Y_1} = g_Y, \text{ and } \sharp(X_1 \cap Y_1) = m\}.$$

Notice that $\dim(\mathcal{M}'_{C_0}) = 3g - 3 - m$. There are natural inclusions

$$\mathcal{M}^{\mathrm{irr}}_{C_0} \hookrightarrow \mathcal{M}_{C_0} \quad \text{and} \quad \mathcal{M}'_{C_0} \hookrightarrow \mathcal{M}_{C_0} ,$$

so that we can form the following fiber products:

$$\mathcal{W}^{\mathrm{irr}} = \mathcal{W} \times_{\mathcal{M}_{C_0}} \mathcal{M}^{\mathrm{irr}}_{C_0} \longrightarrow \mathcal{M}^{\mathrm{irr}}_{C_0} \text{ and } \mathcal{W}' = \mathcal{W} \times_{\mathcal{M}_{C_0}} \mathcal{M}'_{C_0} \longrightarrow \mathcal{M}'_{C_0} ,$$

where $\mathcal{W}$ is the Weierstrass scheme (Definition 2.6). Now, by the very definition of $\mathcal{W}$ we have the inclusions:

$$\mathcal{W}' \subset \overline{\mathcal{W}^{\mathrm{irr}}} = \mathcal{W} .$$

Now, by [1], $\mathcal{W}^{\mathrm{irr}}$ is irreducible so that the same holds for its closure $\overline{\mathcal{W}^{\mathrm{irr}}}$. Hence:

$$\dim(\mathcal{W}') < \dim(\mathcal{W}^{\mathrm{irr}}) = 3g - 3 , \tag{5}$$

where the last equality follows from the fact that $\dim(\mathcal{M}^{\mathrm{irr}}_{C_0}) = 3g - 3$ and that on a irreducible curve the limit Weierstrass scheme (*i.e.* the subscheme of Weierstrass points) is zero dimensional (it is fixed!). But, by the assumption, we have that:

$$\dim(\mathcal{W}') = \dim(\mathcal{W}_0) + \dim(\mathcal{M}'_{C_0}) = m + (3g - 3 - m) = 3g - 3 ,$$

and this contradicts inequality (5). We have hence proven that, necessarily, $\dim(\mathcal{W}_0) \leq$ $\leq m - 1$.

STEP 2. We prove now that $\dim(\mathcal{W}_0) \geq m-1$ by showing that if $P_1, \ldots, P_m$ are smooth points of C_0 arbitrarily chosen, then there exists a 1-parameter family $\mathcal{X} \longrightarrow \mathrm{Spec}(\mathbb{C}[[T]])$, together with n sections $\sigma_1, \ldots, \sigma_n$, such that $\sigma_i(\eta)$ is a Weierstrass point on $\mathcal{X}_\eta$, for $1 \leq i \leq m-1$, and $(\mathcal{X}_0; \sigma_1(0), \ldots, \sigma_n(0))$ is semistably equivalent to $(C_0; P_1, \ldots, P_{m-1})$.

Now, an ordered $(m-1)$-tuple $(P_1, \ldots, P_{m-1})$ of points of C_0 is nothing but a point of the $(m-1)$-fold product:

$$C_0^{m-1} = \underbrace{C_0 \times \ldots \times C_0}_{(m-1) \text{ times}}$$

of the curve C_0 by itself. We are hence naturally led to consider the $(m-1)$-fold product of $\mathcal{X}_{C_0}$ over $\mathcal{M}_{C_0}$:

$$\mathcal{X}_{C_0}^{m-1} := \underbrace{\mathcal{X}_{C_0} \times_{\mathcal{M}_{C_0}} \ldots \times_{\mathcal{M}_{C_0}} \mathcal{X}_{C_0}}_{(m-1) \text{ times}} .$$

Let $\mathrm{pr}_i : \mathcal{X}_{C_0}^{m-1} \longrightarrow \mathcal{X}_{C_0}$ be the projection onto the i-th factor and

$$\rho : \mathcal{X}_{C_0}^{m-1} \longrightarrow \mathcal{M}_{C_0} \quad (\pi \circ \mathrm{pr}_i = \rho)$$

be the structural morphism induced by $\pi : \mathcal{X}_{C_0} \longrightarrow \mathcal{M}_{C_0}$. A point of $\mathrm{pr}_i^{-1}(Z(\mathbb{W}_\pi))$ is the locus of $(m-1)$-tuples $(A_1, \ldots, A_{m-1}) \in \mathcal{X}_{C_0}^{m-1}$ such that A_i is in the zero locus of the wronskian section on $\mathcal{X}_{\pi(A_i)}$. In particular, if $\mathcal{X}_{\pi(A_i)}$ is irreducible, A_i is precisely a Weierstrass point of $\mathcal{X}_{\pi(A_i)}$. Let us consider the intersection:

$$\cap_{i=1}^{m-1} \mathrm{pr}_i^{-1}(Z(\mathbb{W}_\pi)) \subset \mathcal{X}_{C_0}^{m-1} ,$$

whose fiber over the generic point η of $\mathcal{M}_{C_0}$ is a $(m-1)$-tuple of Weierstrass points. Since for each component of C_0 there are non zero sections of the dualizing sheaf ω_{C_0} which vanish identically along that component, it follows that C_0 is entirely contained in $Z(\mathbb{W}_\pi)$, so that, in particular:

$$(P_1, \ldots, P_{m-1}) \in \cap_{i=1}^{m-1} \mathrm{pr}_i^{-1}(Z(\mathbb{W}_\pi)) .$$

Let $\mathcal{D}$ be an irreducible component of $\cap_{i=1}^{m-1} \mathrm{pr}_i^{-1}(Z(\mathbb{W}_\pi))$ containing $(P_1, \ldots, P_{m-1})$. Then:

$$\dim(\mathcal{D}) \geq 3g-3 ,$$

while

$$\dim(\rho(\mathcal{D}) = \dim(\mathcal{D}) - (m-1) \geq (3g-3-m) - 1 \geq \dim(\mathcal{M}'_{C_0}) .$$

This means that $\rho(\mathcal{D}) \not\subseteq \mathcal{M}'_{C_0}$. Let η'_0 be the generic point of $\rho(\mathcal{D})$. Then $\mathcal{X}_{\eta'_0}$ is a generalization of C_0 smoothing at least one node (because C_0 is general). In particular $\mathcal{X}_{\eta'_0}$ is irreducible because otherwise the numerical compatibility between the genus of $\mathcal{X}_{\eta'_0}$ $(g = g_X + g_Y + m - 1)$ and the number of nodes would be violated. So we came to prove that the pointed curve:

$$(C_0; P_1, \dots, P_{m-1})$$

is in the closure of the locus of n-pointed *irreducible* curves

$$\{(C, A_1, \dots, A_{m-1})\} \subset \mathcal{M}_{C_0, m-1}$$

such that A_i is a Weierstrass point, for each $1 \leq i \leq n$. By using Lemma 3.1 we conclude that $(C_0, P_1, \dots, P_{m-1}) \in \overline{\mathcal{W}}_{m-1}$ (by transitivity of the relation of inclusion). By using Proposition 2.2, we have hence proven that there exists a stable curve $\pi : \mathcal{X} \longrightarrow S$ over $S = \mathrm{Spec}(\mathbb{C}[[T]])$, with $\mathcal{X}$ a smooth surface, together with $m-1$ sections $\sigma_1, \dots, \sigma_{m-1}$ such that $\sigma_i(\eta)$ is a Weierstrass point on $\mathcal{X}_\eta$ and $(\mathcal{X}_0, \sigma_1(0), \dots, \sigma_{m-1}(0))$ is semistably equivalent to $(C_0, P_1, \dots, P_{m-1})$. In other words we proved that $(P_1, \dots, P_{m-1}) \in \mathcal{W}_0$, so that $\dim(\mathcal{W}_0) \geq m-1$. Patching such an equality with the one found in Step 1, we get finally:

$$\dim(\mathcal{W}_0) = m - 1 \ ,$$

as desired. □

4. Aspects of limits Weierstrass schemes

4.1. Definition Let C_0 be a stable curve of genus g; let $\mathcal{Z}$ be a limit Weierstrass scheme of C_0 and let X be an irreducible component of C_0. Let $X_{\mathrm{smooth}} = X \setminus (X \cap \cap \overline{(C_0 \setminus X)})$. The restriction $\mathcal{Z}_{|X_{\mathrm{smooth}}}$ as a scheme is called the X-aspect of $\mathcal{Z}$; it is denoted by $\mathcal{Z}_X$.

4.2. Let C_0 be a stable curve of genus $g > 2$. A *good family* for C_0 is a semi-stable generically smooth curve $\pi : \mathcal{X} \longrightarrow S$ over $S = \mathrm{Spec}(\mathbb{C}[[T]])$, such that:

1. $\mathcal{X}$ is a smooth surface;
2. the geometric generic fiber is a non-hyperelliptic smooth curve of genus $g > 2$;
3. the central fiber $\mathcal{X}_0$ has C_0 as a stable model.

4.3. Assume that each component of C_0 is smooth. Let $\mathcal{Z}$ be a limit Weierstrass scheme for C_0. As it follows from Theorem 3.2 it is possible to find a good family $\pi : \mathcal{X} \longrightarrow S$ for C_0 such that the associated limit of Weierstrass scheme $\mathcal{Z}'$ on $\mathcal{X}_0$ satisfies $\mathcal{Z}'_X = \mathcal{Z}_X$ for each irreducible component X of $\mathcal{X}_0$ corresponding to an irreducible component – also denoted by X – of C_0. Let D be a divisor of $\mathcal{X}$ supported on $\mathcal{X}_0$. It turns out that the O_S-module $\pi_*(\omega_\pi(D))$ is a bundle of rank g. In fact $\pi_*(\omega_\pi(D))$ is a finitely generated torsion free $\mathbb{C}[[T]]$-module and hence free by [23, p. 147]. Associated to the rank g-bundle $\pi_*(\omega_\pi(D))$ we have a wronskian on the generic fiber which is equal to the wronskian $\mathbb{W}_\eta$ of $\pi_*(\omega_\pi)$ on the generic fiber. Consider $\pi_*(\omega_\pi(D))(0)$, the fiber of $\pi_*(\omega_\pi(D))$ to the closed point on S; it is isomorphic to $H^0(\mathcal{X}_0; \omega_\pi(D)_{|\mathcal{X}_0})$. If the restriction map:

$$\rho_X : H^0(\mathcal{X}_0, \omega_\pi(D)_{|\mathcal{X}_0}) \longrightarrow H^0(X, \omega_\pi(D)_{|X}) \ , \tag{6}$$

is injective, then $\omega_\pi(D)$ defines a linear system of dimension $g-1$ on X. This linear system has an inflection divisor $\mathcal{R}$ on X.

4.4. PROPOSITION. $\mathcal{R} \cap X_{\text{smooth}} = \mathcal{Z}_X$.

PROOF. One constructs the wronskian of the family $\mathcal{X} \longrightarrow S$ with respect to the sheaf $\omega_\pi(D)$. In other words, one considers the wronskian section $\mathbb{W}_D$ of the bundle $\wedge^g J_\pi^{g-1}\omega_\pi(D) \otimes \left(\wedge^g \pi^*(\pi_*\omega_\pi(D))\right)^\vee$. Such a wronskian defines a divisor $Z(\mathbb{W}_D)$ on $\mathcal{X}$. The closure of $Z(\mathbb{W}_D) \setminus \mathcal{X}_0$ restricted to $\mathcal{X}_0$ is the limit Weierstrass scheme $\mathcal{Z}'$ on $\mathcal{X}_0$. Since ρ_X is injective on $\pi_*(\omega_\pi(D))(0)$, the curve X is not contained in $Z(\mathbb{W}_D)$ and $Z(\mathbb{W}_D) \cap X = \mathcal{R}$. Also, at points of X_{smooth} we have $Z(\mathbb{W}_D)_{|X} = \mathcal{Z}'$. Hence $\mathcal{R} \cap X_{\text{smooth}} = Z(\mathbb{W}_D) \cap X_{\text{smooth}} = \mathcal{Z}' \cap X_{\text{smooth}} = \mathcal{Z} \cap X = \mathcal{Z}_X$. □

4.5. DEFINITION Let $\pi : \mathcal{X} \longrightarrow S$, X and D as in 4.3 and assume that the map ρ_X of formula (6) is injective. Then D will be said to be *suited* for the component X in the family π.

4.6. EXAMPLE *The union of an elliptic curve X and a smooth irreducible curve Y with* $g_Y > 0$ *meeting at m points*. The general situation we start with is as follows: $C_0 = X \cup Y$, X and Y smooth, $g_X = 1$ and $g_Y > 0$. Set $X \cap Y = \{Q_1, \dots, Q_m\}$. Moreover the following notation shall be used: $Q_i = Q_{i,Y}$ in Y and $Q_i = Q_{i,X}$ in X.

Take $P_1, \dots, P_{m-1}$ general on Y. From Theorem 3.2 we know that there exists a limit Weierstrass scheme $\mathcal{Z}$ on C_0 with $\{P_1, \dots, P_{m-1}\} \subset \mathcal{Z}$. For this $\mathcal{Z}$, let $\pi : \mathcal{X} \longrightarrow S$ be as at the end of Theorem 3.2. Let us choose a point in the $(m-1)$-tuple $(P_1, \dots, P_{m-1})$, say P_1. Using the theory of admissible coverings (see [21] and [4]) there is a family $\pi' : \mathcal{X}' \longrightarrow S'$ of nodal curves associated to π such that for the special fiber $\mathcal{X}'_0$ there is an admissible covering $f : \mathcal{X}'_0 \longrightarrow D_0$ of degree $g = m + g_Y$, with D_0 of genus 0 and having a total ramification point P'_1 on some component Y' of $\mathcal{X}'_0$ corresponding to the point P_1 on $Y \subseteq C_0$. Let $f_Y : Y \longrightarrow \mathbb{P}^1$ be the associated covering for Y. Since no points of Y are identified in C_0 and since C_0 contains no rational curves, P_1 is a total ramification point for f_Y. In case $\deg(f_Y) < m + g_Y$, then the way to link $P_1 \in Y \subset \mathcal{X}_0$ with Y' is by means of a chain of rational curves. These rational curves are connected to all components of $f^{-1}(f(Y))$. Since P is a smooth point on C_0, all those curves have to disappear in the stable model of $\mathcal{X}_0$. Hence the components of $f^{-1}(f(Y))$ different from Y are rational curves, not linked to Y. Suppose $f_Y(Q_{1,Y}) \neq f_Y(Q_{2,Y})$. Take some rational curve L with $f(L) = f(Y)$ and $Q \in f^{-1}(f(Q_{1,Y}) \cap L)$. Using a chain of rational curves, $Q_{1,Y}$ must be connected to $Q_{1,X}$. In the admissible covering, Q is connected to some Q' on a rational curve L' with $f(X) = f(L')$. Arguing as above, using a point on $f^{-1}(f(Q_{2,Y})) \cap L$, one finds that the chain of rational curves connected to P would not be a tree. This is a contradiction, hence $f_Y(Q_{1,Y}) = \dots = f_Y(Q_{m,Y})$. But then $m'P \sim Q_{1,Y} + \dots + Q_{m,Y} + F$ for some effective divisor F on Y and some $m' < m + g_Y$. Because P_1 is general with respect to $\{Q_{1,Y}, \dots, Q_{m,Y}\}$ this is impossible. Therefore $\deg(f_Y) = m + g_Y$. Assume $f_Y(Q_{1,Y}) \neq f_Y(Q_{2,Y})$ Because P_1 is general with respect to $\{Q_{1,Y}, \dots, Q_{m,Y}\}$

it follows that $f_Y^{-1}(f_Y(Q_{1,Y}))$ contains a point $Q \notin \{Q_{1,Y}, \dots, Q_{m,Y}\}$. Hence $Q_{1,Y}$ (resp. $Q_{2,Y}$) is linked to $Q_{1,X}$ (resp. $Q_{2,X}$) by means of a chain of rational curves. Using f it follows that Q is linked to a component of $f^{-1}(f(X))$. Since Q is not linked to a point of X, it is linked to a rational curve L in $f^{-1}(f(X))$. Using the chain of rational curves linking $Q_{2,Y}$ to $Q_{2,X}$, one finds that some point of L is linked to a point $f_Y^{-1}(f_Y(Q_{2,Y})) \subset Y$. This contradicts again the fact that C_0 is the stable model of $\mathcal{X}'_0$. It follows that $Q_{1,Y} + \dots + Q_{m,Y}$ belongs to some fiber of f_Y and because of the general choice of P_1 with respect to $\{Q_{1,Y}, \dots, Q_{m,Y}\}$, we find $(m + g_Y)P_1 \sim Q_{1,Y} + \dots + Q_{m,Y} + F$, for some effective divisor F of degree g_Y with $Q_{i,Y} \notin F$. In the admissible covering, identifying $Q_{i,Y}$ with $Q_{i,X}$ using a chain of rational curves, the points of F are connected to points on lines L with $f(L) = f(X)$. The union of those rational curves has to be a disjoint union of trees since they have to disappear in the stable model of C_0. We obtain a picture for the admissible covering as follows, drawn in the case that the support of the divisor F is made by distinct points. Were it not so, the picture would be drawn with the obvious modification (adding ramifications to the rational components of self-intersection -1 attached to F). It is also possible that some more trees of $\mathbb{P}^1$'s are attached to X or Y due to non ordinary ramification points).

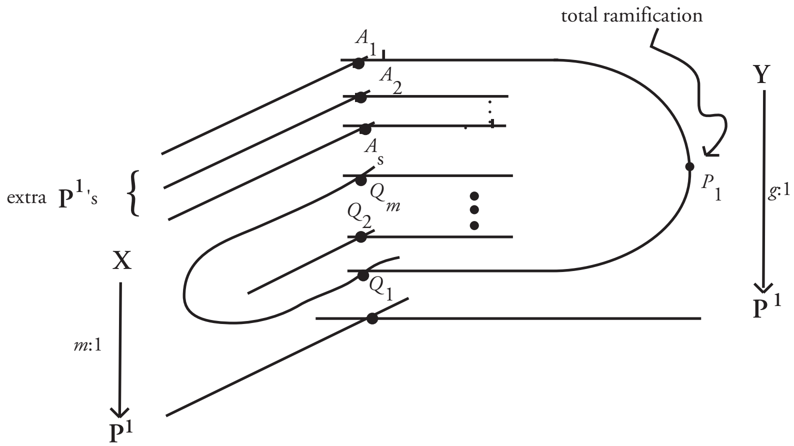

Fig. 2.

From the description at the end of the proof of Theorem 4 in [21], we know that, for each point Q_i there is some $t \geq 0$ such that $\mathcal{X}$ has an A_t singularity around Q_i. The t may be not the same for all the Q_i's. However, looking at the admissible covering of fig. 2, we see that at each node there is a simple ramification. This says, following [4, pp. 48-49], that the above admissible covering may be smoothed in a family:

(7)
$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{P} \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

such that around each Q_i, the surface $\mathcal{X}$ is formally given by $xy - t = 0$. Hence our good family smoothing the curve C_0 such that P_1 is a limit of a Weierstrass point admits a *smooth* total space. We may hence safely use the intersection theory on a smooth surface where all the Weil divisors are Cartier.

Let $D = X$, and consider the sheaf $\omega_\pi(X)$. Then:

$$\omega_\pi(X)_{|_Y} = \omega_Y(2(Q_{1,Y} + \ldots + Q_{m,Y})) \ ,$$

and

$$\omega_\pi(X)_{|_X} = O_X \, .$$

Suppose $s \in H^0(\mathcal{X}_0, \omega_{\pi"}(D))_{|\mathcal{X}_0}$ and $s_{|_Y} = 0$. Then, since $s(Q_i) = 0$ and $\deg(s_{|_X}) = 0$, it follows that $s_{|_X} = 0$, *i.e.* $s = 0$. Hence X is suited for Y in the family π.

5. Determining aspects on stable curves

5.1. Let then $X \cup Y$ be a stable curve union of two irreducible smooth components of genus $g_X = 1$ and $g_Y > 0$ respectively, intersecting transversally at $m \geq 2$ points $Q_1, \ldots, Q_m$ as in Example 4.6 and let $\pi : \mathcal{X} \longrightarrow S$ be any good smoothing family. Then the divisor D of Example 4.6 is suited with respect to the component Y. Hence $H^0(\mathcal{X}_0, \omega_\pi(D)_{|\mathcal{X}_0})$ is, via the restriction map ρ_Y a $g = (g_Y + m)$-dimensional vector subspace of $H^0(Y, \omega_Y(2(Q_1 + \ldots + Q_m)))$, hence a g^{g-1}_{2g-2} on Y.

5.2. Proposition. *The g^{g-1}_{2g-2} on Y so defined is a base point free linear system on Y.*

Proof. As already said $\pi_*(\omega_\pi(D))(0)$ is a $g = (g_Y + m)$-dimensional vector subspace of:

$$H^0(Y, \omega_Y(2(Q_1 + \ldots + Q_m))) = H^0(Y, \omega_\pi(D)_{|_Y}) \, .$$

Take a global section σ of $\pi_*(\omega_\pi(-Y))(0)$ such that $\sigma(Q_i) = 0$ for some i. Since $\deg(\sigma_{|_X}) = 0$ it follows that σ is identically zero on X, so that $\sigma(Q_i) = 0$ for all i. Hence the $g^{g-1}_{2g-2}(Y)$ is determined by a g dimensional subspace of $H^0(Y, \omega_Y(2(Q_1 + \ldots + Q_m)))$ whose sections enjoy the property that the vanishing at any of the Q_i's implies the vanishing at all of them. All such g-dimensional subspaces of $H^0(Y, \omega_Y(2(Q_1 + \ldots + Q_m)))$ can be gotten as follows. Let $Z(Q_1, \ldots, Q_m)$ be the subspace of $H^0(Y, \omega_Y(2(Q + + \ldots + Q_m)))$ spanned by all the sections vanishing simultaneously at $Q_1, \ldots, Q_m$. This is a $g - 1$ dimensional subspace of $H^0(Y, \omega_Y(2Q_1 + \ldots + Q_m))$ and, as a matter of fact, it turns out to be:

$$\mathrm{Im}(H^0(Y, \omega_Y(Q_1 + \ldots + Q_m))) \longrightarrow H^0(Y, \omega(2(Q_1 + \ldots + Q_m))) \ ,$$

or, what is the same, the g^{g-2}_{2g-2} linear series with base points:

$$|\omega_Y(Q_1 + \ldots + Q_m)| + Q_1 + \ldots + Q_m \, .$$

Let $\sigma_0, \ldots, \sigma_{g-2}$ be such that $Z(Q_1, \ldots, Q_m) = \mathrm{span}\{\sigma_0, \ldots, \sigma_{g-2}\}$. Then $\pi_*(\omega_\pi(-Y))(0)$

is necessarily of the form span$\{\sigma_0, \dots, \sigma_{g-2}, \sigma_{g-1}\}$, where $\sigma_{g-1}(Q_i) \neq 0$. Each such linear system is clearly base point free. $\square$

5.3. Theorem. *For the good family* $\pi : \mathcal{X} \longrightarrow S$ *as above, the ramification locus of the* $g^{g-1}_{2g-2}(Y)$ *defined by* $\pi_*(\omega_\pi(D))(0)$, *is contained in the* Y*-aspect* $\mathcal{W}_0(\pi)_Y$ *of the LWS.*

Before proving the claim it is worth to spend few words of comment. We already proved that if P_η is a generic Weierstrass point degenerating to $P_0 \in Y_{\text{smooth}}$, then it is in the support of the ramification scheme of the g^{g-1}_{2g-2}. Conversely, Theorem 5.3 will show that each smooth ramification point of the $g^{g-1}_{2g-2}(Y)$ is limit of a WP. However one cannot exclude a priori that also some (or all of them) of the Q_i's (*i.e.*; the nodes of C_0) occur in such ramification locus. The proposition says that, in this case, there are sections of WP's degenerating to such points. We shall show that, for general families, all the ramification locus of the $g^{g-1}_{2g-2}(Y)$ lies in the smooth locus of Y.

Proof. We use the fact that the locally free rank g sheaf $\pi_*\omega_\pi(D)$ induces a morphism ϕ of $\mathcal{X}$ to $\mathbb{P}^{g-1}$ over S. Such a morphism ϕ «contracts» the component X to a point of $\mathbb{P}^{g-1}$, so that the image of Y through the morphism $\phi_{0,Y}$ induced by ϕ is an irreducible curve of geometric genus g_Y with a m-branched singular point. What we get by means of ϕ is a flat family of projective curves over S whose generic fiber is a geometrically smooth curve of genus g. The arithmetic genus of $\phi_{0,Y}(Y)$ is g as well, and the hyperplane series coincides with the linear series defined by the dualizing sheaf of $\phi_{0,Y}(Y)$, which is hence invertible. In other words, $N = \phi_{0,Y}(Q_i)$ is a Gorenstein singularity having $m = \sharp(X \cap Y)$ as δ invariant. Now, by general semicontinuity arguments, the inflection points of the hyperplane series on the geometric generic fiber degenerate to inflection points of the same series on the special fiber. By the general theory of Weierstrass points on Gorenstein curves (cf. [24]), it follows that there are at least $mg(g-1)$ smooth Weierstrass points of $\phi(\mathcal{X})_{\overline{\eta}}$, degenerating to the singular point N. The total weight of the other limits is due to the sum of:

1. the total weight of the ramification points of Y_{smooth} with respect to the hyperplane series and
2. some extraweight E (cf. [15]) at the singular points which arises if some of the branches is inflectional with respect to the hyperplane series.

Such a total weight is precisely the total weight of the ramification points of the $g^{g-1}_{2g-2}(Y)$ [15] which is, by applying the Brill-Segre formula, exactly $g(g-1)(g_Y+1)$. Notice that $mg(g-1) + g(g-1)(g_Y+1) = (g-1)g(g+1)$, as it should be, which means, in particular, that the part of the Y-aspect of the LWS contained in Y_{smooth} is completely contained in the ramification locus of the $g^{g-1}_{2g-2}(Y)$. $\square$

5.4. Corollary. *The total weight of the Weierstrass points degenerating onto* X_{smooth} *is bounded by* $mg(g-1) + E$.

PROOF. The proof is obvious. We only remark here that some of the generic Weierstrass points degenerating on X could end at the nodes $Q_1, \dots, Q_m$, possibility which occurs in many interesting cases. Of course, if the curve $X \cup_{Q_1}, \dots, \cup_{Q_m} Y$ has general moduli, then $E = 0$. □

5.5. Thanks to Example 4.6, we know that Theorem 5.3 is not empty, *i.e.* that there exists a family of semi-stable curves $\pi : \mathcal{X} \longrightarrow S$ whose special fiber has $X \cup Y$ as a stable model, with $g_X = 1$, and smooth total space $\mathcal{X}$. This can be easily seen by considering the smoothing coming from an admissible cover. Choose $P_0 \in Y \setminus \{Q_1, \dots, Q_m\}$ arbitrarily. Then one can smooth $X \cup Y$ in a family such that $P_0 \in Y \setminus \{Q_1, \dots, Q_m\}$ is the limit of a Weierstrass point on nearby smooth curves by considering the admissible covering drawn in fig. 2, modulo some extra rational components coming from possible non-ordinary ramifications.

We should notice, now, that Theorem 5.3 shows that the $g^{g-1}_{2g-2}(Y)$ defined by $\pi_* \omega_\pi(-Y)(0)$ is a g-dimensional vector space contained in:

$$H^0(Y, \omega_Y(2(Q_1 + \ldots + Q_m))) ,$$

and containing the $g^{g-2}_{2g-2}(Y) := |\omega_Y(Q_1 + \ldots + Q_m)| + Q_1 + \ldots + Q_m$. All the vector subspaces of $H^0(Y, \omega_Y(2(Q_1 + \ldots + Q_m)))$ enjoying this property form a $(m-1)$-dimensional family, as can be easily checked. This result is compatible with Theorem 3.2. The natural question to ask is now if it is true that any such a vector subspace is the $g^{g-1}_{2g-2}(Y)$ associated to some good family. The affirmative answer is provided by the following:

5.6. PROPOSITION. *Let V be any vector subspace of*

$$H^0(Y, \omega_Y(2(Q_1 + \ldots + Q_m)))$$

containing the g^{g-2}_{2g-2}:

$$|\omega_Y(Q_1 + \ldots + Q_m)| + Q_1 + \ldots + Q_m .$$

Then there exists a good family $\pi : \mathcal{X} \longrightarrow S$ such that the ramification locus of such g^{g-1}_{2g-2} is contained in the Y-aspect of $\mathcal{W}_0(\pi)$.

PROOF.

1. Pick $m-1$ distinct points on Y, $P_1, \dots, P_{m-1}$ in general position. Then there exists one and only one g^{g-1}_{2g-2} as in the statement, admitting them as ramification points. Since we are working in characteristic zero, we can use a simple wronskian argument. Choose a basis $\{v_1, \dots, v_{g_Y+m-1}, w_1, \dots, w_m\}$ of $H^0(Y, \omega_Y(2(Q_1 + \ldots + Q_m)))$, such that:

$$\mathrm{Im}\left(H^0\left(Y, \omega_Y\left(\sum Q_i\right)\right) \longrightarrow H^0\left(Y, \omega_Y\left(2\left(\sum Q_i\right)\right)\right)\right) = \mathrm{span}\{v_1, \dots, v_{g_Y+m-1}\}.$$

Then any V as in the statement can be expressed as the

$$\mathrm{span}\{v_1, \dots, v_{g_Y+m-1}, w\} ,$$

where:

$$w = a_1 v_1 + \ldots + a_{g_Y+m-1} v_{g_Y+m-1} + b_1 w_1 + \ldots + b_m w_m ,$$

and at least one of the b_i's is not zero.
Let us denote by $\mathrm{Wr}(v_1, \ldots, v_{g_Y+m-1}, w)$ the wronskian determinant associated to the given basis of V. By linearity and antisimmetry, it can be expressed as:

$$\mathrm{Wr}(v_1, \ldots, v_{g_Y+m-1}, w) = b_1 \mathrm{Wr}(v_1, \ldots, v_{g_Y+m-1}, w_1) + \ldots + b_m \mathrm{Wr}(v_1, \ldots, v_{g_Y+m-1}, w_m).$$

Since the zero locus of each $\mathrm{Wr}(v_1, \ldots, v_{g_Y+m-1}, w_i)$ is a finite set of points, the w_i's can be chosen in such a way that:

$$\mathrm{Wr}(v_1, \ldots, v_{g_Y+m-1}, w_i)(P_j) \neq 0 \quad 1 \leq i \leq m \quad \text{and} \quad 1 \leq j \leq m-1 .$$

It is hence meaningful to solve the linear system in the unknowns b_i:

$$\sum_{i=1}^{m} b_i \mathrm{Wr}(v_1, \ldots, v_{g_Y+m-1}, w_i)(P_j) = 0 ,$$

which in our general situation has maximal rank. In fact, assume that

$$\sum_{i=1}^{m} b_i \mathrm{Wr}(v_1, \ldots, v_{g_Y+m-1}, w_i)(P_j) = 0 ,$$

are independent relations for $1 \leq j \leq j_0 \leq m-1$. The relations give rise to linear systems of type V having ramification in $P_1, \ldots, P_{j_0}$. Since a general point P_{j_0+1} is not a ramification point of a chosen such linear system V, it follows that:

$$\sum_{i=1}^{m} \mathrm{Wr}(v_1, \ldots, v_{g_Y+m-1}, w)(P_{j_0+1}) = 0 ,$$

is one more independent equation.
As a consequence, the above system yields a m-tuple $(b_1, \ldots, b_m) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ uniquely defined up to a multiplication by a non-zero complex number. This m-tuple defines uniquely

$$V = \mathrm{span}\{v_1, \ldots, v_{g_Y+m-1}, w\} ,$$

i.e. a g^{g-1}_{2g-2} which ramifies in $P_1, \ldots, P_{m-1}$, as required. This concludes the proof of Step 1.

2. By Theorem 3.2 we know that there exists $\pi : \mathcal{X} \longrightarrow S$ such that $\{P_1, \ldots, P_{m-1}\} \subset \subset \mathcal{W}_0(\pi)_Y$. We should check that, generically, the total space $\mathcal{X}$ of the family is smooth. To see it, it suffices to consider a Weierstrass point $P_{\overline{\eta}}$ on the geometric generic fiber $\mathcal{X}_{\overline{\eta}}$ degenerating to $P_0 \in Y$. Then $\mathcal{X}_{\overline{\eta}}$ can be exhibited as a $g:1$ ramified covering of $\mathbb{P}^1_{\overline{\mathbb{C}((T))}}$, having $P_{\overline{\eta}}$ as a total ramification in $\mathcal{X}_{\overline{\eta}}$. In the given family, such a covering degenerates to an admissible covering on the special fiber, having a total ramification in P_0. The special fiber of the family of admissible coverings so constructed has to be of the form of Pict. 3.1, as shown in Example 4.6. □

Theorem 5.3 together with Proposition 5.6 implies then:

5.7. THEOREM. *Let* $C_0 = X \cup Y$ *be a stable curve over* $\mathrm{Spec}(\mathbb{C})$, *such that* $g_X = 1$ *and* $\sharp(X \cap Y) = m \geq 2$. *Then a subscheme of* Y_{smooth} *of length* $2g(g-1) + (g_Y - 1)g(g-1)$ *is contained in a LWS* $\mathcal{W}_0(\pi)$ *of a suitable good family* $\pi : \mathcal{X} \longrightarrow S$, *if and only if it coincides with the ramification locus of a* g^{g-1}_{2g-2} *contained in* $|\omega_Y(2(Q_1 + \ldots + Q_m))|$ *and containing* $|\omega_Y(Q_1 + \ldots + Q_m)| + Q_1 + \ldots + Q_m$.

To better show the content of the above propositions, it seems worth, now, to discuss two examples, which show that there are types of stable curves whose nodes are always limits of WP's. This fact should be contrasted with what happens for stable curves of compact type: if such curves are «sufficiently general», the nodes are not limits of Weierstrass points (see *e.g.* [4, 8]).

5.8. Let $C_0 = X \cup Y$ be a stable curve of genus $m+1$, which is the union of two elliptic curves meeting transversally in $m \geq 2$ points. We already know that given any g^{g-1}_{2g-2} on Y as in Theorem 5.3 then there exists a good family $\pi : \mathcal{X} \longrightarrow S$ such that $(\pi_* \omega_\pi(D)(0) \subseteq H^0(Y, O_Y(2(Q_1 + \ldots + Q_m)))$ coincides with such series. Also we know that the same family induces (uniquely!) on X a $g^{g-1}_{2g-2}(X)$ having the same features. Generically, a g^{g-1}_{2g-2} on Y can be chosen in such a way that it does not ramify at the nodes $Q_1, \ldots, Q_m$. Then on the smooth locus of X and Y, if the family $\pi : \mathcal{X} \longrightarrow S$ is chosen in a sufficiently general way, degenerate $2m(m+1)$ generic Weierstrass points. The total weight of the *LWS* is $m(m+1)(m+2)$, so that $m(m+1)(m-2)$ Weierstrass points in the generic fiber must degenerate onto the nodes. This can be seen also by considering the embedding of $\pi : \mathcal{X} \longrightarrow S$ in $\mathbb{P}^m_S$ by means, for instance, of $\pi_* \omega_\pi(D)$. The image of the special fiber in $\mathbb{P}^m_{\mathbb{C}}$ is an integral Gorenstein curve of arithmetic genus $m+1$ and a m-branched singular point having m as δ-invariant. Hence, the total weight of the Weierstrass points degenerating on X is given by the weight of such a node, which is $m^2(m+1)$. Since only $2m(m+1)$ Weierstrass points degenerate on X_{smooth}, it follows that $m(m+1)(m-2)$ should degenerate onto $Q_1, \ldots, Q_m$.

Notice that in the above example, once we choose a g^{g-1}_{2g-2} on Y, we can find a good family smoothing it which induces uniquely a $g^{g-1}_{2g-2}(X)$ on X. However it is not yet clear to us the shape of such $g^{g-1}_{2g-2}(X)$, beside the already mentioned properties. Clearly, the space of the $g^{g-1}_{2g-2}(X)$ which fits in the same LWS together with the fixed g^{g-1}_{2g-2} is zero dimensional. But it is not clear the link between the two. In spite of this fact, there are examples where our knowledge of the aspects of the LWS is more precise, as the following example shows.

5.9. EXAMPLE In this example $C_0 = X \cup Y$ (as in fig. 3) is such that $X \cap Y = \{A, B\}$, $g_X = 1$, $g_Y = 2$, and $A + B$ is not a canonical divisor for Y and that $A - B$ is not 2-torsion on X. Let $P_0 \in Y \setminus \{A, B\}$ be a point arbitrarily chosen. Then we claim that:

1. (*a*) There exists a good family $\pi : \mathcal{X} \longrightarrow S$ such that $P_0 \in Y$ is the limit of a WP $P_{\overline{\eta}}$ on $\mathcal{X}_{\overline{\eta}}$.

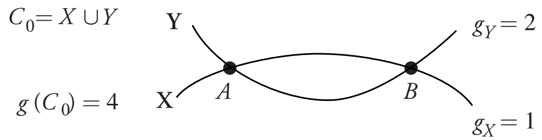

Fig. 3.

(b) All the limits of the Weierstrass points in the smooth locus of Y are the ramification points of the unique $g_6^3(Y)$ contained in $|\omega_Y(2A + 2B)|$, containing $|\omega_Y(A+B)| + A + B$ and ramifying in P (existing by Proposition 5.6 and the remarks preceding it).

2. The X-aspect of the LWS contained in X_{smooth} does not depend on the choice of the point $P_0 \in Y$ and hence on the smoothing family choosen. It coincides with the ramification locus of the linear system $|2A + 2B|$.
3. The total weight of the Weierstrass points degenerating to A and B is 8.

Proof of 1. To prove 1, we first construct a $4:1$ admissible covering of a reducible rational curve D_0 by means of C_0, such that $P_0 \in Y$ is a total ramification point. In order to do that, notice that, by the Riemann-Roch formula, one has:

$$h^0(Y, 4P_0 - A - B) = 1 - 2 + 2 + h^0(\omega_Y(A+B) - 4P_0) \geq 1 ,$$

so that there exists $P \neq Q$, such that $4P_0 \sim A + B + P + Q$. Hence we may cover $\mathbb{P}^1$ with a map $f_Y : Y \longrightarrow \mathbb{P}^1_Y$ such that $f_Y^{-1}(\infty) = 4P_0$ and $f_Y^{-1}(0) = A + B + P + Q$. Moreover, we have:

$$h^0(X, A+B) = 2 ,$$

meaning that there exists a holomorphic function of degree 2, $f_X : X \longrightarrow \mathbb{P}^1_X$ such that $f_X^{-1}(0) = A + B$. Adding extra rational components lying over $\mathbb{P}^1_X$, setting $D_0 = \mathbb{P}^1_X \cup \mathbb{P}^1_Y$, one gets the admissible covering of fig. 4.

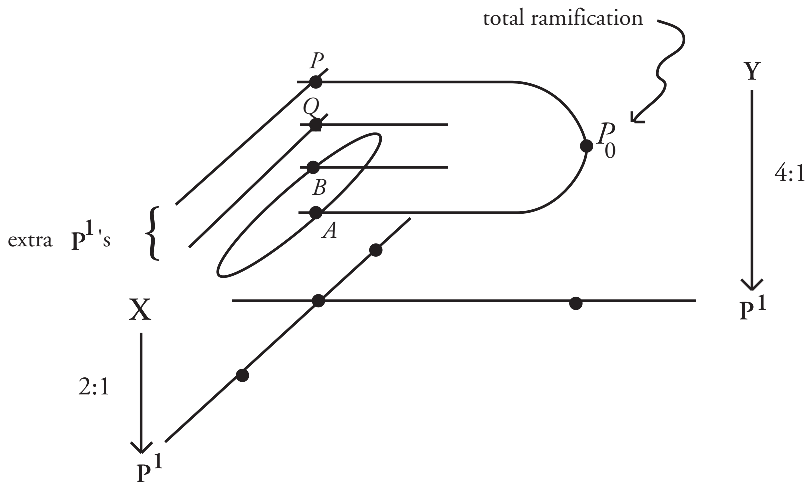

Fig. 4.

We claim that such an admissible covering can be smoothed in a family:

(8)
$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{P} \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

with $\mathcal{X}$ a smooth surface. This follows from the analysis performed in [4, pp. 48-49], where the Author investigates under which conditions the smoothing of admissible covering gets an A_t $(t \geq 1)$ singularity around the nodes of the special fiber. In our case the smoothing surface, around each node A and B of the special fiber is formally given by $xy - t = 0$ (cf. [4, pp. 48-49). That there is a unique $g_6^3(Y)$ fulfilling the property 1 b above, has already been proven in Proposition 5.6. Notice that the divisor X is suited with respect to Y. In fact $\omega_\pi(X)_{|X} = 0$, and twisting ω_π by $O_{\mathcal{X}}(X)$ and restricting it on the special fiber is meaningful, because we checked that $\mathcal{X}$ is smooth. It follows that the aspect $g_6^3(Y)$ is a sublinear series of $|\omega_Y(2A + 2B)|$ containing $|\omega_Y(A + B) + A + B|$ (cfr. proof of Proposition 5.6) and it ramifies at limits of WP's (by Theorem 5.3). Hence it ramifies at P_0. But there is only one such $g_6^3(Y)$, so that it is the same we started with. A simple application of the Brill-Segre formula tells us that the total weight of the Y-aspect of the LWS is 36. This concludes the proof of Claim 1.

PROOF OF 2. To prove 2, first of all recall that we may freely use the intersection theory on a smooth surface, since the family smoothing the admissible covering of fig. 4, has a smooth surface as a total space. Now notice that $\deg(\omega_\pi(-2X)_{|X}) = \deg(O_X(3A + 3B)) = 6$. Hence, $\deg(\omega_\pi(-2X)_{|Y}) = \deg(\omega_Y(-A - B)) = 0$. Moreover, since $A + B \not\sim \omega_Y$, we have $H^0(Y, \omega_Y(-A - B)) = 0$. Now, $\pi_*\omega_\pi(-2X)(0)$ can be identified with a vector subspace of $H^0(X, O_X(3A + 3B))$. But any section $\sigma \in \pi_*\omega_\pi(-2X)(0)$ is such that $\sigma_{|Y} = 0$, so that $\sigma(A) = \sigma(B) = 0$. This means that our $g_{2g-2}^{g-1}(X)$ has a base point in A and a base point in B. Hence it coincides, by dimension reasons, with $|2A + 2B| + A + B$ on X. Notice that neither A nor B is a ramification point of the base point free linear series $|2A + 2B|$ on X. For if, say, B were such a ramification point we would have $2A \sim 2B$, contradicting the hypothesis according which $A - B$ is not 2-torsion. By Theorem 5.7, $Q \in X_{\text{smooth}}$ is a limit of a WP on $\mathcal{X}_{\overline{\eta}}$ if and only if it is a ramification point of $|2A + 2B|$ proving the claim 2. Notice that the base points A and B of the $g_{2g-2}^{g-1}(X)$ contributes each by 4 to the total weight of its ramification locus. Nevertheless we cannot conclude yet, for this reason, that some Weierstrass points on the smooth fiber degenerate into the nodes of C_0.

PROOF OF 3. As for Claim 3 we argue as follows: $\pi_*\omega_\pi(X)$ induces a morphism ϕ of $\mathcal{X}$ in $\mathbb{P}^3$ over S. The image of the geometric generic fiber is a canonical smooth curve of genus 4, while the image of the special fiber is a curve of degree 6 with a 2-branched singular point whose δ-invariant (by genus reasons) is equal to 2. In other words, the morphism ϕ contracts the component X (where the degree of the restriction of $\omega_\pi(X)$ is 0). What one gets, in the special fiber, is a Gorenstein curve of arithmetic genus 4. The contribution of the singular point to the total weight is $2 \cdot 4 \cdot 3 = 24$ (see, *e.g.* [24]), which corresponds to the total weight of WP's degenerating onto the component X. But on X we found only 16 limits, located in the smooth locus. Hence 8 WP's on $\mathcal{X}_{\overline{\eta}}$ should degenerate into the nodes. One may expect that in the general situation the limits falling into the nodes distribute uniformly into the two nodes, namely 4 on one node and 4 to the other one. This is actually the case, for general smoothing, as it is shown by Esteves in [10].

Some remarks and few words of warning seem to be necessary. First of all notice that the limit positions of Weierstrass points in the good family we started with in the previous example, have been completely determined. Also, such limits are parametrized by a 1-parameter family of g^3_6 on Y, and this is compatible with Theorem 3.2. On the other hand the reader may wonder about the behaviour of the limits on the component X. In fact, they are fixed, apparently contradicting the well known fact that each point of X can be limit of a Weierstrass point. As a matter of fact, if $P \in X$ is arbitrarily chosen, we can easily construct either the admissible covering 5*a* or the admissible covering 5*b*, by imitating the procedure shown to construct that of fig. 4.

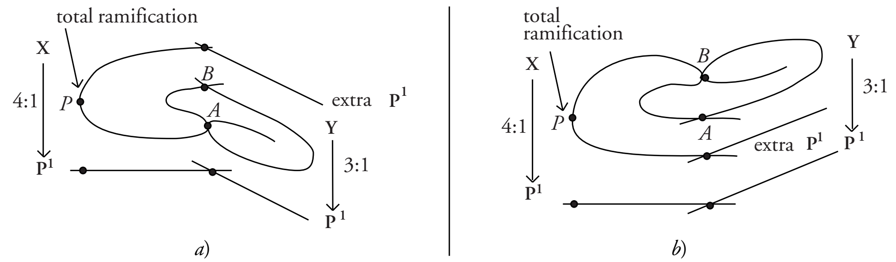

Fig. 5.

However, the surface arising from the smoothing of the admissible cover drawn in fig. 5*a* has an A_1 singularity at A, as well as, similarly, the surface arising from smoothing the admissible cover of fig. 5*b* has an A_1 singularity at B. This follows from the local analysis that, again, may be found in [4, pp. 48-49].

This means that the above procedure of twisting the dualizing sheaf does not make sense in this case, because the divisors involved are not Cartier. Before using the same technique, one should blow up the surface at the singular point. One gets in such a way a new $\pi : \mathcal{X} \longrightarrow S$ with same geometric generic fiber, with $\mathcal{X}$ smooth but with special fiber a semistable model of C_0, namely:

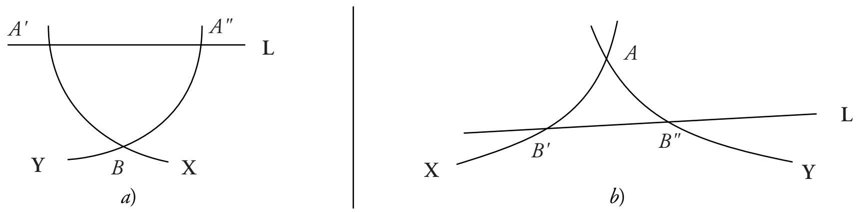

Fig. 6.

gotten by inserting the rational component L at one of the nodes.

It is worth of remarking, before concluding the paper, that we implicitly proved that, in general, the space of the limit Weierstrass scheme is not irreducible. This is because there is a 1-dimensional family of Weierstrass schemes all whose X-aspects are the ramification locus of $|2A + 2B|$, which is a component of the space of Weierstrass schemes (because of Theorem 3.2), so there need to be another 1-dimensional component such that a general element of it contains a general point P of X.

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