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Some results on critical groups for a class of functionals defined on Sobolev Banach spaces

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Analisi matematica. — *Some results on critical groups for a class of functionals defined on Sobolev Banach spaces.* Nota di Silvia Cingolani e Giuseppina Vannella, presentata (*) dal Socio A. Ambrosetti.

Abstract. — We present critical groups estimates for a functional f defined on the Banach space $W_0^{1,p}(\Omega)$, Ω bounded domain in $\mathbb{R}^N$, $2 < p < \infty$, associated to a quasilinear elliptic equation involving p-laplacian. In spite of the lack of an Hilbert structure and of Fredholm property of the second order differential of f in each critical point, we compute the critical groups of f in each isolated critical point via Morse index.

Key words: p-laplacian; Critical groups estimates; Morse index.

Riassunto. — *Alcuni risultati sui gruppi critici per una classe di funzionali definiti su spazi di Sobolev Banach.* Presentiamo stime di gruppi critici per un funzionale f definito sullo spazio di Banach $W_0^{1,p}(\Omega)$, Ω dominio limitato in $\mathbb{R}^N$, $2 < p < \infty$, associato a una equazione ellittica che coinvolge il p-laplaciano. Nonostante la mancanza di una struttura di Hilbert e di proprietà di Fredholm del differenziale secondo di f nei punti critici, valutiamo i gruppi critici di f in ogni punto critico isolato mediante l'indice di Morse.

1. Introduction and statement of the results

In this *Note* we outline some results discussed in [4] in a more complete form. We present critical groups computations for some functionals associated to a class of quasilinear elliptic problems, involving p-laplacian. Precisely, we shall consider the functional $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by setting

$$f(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \int_\Omega G(u)\, dx \tag{1.1}$$

where $2 < p < \infty$ and Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$, with sufficiently regular boundary $\partial\Omega$. Here $G(t) = \int_0^t g(s)\, ds$ and $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following assumption:

(g) $|g'(t)| \leq c_1 |t|^q + c_2$ *with* c_1, c_2 *positive constants and* $0 \leq q < p^* - 2$, $p^* = Np/(N-p)$ *if* $N > p$, *while q is any positive number, if* $N = p$.

Otherwise, if $N < p$, no restrictive assumption on the growth of g is required.

We point out that the computation of critical groups classically requires an Hilbert space structure. In particular we recall that, if H is an Hilbert space and $f : H \to \mathbb{R}$ is a smooth functional, a critical point u of f is said to be non degenerate if the second order differential $f''(u) : H \to H^*$ is an isomorphism.

When u is a non degenerate critical point in H, the Morse splitting lemma holds. As a consequence the local behaviour of the functional near u is quite clear and computing the critical groups is possible via the Morse index $m(f, u)$, namely the supremum of the dimensions of the subspaces on which $f''(u)$ is negative definite. In the case

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in which $f''(u)$ is a Fredholm operator from H to H^* (so even if $f''(u)$ is not an isomorphism) a generalized Morse lemma, due to Gromoll and Meyer, holds, giving the basic tool for the effective computation of the critical groups. Therefore we can say that critical groups estimates seem to require an Hilbert space structure and the presence of Fredholm operators. We emphasize that a lot of difficulties arise from the fact that the functional (1.1) is defined on a Banach (not Hilbert) space.

First of all, it is not at all clear what can be a reasonable definition of non degenerate critical point in this setting. In fact, using the classical definition given for Hilbert spaces, any critical point u of f is degenerate, as, being $p > 2$, $W_0^{1,p}(\Omega)$ is not isomorphic to the dual space $W^{-1,p'}(\Omega)$ $(1/p + 1/p' = 1)$. Furthermore, in our setting $f''(u)$ can not be a Fredholm operator, so not only the classical Morse Lemma does not hold, but even generalized Morse lemmas of Gromoll-Meyer type fail.

In spite of these difficulties, we are able to obtain critical groups estimates for functional f in u.

Before stating the main results, we introduce some notations. For any $a \in \mathbb{R}$, we denote by f^a the set $\{x \in W_0^{1,p}(\Omega) : f(x) \leq a\}$. If u is an isolated critical point of f and $c = f(u)$, then we denote by $m(f, u)$ the Morse index of f in u and by $m^*(f, u)$ the sum of $m(f, u)$ and the dimension of the kernel of $f''(u)$ in $W_0^{1,p}(\Omega)$. Moreover $C_q(f, u)$ denotes the q-th critical group of f in u with respect to a field $\mathbb{K}$, defined by

$$C_q(f, u) = H^q(f^c, f^c \setminus \{u\}) ,$$

$q = 0, 1, 2, \ldots$, where $H^q(A, B)$ stands for the q-th Alexander-Spanier cohomology group of the pair (A, B) with coefficients in $\mathbb{K}$.

The first result we state is the following theorem.

THEOREM 1.1. *Let u be an isolated critical point of the functional* (1.1) *such that $f''(u)$ is injective. Then $m(f, u)$ is finite and*

$$\begin{aligned} C_q(f, u) &\cong \mathbb{K}, && \text{if} \quad q = m(f, u) , \\ C_q(f, u) &= \{0\}, && \text{if} \quad q \neq m(f, u) . \end{aligned}$$

This theorem extends a classical result in Hilbert spaces for non degenerate critical points, showing that the critical groups of f in u depend only upon its Morse index. It is interesting to observe that the usual non degeneracy condition, namely $f''(u)$ is an isomorphism, can be weakened by requiring only the injectivity. This suggests, in the setting of functional (1.1), a new definition of non degenerate critical point, *i.e.*

u is a non degenerate critical point of f if $f''(u) : W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$ is injective.

We mention that in literature some authors have introduced different weaker non degeneracy conditions for the critical points of functionals defined on a Banach space (see *e.g.* [2, 3, 9, 10]). However these non degeneracy conditions seem to be rather involved and in general not easy to be verified.

In the case in which $f''(u)$ is not injective, we shall prove that the number of non trivial critical groups of f in u is finite. Precisely, we state the following result.

THEOREM 1.2. *Let u be an isolated critical point of the functional* (1.1). *Then $m(f, u)$ and $m^*(f, u)$ are finite and*

$$C_q(f, u) = \{0\}$$

for any $q \leq m(f, u) - 1$ and $q \geq m^*(f, u) + 1$.

We remark that the case $q \geq m^* + 1$ corresponds to study $f''(u)$ on an infinite dimensional subspace. In order to overcome this difficulty, we obtain a suitable reduction to finite dimension. We quote that in a recent paper by Lancelotti [6], a finiteness result on the non trivial critical groups is obtained for a class of continuous functionals defined on Hilbert spaces via a finite dimensional reduction.

In a forthcoming paper, the critical groups estimates, obtained in Theorems 1.1 and 1.2, will be applied to get a multiplicity result for a quasilinear elliptic problem arising in the mathematical description of solitons propagation phenomena (see, for example, [1]).

2. SKETCH OF THE PROOF OF THEOREMS 1.1 AND 1.2

In what follows, we denote by $(\cdot|\cdot)$ the scalar product in $\mathbb{R}^N$, by $\|\cdot\|$ the usual norm in $W_0^{1,p}(\Omega)$. Let us denote $B_r(u) = \{v \in W_0^{1,p}(\Omega) : \|v - u\| < r\}$, where $u \in W_0^{1,p}(\Omega)$ and $r > 0$. Moreover we denote by $\langle \cdot , \cdot \rangle : W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega) \to \mathbb{R}$ the duality pairing.

Standard arguments prove that f is a C^2 functional on $W_0^{1,p}(\Omega)$ and it is easy to prove that the second order differential of f in u is given by

$$\langle f''(u)v, w\rangle = \int_\Omega (1 + |\nabla u|^{p-2})(\nabla v|\nabla w)\, dx + \\ + \int_\Omega (p-2)|\nabla u|^{p-4}(\nabla u|\nabla v)(\nabla u|\nabla w)\, dx + \int_\Omega g'(u)vw\, dx$$

for any $v, w \in W_0^{1,p}(\Omega)$.

Let us fix an isolated critical point $u \in W_0^{1,p}(\Omega)$ of f and set $c = f(u)$. By [7, 8], we can infer that $u \in C^1(\overline{\Omega})$. Let $b(x) = |\nabla u(x)|^{(p-4)/2}\nabla u(x) \in L^\infty(\Omega)$. Let H_b be the closure of $C_0^\infty(\Omega)$ under the scalar product

$$(v, w)_b = \int_\Omega (1 + |b|^2)(\nabla v|\nabla w)\, dx + (p-2)(b|\nabla v)(b|\nabla w)\, dx\, .$$

We emphasize that the space H_b is $W_0^{1,2}(\Omega)$ equipped by an equivalent Hilbert structure, which depends on the critical point u, being suggested by $f''(u)$ itself. In such a way $W_0^{1,p}(\Omega) \subset H_b$ continuously.

Now let us denote H_b^* the dual space of H_b and $\langle \cdot , \cdot \rangle_b : H_b^* \times H_b \to \mathbb{R}$ the duality pairing.

Being $u \in C^1(\overline{\Omega})$, $f''(u)$ can be extended to a Fredholm operator $L_b : H_b \to H_b^*$ defined by setting

$$\langle L_b v, w\rangle_b = (v, w)_b + \int_\Omega g'(u)vw\, dx$$

for any $v, w \in H_b$. L_b is a Fredholm operator with index zero, as it is a compact perturbation of the Riesz isomorphism from H_b to H_b^*. Therefore we can consider the splitting

$$H_b = H^- \oplus H^0 \oplus H^+$$

where H^-, H^0, H^+ are, respectively, the negative, null, and positive spaces, according to the spectral decomposition of L_b in $L^2(\Omega)$.

Since $u \in C^1(\overline{\Omega})$, we can deduce from standard regularity theory that

$$H^- \oplus H^0 \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Consequently, denoted by $W = H^+ \cap W_0^{1,p}(\Omega)$ and $V = H^- \oplus H^0$, we get the splitting

$$W_0^{1,p}(\Omega) = V \oplus W .$$

Furthermore, denoting by $\|\cdot\|_b$ the norm induced by $(\cdot,\cdot)_b$, it is obvious that there exists $c > 0$ such that

$$\langle L_b v, v\rangle_b + c\int_\Omega v^2\, dx \geq \|v\|_b^2 \quad \forall\ v \in H_b .$$

Therefore one can easily show that

$$\exists\, \mu > 0 \ \text{ s.t. } \langle L_b v, v\rangle_b \geq \mu\|v\|_b^2 \quad \forall\ v \in H^+ . \tag{2.1}$$

and, by (2.1) we infer

$$\langle f''(u)v, v\rangle \geq \mu\|v\|_b^2 \quad \forall\ v \in W . \tag{2.2}$$

In particular $m^*(f, u) = \dim V$ is finite.

Note that (2.2) does not assure that f is convex in u along the direction of W, as $\|\cdot\|_b$ is weaker than the norm of $W_0^{1,p}(\Omega)$. Furthermore in general (2.2) does not guarantee an «uniform weak convexity» of f near u along the direction of W.

Nevertheless we are able to prove a sort of local convexity in the bounded sets of $L^\infty(\Omega)$ along the direction of W.

This allows to obtain a finite dimensional reduction. More precisely we get the following crucial result.

LEMMA 2.1. *There exist $r > 0$ and $\rho \in]0, r[$ such that for any $v \in V \cap \overline{B}_\rho(0)$ there exists one and only one $\overline{w} \in W \cap B_r(0) \cap L^\infty(\Omega)$ such that for any $z \in W \cap \overline{B}_r(0)$ we have*

$$f(v + \overline{w} + u) \leq f(v + z + u) .$$

So we can introduce the map $\psi : v \in V \cap \overline{B}_\rho(0) \mapsto \overline{w} \in W \cap B_r(0)$ where $\overline{w}$ is the unique minimum point of the function $w \in W \cap \overline{B}_r(0) \mapsto f(u + v + w)$, and it is possible to show that ψ is continuous. Furthermore the function $\phi : V \cap \overline{B}_\rho(0) \to \mathbb{R}$ defined by $\phi(v) = f(u + v + \psi(v))$ is a continuous map with $\phi(0) = f(u) = c$.

Now let us introduce the set

$$Y = \{u + v + \psi(v) \ : \ v \in V \cap \overline{B}_\rho(0)\} .$$

Using a suitable pseudogradient flow it can be proved that

$$C_j(f, u) = C_j(f_{|Y}, u).$$

Moreover it is quite simple to show that

$$C_j(\phi, 0) = C_j(f_{|Y}, u).$$

So finally

$$C_j(f, u) = C_j(\phi, 0)$$

where ϕ is defined on a subset of the finite dimensional space V.

In particular, if $f''(u)$ is injective, it can be deduced that 0 is a local maximum of ϕ in $V \cap \overline{B}_\rho(0)$, so that Theorem 1.1 comes.

More generally, not requiring the injectivity of $f''(u)$, it is clear that $C_j(\phi, 0) = \{0\}$ when $j \geq m^*(f, u) + 1 = \dim V + 1$. Finally Theorem 2.6 of [6] assures that $C_j(\phi, 0) = \{0\}$ if $j \leq m(f, u) - 1$ and thus Theorem 1.2 derives.

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