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Inner Carathéodory completeness of Reinhardt domains

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Funzioni di variabili complesse. — *Inner Carathéodory completeness of Reinhardt domains.* Nota di Włodzimierz Zwonek, presentata (*) dal Socio E. Vesentini.

Abstract. — We give a description of bounded pseudoconvex Reinhardt domains, which are complete for the Carathéodory inner distance.

Key words: Carathédory distance; Inner Carathédory distance; Pseudoconvex Reinhardt domain.

Riassunto. — *Completezza di domini di Reinhardt per la distanza interna di Carathéodory.* Si presenta la descrizione di domini di Reinhardt limitati e pseudo-convessi, che sono completi per la distanza interna di Carathéodory.

In the class of bounded pseudoconvex Reinhardt domains the notion of completeness with respect to the Carathéodory, Kobayashi and Bergman distances is completely understood. In the paper we present the description of bounded pseudoconvex Reinhardt domains complete with respect to the inner Carathéodory distance. It turns out that in the class of bounded pseudoconvex Reinhardt domains Carathéodory completeness coincides with the inner Carathéodory completeness. Therefore, this paper sharpens the result from [10]. This more general result is proven with different methods than those used in [10].

Below we recall the definitions that we shall need (for all the necessary information on these functions consult *e.g.* [5]). For a domain $D \subset \mathbb{C}^n$ we put

$$c_D(w, z) := \sup\{p(f(w), f(z)) : f \in \mathcal{O}(D, E)\}, \quad w, z \in D,$$

$$\widetilde{k}_D(w, z) := \inf\{p(\lambda_1, \lambda_2) : \text{ there is } f \in \mathcal{O}(E, D) \text{ with } f(\lambda_1) = w, f(\lambda_2) = z\},$$

$$k_D := \text{the largest pseudodistance} \leq \widetilde{k}_D, \quad w, z \in D,$$

where p denotes the Poincaré distance on the unit disc $E \subset \mathbb{C}$. We call c_D (respectively, k_D) the *Carathéodory* (respectively, *Kobayashi*) *pseudodistance*.

It is well-known that the Carathéodory pseudodistance is, in contrast to the Kobayashi pseudodistance, not inner (see *e.g.* [1, 7, 4]). Therefore, it is natural to consider the *inner Carathéodory pseudodistance* defined as follows:

$$c_D^i(w, z) := \inf\{L_{\gamma_D}(\alpha)\},$$

where the infimum is taken over piecewise C^1-curves $\alpha : [0, 1] \mapsto D$ such that $\alpha(0) = w$, $\alpha(1) = z$ and $L_{\gamma_D}(\alpha) := \int_0^1 \gamma_D(\alpha(t), \alpha'(t))dt$, where γ_D denotes the infinitesimal version of the Carathéodory pseudodistance, *i.e.* the *Carathéodory-Reiffen pseudometric*

(*) Nella seduta del 20 aprile 2001.

defined as follows

$$\gamma_D(z;X) := \sup\{\gamma_E(F(z);F'(z)X) : F \in \mathcal{O}(D, E)\}, \ z \in D, X \in \mathbb{C}^n ,$$

where $\gamma_E(\lambda;X) := \frac{|X|}{1-|\lambda|^2}$, $\lambda \in E$, $X \in \mathbb{C}$.

Let us recall the property of contractivity of all the distances introduced above, namely, $d_G(F(w), F(z)) \leq d_D(w, z)$ for any $F \in \mathcal{O}(D, G)$, $w, z \in D$, where $d = k, c$ or c^i.

The Bergman distance (defined on bounded domains) goes beyond the class of the above considered pseudodistances (it does not have the above mentioned contractivity property). We do not give its definition here because this distance will only be of illustrative meaning for us. The following fundamental inequalities hold on any bounded domain D:

$$c_D \leq c_D^i \leq b_D , \ c_D^i \leq k_D . \tag{1}$$

Recall that D is called *d-complete* if any d_D-Cauchy sequence is convergent in the natural topology of D ($d = c, c^i, k$ or b).

In order to formulate the known results we need to introduce some objects related to the geometry of Reinhardt domains.

Since now on we assume that D is always a bounded pesudoconvex Reinhardt domain (D is called *Reinhardt* when $z \in D$ if and only if $(\lambda_1 z_1, \dots, \lambda_n z_n) \in D$ for any $\lambda_1, \dots, \lambda_n \in \partial E$).

It is well-known that for a bounded pseudoconvex domain the set $\log D := \{x \in \mathbb{R}^n : (e^{x_1}, \dots, e^{x_n}) \in D\}$ is a convex domain containing no straight line. For any point $a \in \log D$ we define following [11] ($\mathbb{R}_+ := [0, \infty)$):

$$\mathfrak{C}(D) := \{v \in \mathbb{R}^n : a + \mathbb{R}_+ v \subset \log D\} ,$$

$$\widetilde{\mathfrak{C}}(D) := \{v \in \mathfrak{C}(D) : \overline{\exp(a + \mathbb{R}_+ v)} \subset D\} ,$$

$$\mathfrak{C}'(D) := \mathfrak{C}(D) \setminus \widetilde{\mathfrak{C}}(D) .$$

It is easy to see that the sets defined above are independent of the choice of $a \in \log D$. For $j = 1, \dots, n$, let $V_j := \{z \in \mathbb{C}^n : z_j = 0\}$. The description of complete bounded pseudoconvex Reinhardt domains with respect to the Bergman, Carathéodory and Kobayashi distances is given in the following theorem.

THEOREM 1. *Let D be a bounded pseudoconvex Reinhardt domain.*

(a) (cf. [3, 8]) *D is Kobayashi complete.*

(b) (cf. [6, 3, 10]) *D is Carathéodory complete if and only if the following condition is satisfied*:

$$\text{for any } j = 1, \dots, n, \ \text{ if } \bar{D} \cap V_j \neq \emptyset , \ \text{ then } D \cap V_j \neq \emptyset . \tag{2}$$

(c) (cf. [9]) *D is Bergman complete if and only if $\mathfrak{C}'(D) \cap \mathbb{Q}^n = \emptyset$.*

The aim of this *Note* is to give the proof of the following result.

THEOREM 2. *Let D be a bounded pseudoconvex Reinhardt domain. Then D is c^i-complete if and only if D is c-complete,* i.e. *D satisfies* (2).

Recall that there are c-complete but not c^i-complete complex analytic spaces (see [2]) although no example of that type is known in the class of bounded domains in $\mathbb{C}^n$. The above result delivers an example of a class of domains in which two notions of completeness coincide.

PROOF OF THEOREM 2. In view of the inequalities (1) and Theorem 1(b) it is sufficient to show that any domain not satisfying property (2) is not c^i-complete. Because of the contractivity properties of c^i, proceeding exactly as in [10], the proof will be completed if we show that D is not c^i-complete, when $D \subset \mathbb{C}_*^n$ ($\mathbb{C}_* := \mathbb{C} \setminus \{0\}$) is a pseudoconvex Reinhardt domain such that

$$\log D = \{0\} \times (\log \delta \, , - \log \delta)^{n-1} + \mathbb{R}_- \gamma \, ,$$

where $\delta \in (0 \, , 1)$, $\gamma = (\gamma_1 \, , \dots \, , \gamma_n) \in (0 \, , \infty)^n$, $\gamma_1 = 1$.

Fix such a D. Without loss of generality we may assume that $n \geq 2$. Note that $\mathfrak{C}(D) = \mathbb{R}_- \gamma$ and $\mathfrak{C}'(D) = (0 \, , \infty)\gamma$.

Let $\alpha \in \mathbb{Z}^n$. It is easy to see that $z^\alpha := z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}$ is bounded on D, if and only if $\langle \alpha \, , \gamma \rangle \geq 0$.

Let us fix for a while some $0 < t < 1$. Below we get some upper bound for $\gamma_D((t^{\gamma_1} \, , \dots \, , t^{\gamma_n}); (\gamma_1 t^{\gamma_1 - 1} \, , \dots \, , \gamma_n t^{\gamma_n - 1}))$.

Fix an $F \in \mathcal{O}(D, E)$. Then using the Laurent expansion of F

$$F(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha \, , \quad z \in D \, ,$$

where

$$a_\alpha = \frac{1}{(2\pi i)^n} \int_{|\zeta_1| = r_1 \, , \dots , |\zeta_n| = r_n} \frac{F(\zeta) d\zeta_1 \dots d\zeta_n}{\zeta^{\alpha + 1}}$$

is independent of $(r_1 \, , \dots \, , r_n) \in D$ and the convergence of the series is locally uniform on D.

Note that

$$|a_\alpha| \leq \frac{1}{r_1^{\alpha_1} \dots r_n^{\alpha_n}} \text{ for any } r \in D \, . \tag{3}$$

It easily follows from (3) that $a_\alpha = 0$ for all $\alpha \in \mathbb{Z}^n$ such that z^α is not bounded on D (*i.e.* $\langle \alpha \, , \gamma \rangle < 0$). Therefore,

$$F(z) = \sum_{\alpha \in \mathbb{Z}^n : \langle \alpha \, , \gamma \rangle \geq 0} a_\alpha z^\alpha \, , \quad z \in D \, .$$

Taking in (3) $r_1 < 1$ arbitrarily large and r_j, $j = 2 \, , \dots \, , n$, arbitrarily close to δ (or $1/\delta$), we get the following estimate

$$|a_\alpha| \leq \delta^{|\alpha_2| + \ldots + |\alpha_n|} \, .$$

On the other hand

$$|F'(t^{\gamma_1}, \dots, t^{\gamma_n})(\gamma_1 t^{\gamma_1 - 1}, \dots, \gamma_n t^{\gamma_n - 1})| =$$

$$= \left| \sum_{\alpha\in\mathbb{Z}^n:\langle\alpha,\gamma\rangle\geq 0} \left(a_\alpha \sum_{j=1}^n \alpha_j t^{\gamma_1\alpha_1} \dots t^{\gamma_{j-1}\alpha_{j-1}} t^{\gamma_j(\alpha_j-1)} t^{\gamma_{j+1}\alpha_{j+1}} \dots t^{\gamma_n\alpha_n} \gamma_j t^{\gamma_j - 1} \right) \right| \leq$$

$$\leq \sum_{\alpha\in\mathbb{Z}^n:\langle\alpha,\gamma\rangle> 0} |a_\alpha| \langle\alpha,\gamma\rangle t^{\langle\alpha,\gamma\rangle - 1} \leq \sum_{\alpha\in\mathbb{Z}^n:\langle\alpha,\gamma\rangle> 0} \delta^{|\alpha_2|+\ldots+|\alpha_n|} \langle\alpha,\gamma\rangle t^{\langle\alpha,\gamma\rangle - 1}.$$

In view of the above estimates, taking the supremum over all $F \in \mathcal{O}(D, E)$ with $F(t^{\gamma_1}, \dots, t^{\gamma_n}) = 0$ we get the following inequality

$$\gamma_D((t^{\gamma_1}, \dots, t^{\gamma_n}); (\gamma_1 t^{\gamma_1 - 1}, \dots, \gamma_n t^{\gamma_n - 1})) \leq \sum_{\alpha\in\mathbb{Z}^n:\langle\alpha,\gamma\rangle> 0} \delta^{|\alpha_2|+\ldots+|\alpha_n|} \langle\alpha,\gamma\rangle t^{\langle\alpha,\gamma\rangle - 1}.$$

Below we prove that $L_{\gamma_D}(\{(0, 1/2] \ni t \mapsto (t^{\gamma_1}, \dots, t^{\gamma_n}) \in D\}) < \infty$, which will finish the proof of non-c^i-completeness of D.

Note that interchanging the integral with the sum we get

$$\int_0^{1/2} \gamma_D((t^{\gamma_1}, \dots, t^{\gamma_n}); (\gamma_1 t^{\gamma_1 - 1}, \dots, \gamma_n t^{\gamma_n - 1}))dt \leq$$

$$\leq \sum_{\alpha\in\mathbb{Z}^n:\langle\alpha,\gamma\rangle> 0} \delta^{|\alpha_2|+\ldots+|\alpha_n|} \int_0^{1/2} \langle\alpha,\gamma\rangle t^{\langle\alpha,\gamma\rangle - 1} =$$

$$= \sum_{\alpha\in\mathbb{Z}^n:\langle\alpha,\gamma\rangle> 0} \delta^{|\alpha_2|+\ldots+|\alpha_n|} \frac{1}{2^{\langle\alpha,\gamma\rangle}} = \sum_{\alpha_2,\dots,\alpha_n\in\mathbb{Z}} \delta^{|\alpha_2|+\ldots+|\alpha_n|} \sum_{\alpha_1\in\mathbb{Z}:\alpha_1>-\langle\alpha',\gamma'\rangle} \frac{1}{2^{\langle\alpha,\gamma\rangle}} \leq$$

$$\leq \sum_{\alpha_2,\dots,\alpha_n\in\mathbb{Z}} \delta^{|\alpha_2|+\ldots+|\alpha_n|} \frac{1}{2^{\langle\alpha',\gamma'\rangle + [-\langle\alpha',\gamma'\rangle]}}$$

($[x]$ denotes the largest integer not exceeding $x \in \mathbb{R}$, $\alpha' := (\alpha_2, \dots, \alpha_n)$, $\gamma' := := (\gamma_2, \dots, \gamma_n)$).

Since $\langle\alpha',\gamma'\rangle + [-\langle\alpha',\gamma'\rangle] \geq -1$ the last number is finite, which finishes the proof. $\square$

REMARK 3. The contractivity property of c^i easily implies that in the case when the domain chosen in the proof is such that $\gamma \in \mathbb{Q}^n$ the above proof may be completed much faster. Namely, in this case one may easily embed the punctured disc in D and the non-c^i-completeness will follow from the non-c^i-completeness of the punctured disc.

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