Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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Pseudo Laguerre and pseudo Hermite polynomials

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **12** (2001), n.2, p. 75–84.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2001_9_12_2_75_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2001.

Rend. Mat. Acc. Lincei
s. 9, v. 12:75-84 (2001)

Funzioni speciali. — *Pseudo Laguerre and pseudo Hermite polynomials.* Nota di
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Abstract. — We start from pseudo hyperbolic and trigonometric functions to introduce pseudo Laguerre and Hermite polynomials. We discuss the link with families of Bessel functions and analyze all the associated problems from a unifying point of view, employing operational tools.

Key words: Hermite polynomials; Laguerre polynomials; Operator.

Riassunto. — *Pseudo polinomi di Hermite e Laguerre.* Si utilizzano le funzioni pseudo trigonometriche e pseudo iperboliche per introdurre pseudo polinomi di Hermite e Laguerre. Si discute il legame con le famiglie di funzioni di Bessel e si analizzano le relative problematiche da un punto di vista unitario che utilizza metodi operazionali.

1. Introduction

The pseudo hyperbolic and pseudo trigonometric functions have been introduced on the eve of seventies by Ricci [1]. This class of functions providing a fairly natural generalization of the ordinary exponential, hyperbolic and trigonometric functions, offers the possibility of exploring, from a more general and unifying point of view, the theory of special functions including generalized cases.

We will show that starting from the functions introduced in [1], we can recover a common thread linking them to non-standard forms of Hermite and Laguerre polynomials and of Bessel functions.

These introductory remarks are aimed at summarizing the theory of pseudo hyperbolic and trigonometric functions by exploiting a formalism and a point of view more convenient for the purposes of the present paper.

The operator $\mathcal{D}_x^{-1}$ defines the inverse of the derivative and once acting on unity yields

$$\mathcal{D}_x^{-m} 1 = \frac{x^m}{m!} . \tag{1.1}$$

The unity will be omitted in the following for the sake of conciseness. It is evident that $\mathcal{D}_x^{-1}$ is essentially an integral operator and the lower integration limit has been assumed to be zero. The following two identities are a fairly direct consequence of the previous considerations, it is indeed easily checked that

$$\mathcal{D}_x^{-j}(\mathcal{D}_x^{-m}) = \frac{x^{m+j}}{(m+j)!} . \tag{1.2}$$

(*) Nella seduta del 15 dicembre 2000.

and that

$$e^x = \sum_{m=0}^{\infty} \mathcal{D}_x^{-m} = \frac{1}{1-\mathcal{D}_x^{-1}} . \tag{1.3}$$

By recalling that

$$\frac{1}{A} = \int_0^{\infty} e^{-sA} ds , \tag{1.4}$$

we can conclude from eqs. (1.3), (1.4) that

$$e^x = \int_0^{\infty} e^{-s} e^{s\mathcal{D}_x^{-1}} ds = \int_0^{\infty} e^{-s} \mathcal{C}_0(sx) ds , \tag{1.5}$$

which follows from the relations

$$e^{\alpha \mathcal{D}_x^{-1}} = \sum_{m=0}^{\infty} \frac{\alpha^m \mathcal{D}_x^{-m}}{m!} = \mathcal{C}_0(\alpha x) , \qquad \mathcal{C}_n(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!(n+r)!} \tag{1.6}$$

with $\mathcal{C}_n(x)$ being the n^{th} order Tricomi function with generating function [2]

$$\sum_{n=-\infty}^{\infty} t^n \mathcal{C}_n(x) = e^{t+\frac{x}{t}} . \tag{1.7}$$

We extend the definition of exponential by introducing in the following a new family of functions, characterized by an integer r

$$E_0(x; r) = \sum_{m=0}^{\infty} \mathcal{D}_x^{-mr} = \sum_{m=0}^{\infty} \frac{x^{mr}}{(mr)!} . \tag{1.8}$$

A slight extension of the formalism leading to eq. (1.5), yields the following integral representation for the $E_0(x; r)$ function

$$E_0(x; r) = \int_0^{\infty} e^{-s} \mathcal{C}_0(x^r s \mid r) ds \tag{1.9}$$

where $\mathcal{C}_0(x \mid r)$ is the 0^{th} order of the Wright function defined by [2]

$$\mathcal{C}_n(x \mid r) = \sum_{k=0}^{\infty} \frac{x^k}{k!(n+kr)!} , \qquad \sum_{n=-\infty}^{\infty} t^n \mathcal{C}_n(x \mid r) = e^{t+\frac{x}{t^r}} . \tag{1.10}$$

We can infer directly from their definition that the functions $E_0(x; r)$, called from now on pseudo hyperbolic, can be complemented by

$$E_j(x; r) = \mathcal{D}_x^{-j} E_0(x; r) = \sum_{m=0}^{\infty} \frac{x^{mr+j}}{(mr+j)!} = x^j \int_0^{\infty} e^{-s} \mathcal{C}_j(x^r s \mid r) ds \tag{1.11}$$

all linearly independent if $j < r$, and satisfying the identities

$$\frac{d}{dx} E_j(x; r) = E_{j-1}(x; r) \tag{1.12}$$

and

$$\frac{d}{dx}E_0(x;r) = E_{r-1}(x;r) \tag{1.13}$$

which can be combined to get

$$\left(\frac{d}{dx}\right)^r E_j(x;r) = E_j(x;r)\,. \tag{1.14}$$

This last relation suggests that the $E_j(x;r)$ functions can be written in terms of the roots of unity, thus getting

$$E_j(x;r) = \frac{1}{r}\sum_{l=1}^{r}\frac{e^{x\rho_l}}{\rho_l^j}\,, \qquad \rho_l = e^{2\pi i \frac{l}{r}}\,. \tag{1.15}$$

The pseudo trigonometric functions can be defined in a fairly similar way and read

$$S_j(x;r) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{kr+j}}{(kr+j)!} \tag{1.16}$$

or

$$S_j(x;r) = \frac{\varepsilon_r^{-j}}{r}\sum_{l=1}^{r}\frac{e^{\varepsilon_r x\rho_l}}{r_l^j} \tag{1.17}$$

where ε_r denotes one of the r^{th} roots of -1.

This class of functions too can be generated by means of the negative derivative operator and can be shown to be related to the Wright function by the integral representation

$$S_j(x;r) = x^j\int_0^{\infty} e^{-s}C_j(x^r s \mid r)ds \tag{1.18}$$

where

$$C_n(x \mid r) = \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!(n+kr)!} \tag{1.19}$$

and

$$\sum_{n=-\infty}^{\infty} t^n C_n(x \mid r) = e^{t-\frac{x}{t^r}}\,. \tag{1.20}$$

Further comments on this family of functions can be found in [1] and in the forthcoming parts of the paper.

It is worth underlying that one of the major conclusions of this introduction is the fact that the functions, defined in [1], exhibit a fairly natural link with Bessel type functions and that the negative derivative operator, along with its associated operational calculus, which already played a central role in the theory of generalized Laguerre functions [3], offer a useful tool to explore their properties.

In [3] it has been shown that the operator $\mathcal{D}_x^{-1}$ plays a central role in the theory of ordinary and generalized Laguerre polynomials. In the forthcoming sections of the paper we will show how a proper combination of the points of view of [1, 3] offers the possibility of developing the theory of pseudo Laguerre and pseudo Hermite polynomials. We will discuss the properties of these new families of polynomials and we will analyze possible developments and applications of the theory.

2. Pseudo Laguerre polynomials

The negative derivative operator has been employed in [3] to derive the theory of ordinary and generalized Laguerre polynomials from an operational point of view. In this section we will employ the pseudo hyperbolic and trigonometric functions and extend the method of [3] to define families of pseudo-Laguerre polynomials. We define therefore the pseudo Laguerre polynomials (PLP) by means of the following operational rule

$$\begin{aligned}\mathcal{L}_n(x,y;r,0) &= (y-\mathcal{D}_x^{-r})^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}y^{n-k}\mathcal{D}_x^{-rk} = \\ &= n!\sum_{k=0}^{n}\frac{(-1)^k y^{n-k}x^{rk}}{(n-k)!k!(rk)!} = y^n\left({}_1F_r\left(-n;\frac{1}{r},\frac{2}{r},\ldots,1;\frac{(rx)^r}{y}\right)\right).\end{aligned} \tag{2.1}$$

It is evident that the above family of PLP reduces to the ordinary Laguerre [4] for $y=r=1$.

The relevant generating function can be obtained by following the method suggested in [3] for the polynomials $\mathcal{L}_n(x,y;1,0)$. From the first two terms of the equalities chain in eq. (2.1) we find ($|\,yt\,|<1$)

$$\begin{aligned}\sum_{n=0}^{\infty}t^n\mathcal{L}_n(x,y;r,0) &= \sum_{n=0}^{\infty}t^n(y-\mathcal{D}_x^{-r})^n = \frac{1}{1-t(y-\mathcal{D}_x^{-r})} = \\ &= \frac{1}{1-yt}\,\frac{1}{1+\frac{t}{1-yt}\mathcal{D}_x^{-r}} = \frac{1}{1-yt}\sum_{k=0}^{\infty}(-1)^k\left(\frac{t}{1-yt}\right)^k\mathcal{D}_x^{-rk}.\end{aligned} \tag{2.2}$$

According to the discussion of the previous section the properties of the negative derivative operator leads to

$$\sum_{n=0}^{\infty}t^n\mathcal{L}_n(x,y;r,0) = \frac{1}{1-yt}S_0\left(\left(\frac{t}{1-yt}\right)^{\frac{1}{r}}x;r\right). \tag{2.3}$$

We can apply an analogous procedure to get the further generating function

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathcal{L}_n(x,y;r,0) = \sum_{n=0}^{\infty}\frac{t^n}{n!}(y-\mathcal{D}_x^{-r})^n = e^{yt}e^{-t\mathcal{D}_x^{-r}} = e^{yt}C_0(x^r t\mid r). \tag{2.4}$$

An obvious extension of the polynomials (2.1) is provided by

$$\mathcal{L}_n(x,y;r,j) = \mathcal{D}_x^{-j}(y-\mathcal{D}_x^{-r})^n = n!\sum_{k=0}^{n}\frac{(-1)^k y^{n-k}x^{rk+j}}{(n-k)!(rk+j)!k!};\quad j<r, \tag{2.5}$$

the relevant generating functions can be derived by exploiting the same means as above, thus getting

$$(2.6)\qquad \sum_{n=0}^{\infty} t^n \mathcal{L}_n(x, y; r, j) = \frac{1}{1-yt}\left[\frac{1-yt}{t}\right]^{\frac{j}{r}} S_j\left(\left(\frac{t}{1-yt}\right)^{\frac{1}{r}} x; r\right) ,$$

and

$$(2.7)\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathcal{L}_n(x, y; r, j) = e^{yt} x^j C_j(x^r t \mid r) .$$

In analogy with the ordinary case we introduce the associated polynomials as

$$(2.8)\quad \mathcal{L}_n^{(m)}(x, y; r, j) = \left(1 - y\frac{\partial^r}{\partial x^r}\right)^m (y - \mathcal{D}_x^{-r})^n\, \mathcal{D}_x^{-j} = \left(1 - y\frac{\partial^r}{\partial x^r}\right)^{m+n} \frac{(-1)^n x^{nr+j}}{(nr+j)!} ,$$

where m is an integer, even though it can be any real value. By combining the methods leading to eqs. (2.3), (2.6) and by noting that [1]

$$(2.9)\qquad \left(\frac{d}{dx}\right)^r S_j(\alpha x; r) = -\alpha S_j(\alpha x; r) ,$$

we obtain the generating function of the PLP as

$$(2.10)\qquad \sum_{n=0}^{\infty} t^n \mathcal{L}_n^{(m)}(x, y; r, j) = \frac{1}{(1-yt)^{m+1}}\left[\frac{1-yt}{t}\right]^{j/r} S_j\left(\left(\frac{t}{1-yt}\right)^{\frac{1}{r}} x; r\right) .$$

It is worth noting that a natural consequence of eq. (2.8) is the identity

$$(2.11)\qquad \sum_{n=0}^{\infty} \mathcal{L}_n^{(-n)}(x, y; r, j) = S_j(x; r) .$$

Further examples displaying the intimate connection between P.L.P. and Ricci's functions will be discussed in the final section of the paper.

Before ending these preliminary considerations let us note that since

$$(2.12)\quad \frac{\partial}{\partial y}\mathcal{L}_n(x, y; r, j) = n\mathcal{L}_{n-1}(x, y; r, j) , \quad (y - \mathcal{D}_x^{-r})\mathcal{L}_n(x, y; r, j) = \mathcal{L}_{n+1}(x, y; r, j)$$

we obtain

$$(2.13)\qquad \left(y\frac{\partial^r}{\partial x^r} - 1\right)\frac{\partial}{\partial y}\mathcal{L}_n(x, y; r, j) = n\frac{\partial^r}{\partial x^r}\mathcal{L}_n(x, y; r, j) .$$

In the forthcoming section we will show a further natural complement of the functions defined in [1] is provided by the pseudo Hermite polynomials.

3. Pseudo Hermite polynomials

The two variable polynomials

$$(3.1)\quad \Delta_n(x, y; r, 0) = n! \sum_{k=0}^{n} \frac{y^{n-k} x^{rk}}{(n-k)!(rk)!} = y^n \left({}_2F_r\left(-n, 1; \frac{1}{r}, \frac{2}{r}, \dots , 1; -\frac{(rx)^r}{y}\right)\right)$$

are easily shown to satisfy the differential equation

$$\frac{\partial}{\partial y}\Delta_n(x, y; r, 0) = \frac{\partial^r}{\partial x^r}\Delta_n(x, y; r, 0), \quad \Delta_n(x, 0; r, 0) = \frac{n!}{(rn)!}x^{rn}. \tag{3.2}$$

Even though not explicitly stated, the polynomials (3.1) belong to the Hermite family, as follows from eq. (3.2) which provides the operational definition

$$\Delta_n(x, y; r, 0) = e^{y\frac{\partial^r}{\partial x^r}}\left[\frac{n!}{(rn)!}x^{rn}\right]. \tag{3.3}$$

By recalling that the Kampè de Feriet polynomials $H_n^{(m)}(x, y)$ are defined through [5]

$$\begin{aligned} &H_n^{(m)}(x, y) = e^{y\frac{\partial^m}{\partial x^m}}[x^n], \\ &H_n^{(m)}(x, y) = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{x^{n-mk}y^k}{(n-mk)!k!}, \end{aligned} \tag{3.4}$$

we can identify the $\Delta_n(x, y; r, 0)$ with

$$\Delta_n(x, y; r, 0) = \frac{n!}{(rn)!}H_{rn}^{(r)}(x, y). \tag{3.5}$$

The generating function of the polynomials (3.1) defined from now on pseudo-Hermite polynomials (PHP), can be derived, by recalling that, according to [6, 7], the following identity holds

$$\sum_{n=0}^{\infty}\frac{t^{rn}}{(rn)!}H_{rn}^{(m)}(x, y) = \frac{1}{r}\sum_{l=1}^{r}e^{x\tau_l + y\tau_l^m}, \quad \tau_l = te^{2\pi i\frac{l}{r}}. \tag{3.6}$$

Accordingly we get

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\Delta_n(x, y; r, 0) = e^{yt}E_0(xt^{\frac{1}{r}}; r). \tag{3.7}$$

It goes by itself that we can introduce the polynomials

$$\Delta_n(x, y; r, j) = \mathcal{D}_x^{-j}\Delta_n(x, y; r, 0) = n!\sum_{k=0}^{n}\frac{y^{n-k}x^{rk+j}}{(n-k)!(rk+j)!}, \tag{3.8}$$

which in terms of Kampè de Feriet read

$$\Delta_n(x, y; r, j) = \frac{n!}{(rn+j)!}H_{rn+j}^{(r)}(x, y) \tag{3.9}$$

and the relevant generating function can be written in terms of pseudo hyperbolic function as

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\Delta_n(x, y; r, j) = t^{-j/r}e^{yt}E_j(xt^{\frac{1}{r}}; r). \tag{3.10}$$

This last identity completes the first part of the paper. We have indeed proved the existence of new families of polynomials linked to pseudo-hyperbolic and trigonometric polynomials functions in the same way in which ordinary polynomials Laguerre and Hermite are linked to the hyperbolic and circular functions.

Before concluding this section we discuss the addition theorem for the polynomials (3.1), which is a direct consequence of the addition theorem holding for the pseudo hyperbolic functions and proved in [1], according to which we get

$$\begin{aligned} E_0\,(x+y;r) &= E_0(x;r)E_0(y;r) + E_1(x;r)E_{r-1}(y;r) + \ldots + E_{r-1}(x;r)E_1(y;r) \\ E_1\,(x+y;r) &= E_1(x;r)E_0(y;r) + E_2(x;r)E_{r-1}(y;r) + \ldots + E_0(x;r)E_1(y;r) \\ E_{r-1}(x+y;r) &= E_{r-1}(x;r)E_0(y;r) + E_0(x;r)E_{r-1}(y;r) + \ldots + E_{r-2}(x;r)E_1(y;r)\,. \end{aligned} \tag{3.11}$$

The above identities can be exploited to get the following addition theorem for PHP

$$\begin{aligned} &\Delta_n(x+z,\, y+w;\, r,\, 0) = \\ &= \sum_{s=0}^{n}\Big[n\Big(B_{n-1}^{(1,r-1)}(x,y;z,w\mid r) + \ldots + B_{n-1}^{(r-1,1)}(x,y;z,w\mid r)\Big) + B_n^{(0,0)}(x,y;z,w\mid r)\Big] \end{aligned} \tag{3.12}$$

where

$$B_n^{(\alpha,\beta)}(x,\,y;\,z,\,w\mid r) = \sum_{s=0}^{n}\binom{n}{s}\Delta_{n-s}(x,\,y;\,r,\,\alpha)\Delta_s(z,\,w;\,r,\,\beta) \tag{3.13}$$

In Appendix we will discuss the possibility of further exploiting the pseudo hyperbolic nature of $E_j(x;r)$ functions to draw further consequences relevant to the PLP.

We believe interesting to consider a further example relevant to the family of polynomials

$$\Delta_n^{(m)}(x,\,y;\,r,\,j) = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{y^{n-mk}x^{rk+j}}{(n-mk)!(rk+j)!}\,, \quad m<r \tag{3.14}$$

satisfying the differential equation

$$\frac{\partial^m}{\partial y^m}\Delta_n^{(m)}(x,\,y;\,r,\,j) = \frac{\partial^r}{\partial x^r}\Delta_n^{(m)}(x,\,y;\,r,\,j)\,. \tag{3.15}$$

According to the so far developed discussion it is easily realized that the polynomials (3.14) too are linked to the pseudo hyperbolic functions by the generating function

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\Delta_n^{(m)}(x,\,y;\,r,\,j) = t^{-\frac{mj}{r}}\,e^{yt}E_j(xt^{\frac{m}{r}};r) \tag{3.16}$$

which can be used as a useful starting point to study the properties of this family of polynomials.

4. Concluding remarks

The possibilities offered by the functions introduced in [1] and by the associated properties are so wide that all the implications cannot be discussed in the space of a paper. This concluding section is therefore aimed at presenting possible directions where the research may develop.

We must emphasize that the polynomials (2.5) can also be identified as different forms of Konhauser polynomials $Z_n^{(j)}(x, r)$ [8] and indeed we have

$$Z_n^{(j)}(x, r) = \frac{(rn+j)!}{n!} x^{-j} \mathcal{L}_n(x, 1; r, j) \tag{4.1}$$

(within the present context the condition of j integer may be relaxed, in this case $\mathcal{L}_n(x, 1; r, j)$ is a function). The method employed in this paper can be therefore exploited to investigate further properties of the $Z_n^{(j)}(x, r)$ family and of its associated biorthogonal partners [8].

Furthermore Ben Cheikh discussed polynomials similar to those considered in this paper in [9, 10].

The first example of polynomials generalizing those discussed in this paper

$$\mathcal{L}_n(x, y; r, 0, p) = n! \sum_{s=0}^{n} \frac{(-1)^s y^{n-s} x^{rs}}{(n-s)!((rs)!)^p} \tag{4.2}$$

which for $r = 1$ reduce to the Laguerre forms discussed in [11, 12] and for $r = 1$, $p = 2$, $y = 1$ to the ordinary Laguerre.

By limiting ourselves to the case $p = 2$ we note that this family of polynomials satisfies the equation

$$\frac{\partial}{\partial y} \mathcal{L}_n(x, y; r, 0, 2) = -\frac{\partial^r}{\partial x^r} x^r \frac{\partial^r}{\partial x^r} \mathcal{L}_n(x, y; r, 0, 2) \tag{4.3}$$

and the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}_n(x, y; r, 0, 2) = e^{yt} E_0[t^{\frac{1}{r}} \mathcal{D}_x^{-1}; r] \tag{4.4}$$

where

$$E_0(\alpha \mathcal{D}_x^{-1}; r) = \frac{1}{r} \sum_{l=1}^{r} C_0(\alpha x e^{2i\pi \frac{l}{r}}) . \tag{4.5}$$

A further interesting case is offered by

$$\mathcal{L}_n^{(q)}(x, y; r, 0, p) = n! \sum_{s=0}^{[\frac{n}{q}]} \frac{(-1)^s y^{n-qs} x^{rs}}{(n-qs)!((rs)!)^p} \tag{4.6}$$

which can be viewed as a generalized form of Legendre polynomials [4], we get indeed

$$\mathcal{L}_n^{(2)}\left(\frac{1}{4}(1-y^2), y; 1, 0, 2\right) = P_n(y) . \tag{4.7}$$

The properties of the polynomials (4.6) can be studied by exploiting the formalism developed in this paper, but this aspect of the problem will be studied in a forthcoming investigation.

We must underline that the polynomials we have introduced as well as the functions of [1] can be framed within the more general context of the theory of hypergeometric functions. The method we have developed offers, however, specific means of investigations more convenient for the problem under study.

The present paper has provided the basic elements to study the theory of pseudo Laguerre and pseudo Hermite polynomials, but we are far from having clarified their properties which can be exploited in many fields of research as in classical and quantum optics.

The evolution of the optical field of a high gain free electron laser [13] is governed by pseudo trigonometric functions $S_j(x;3)$, while polynomials of the type (3.1) play a central role in the theory of squeezed states of light [6], they are indeed the appropriate polynomial forms describing quantum states with reduced optical fluctuations.

Appendix

The following theorem has been proved in [1] $(\alpha = 0, 1, \ldots, r-1)$

$$\text{(A1)} \qquad \sum_{s=1}^{r} E_\alpha(p_s; r) = rE_\alpha\left(\frac{1}{r}\sum_{q=1}^{r} p_q; r\right) E_0\left(\frac{1}{r}\sum_{q=1}^{r} \rho_{r-q}p_q; r\right)$$

where only two of the p_r (say $p_{h,k}$, $k < h$) are arbitrary and the others satisfy the $r-2$ conditions

$$\text{(A2)} \qquad \begin{pmatrix} 1 & \rho_h & p_h \\ 1 & \rho_k & p_k \\ 1 & \rho_j & p_j \end{pmatrix} = 0\,, \qquad i = 1, 2, \ldots, h-1, h+1, \ldots, k-1, k+1, \ldots, r$$

with ρ_α being one of the roots of unity. This theorem yields the following addition theorem for PHP

$$\text{(A3)} \qquad \sum_{s=1}^{r} \Delta_n(p_s, y; r, j) = r\sum_{s=0}^{n}\binom{n}{s}\Delta_{n-s}\left(\frac{1}{r}\sum_{q=1}^{r} p_q, \frac{y}{2}; r, j\right)\Delta_s\left(\frac{1}{r}\sum_{q=1}^{r}\rho_{r-q}p_q, \frac{y}{2}; r, 0\right).$$

From identities of the type [1]

$$\text{(A4)} \qquad E_0(\alpha+\rho_1\beta; r) = E_0(\alpha; r)E_0(\beta; r) + \rho_{r-1}E_1(\alpha; r)E_{r-1}(\beta; r) + \ldots + \rho_1 E_{r-1}(\alpha; r)E_1(\beta; r)$$

we get

$$\text{(A5)} \qquad \begin{aligned} &\Delta_n(x+\rho_1 z, y; r, 0) = \\ &= \Bigg[B_{n-1}^{(0,0)}\left(x, \frac{y}{2}; z, \frac{y}{2} \mid r\right) + n\Bigg(\rho_{r-1}B_{n-1}^{(1,r-1)}\left(x, \frac{y}{2}; z, \frac{y}{2} \mid r\right) + \ldots + \\ &\qquad + \rho_1 B_{n-1}^{(r-1,1)}\left(x, \frac{y}{2}; z, \frac{y}{2} \mid r\right)\Bigg)\Bigg]. \end{aligned}$$

Analogous identities can be obtained for the pseudo Laguerre families, but this aspect of the problem will be discussed elsewhere.

Acknowledgements

The author expresses his sincere appreciation to Prof. P.E. Ricci for many enlightening discussions on the theory of pseudo hyperbolic and trigonometric functions, for kind interest and encouragements. It is also a pleasure to recognize the careful critical analysis of Dr. S. Lorenzutta which significantly improved the quality of the manuscript.

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Pervenuta il 17 ottobre 2000,
in forma definitiva il 22 gennaio 2001.

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