Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Paola Loreti

Exact controllability of shells in minimal time

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **12** (2001), n.1, p. 43–48.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2001_9_12_1_43_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2001.

Rend. Mat. Acc. Lincei
s. 9, v. 12:43-48 (2001)

Teoria dei controlli. — *Exact controllability of shells in minimal time.* Nota di Paola Loreti, presentata (*) dal Socio C. Baiocchi.

Abstract. — We prove an exact controllability result for thin cups using the Fourier method and recent improvements of Ingham type theorems, given in a previous paper [2].

Key words: Shells; Fourier method; Ingham type inequalities.

Riassunto. — *Controllabilità esatta di calotte in tempo minimo.* Dimostriamo un risultato di controllabilità esatta per calotte sottili, utilizzando il metodo di Fourier e miglioramenti recenti di teoremi di tipo Ingham, dati in un precedente articolo [2].

1. Introduction and formulation of the main result

Since the introduction of the Hilbert Uniqueness Method by J.-L. Lions in 1986, see [14, 15], many works were devoted to the controllability and stabilizability of different plate models, see *e.g.* [12, 13]. The similar study of the more complex shell models is more recent, see *e.g.* [4-7], etc. The purpose of this paper is to prove optimal results for spherical shells with a central hole.

By the Love-Koiter linear shell theory [17, 19] we can formulate the mathematical model of a spherical cup of opening angle $0 < \theta_0 < \pi$ with a hole of opening angle $0 < \theta_1 < \theta_0$. In the case $\theta_0 = \frac{\pi}{2}$ a similar analysis can be done also in the absence of a hole, see [16]. We only consider axially symmetric deformations. Then the meridional and radial displacements $u(\theta\,, t)$ and $w(\theta\,, t)$ of a point P, belonging to the middle surface of the shell, satisfy in $(\theta_1\,, \theta_0) \times \mathbb{R}$ the following coupled system of partial differential equations:

$$
(1.1)\qquad \begin{cases} du_{tt} - \mathcal{L}(u) + (1+\nu)w' - e\mathcal{L}(u+w') = 0\,, \\ dw_{tt} - \dfrac{1+\nu}{\sin\theta}(u\sin\theta)' + \dfrac{e}{\sin\theta}[\mathcal{L}(u+w')\sin\theta']' + 2(1+\nu)w = 0\,, \end{cases}
$$

where $'$ and the subscript t stand for the derivatives with respect to θ and t,

$$\mathcal{L}(v) := v'' + v'\cot\theta - (\nu + \cot^2\theta)v\,,$$

and d, c, ν are given constants. More precisely, denoting by R and h the radius and the half-thickness of the middle surface, by λ and η the Lamé constants, by d_0 the density and by E the Young modulus, we have

$$c = \frac{h^2}{3R^2}\,, \qquad \nu = \frac{\lambda}{\lambda + 2\eta} \quad \text{and} \quad d = \frac{d_0 E}{1-\nu^2}R^2.$$

Note that $-1 < \nu < 1/2$ and $c\,, d > 0$.

(*) Nella seduta del 15 dicembre 2000.

According to the Hilbert Uniqueness method, the exact controllability of this system holds true in suitable function spaces provided a special uniqueness property is satisfied. This was explained for the present context in [6], so that in this paper we only study the required uniqueness of the solutions of (1.1) completed by the following boundary and initial conditions:

$$\text{(1.2)} \qquad \begin{cases} u(\theta_0\,, t) = u(\theta_1\,, t) = 0\,, \\ w'(\theta_0\,, t) = w'(\theta_1\,, t) = 0\,, \\ \mathcal{L}(u + w')(\theta_0\,, t) = \mathcal{L}(u + w')(\theta_1\,, t) = 0\,, \end{cases} \qquad t \in \mathbb{R}\,,$$

$$\text{(1.3)} \qquad \begin{cases} u(\theta\,, 0) = u_0\,, \quad u_t(\theta\,, 0) = u_1\,, \\ w(\theta\,, 0) = w_0\,, \quad w_t(\theta\,, 0) = w_1\,, \end{cases} \qquad \theta_1 < \theta < \theta_0.$$

It follows from more general results established in [7] that the problem (1.1), (1.2), (1.3) is well posed in the Hilbert space $\mathcal{V} \times \mathcal{H}$ defined by

$$\mathcal{V} := H_0^1(\theta_1\,, \theta_0) \times (H^2 \cap H_0^1)(\theta_1\,, \theta_0)$$

and

$$\mathcal{H} := L^2(\theta_1\,, \theta_0) \times L^2(\theta_1\,, \theta_0).$$

In [7] more complex spaces are used, but under the present assumption $\theta_1 > 0$ they are equivalent to the above ones. Our main result is the following:

Theorem 1.1. *For all but countably many exceptional values of c, the following uniqueness property holds true. If a solution of* (1.1)-(1.3) *satisfies*

$$\text{(1.4)} \qquad w(\theta_0\,, t) = 0\,, \quad 0 < t < T,$$

for some $T > 2\sqrt{d}(\theta_0 - \theta_1)$, *then in fact* $v = (u\,, w)$ *vanishes identically in* $(\theta_1\,, \theta_0) \times \mathbb{R}$.

Remark. The same conclusion was obtained in [4] for the particular case of the half-sphere ($\theta_0 = \pi/2$, $\theta_1 = 0$), for some very particular choices of the parameters. The proof had two important ingredients:

- thanks to the particular choice of the angles the eigenfunctions of the infinitesimal generator $\mathcal{A}$ of the corresponding semigroup have an explicit representation by Legendre polynomials;
- thanks to the choice of the parameters the spectrum of $\mathcal{A}$ satisfies a crucial gap condition, enabling one to apply a classical generalization of Parseval's equality, due to Ingham [8].

In order to treat the present general case, we have to modify substantially our approach:

- without determinig explicitly the eigenfunctions and eigenvalues of $\mathcal{A}$, we can establish the existence of a Riesz basis of $\mathcal{V} \times \mathcal{H}$, formed by eigenfunctions of $\mathcal{A}$, and we can obtain a sufficiently precise information on the distribution of the corresponding eigenvalues by applying the spectral theory of ordinary differential operators as exposed by Titchmarsh in [20].

- The study of the eigenvalues shows that the gap condition needed for the application of Ingham's theorem is not satisfied in general. However, a weaker gap condition still holds, and this is still sufficient for our purposes because we may apply a recent generalization of Ingham's theorem, given in [1] (see also [2, 9]) which also extends a celebrated theorem of Beurling [3].

2. Representation of the solutions

Let us clarify the structure of the solutions of (1.1)-(1.3). We refer to [18] for the study of the spectrum in the general case. In the present particolar case, following [19], it is useful to introduce a primitive s of u with respect to θ and to use the differential operator

$$\mathcal{D}(s) = s'' + s' \cot\theta + 2s.$$

Then, setting also

$$k := (1 + c)(1 + \nu)$$

for brevity, (1.1) can be rewritten in a more convenient form:

$$\begin{cases} ds_{tt} = \mathcal{D}(s) + \big(c\mathcal{D} - k\big)(s + w)\,, \\ dw_{tt} = (1+\nu)\mathcal{D}(s) - \big(c\mathcal{D}^2 - c(3+\nu)\mathcal{D} + 2k\big)(s + w). \end{cases} \tag{2.1}$$

Consider the following eigenvalue problem:

$$\begin{cases} -\mathcal{D}(f_j) = \alpha_j f_j \quad \text{in} \quad (\theta_1\,, \theta_0)\,, \\ f_j'(\theta_0) = f_j'(\theta_1) = 0. \end{cases} \tag{2.2}$$

Thanks to our assumption $0 < \theta_1 < \theta_0 < \pi$ the coefficients of $\mathcal{D}$ are continuous on the compact interval $[\theta_1\,, \theta_0]$. (The assumption on the existence of a hole is crucial here.) We may therefore apply the spectral theory as developed in the first chapter of Titchmarsh's book [20]. Thus there exists a Riesz basis $f_0\,, f_1\,, \dots$ of $L^2(\theta_1\,, \theta_0)$, formed by eigenfunctions of the problem (2.2). Furthermore, the following asymptotic relations are satisfied as $j \to \infty$:

$$\sqrt{\alpha_j} = \frac{j\pi}{\theta_0 - \theta_1} + O\left(\frac{1}{j}\right)\,, \tag{2.3}$$

$$f_j = \sqrt{\frac{2}{\theta_0 - \theta_1}} \cos\left(\frac{j\pi\theta}{\theta_0 - \theta_1}\right) + O\left(\frac{1}{j}\right). \tag{2.4}$$

Rewriting (2.1) in the operational form

$$dv_{tt} = \mathcal{A}v\,, \quad v = (s\,, w)$$

and using these eigenfunctions we can find a Riesz basis of $\mathcal{V} \times \mathcal{H}$, formed by eigenfunctions of the form $(\omega_j f_j\,, f_j)$ of $\mathcal{A}$. Indeed, the equation $\mathcal{A}(\omega_j f_j\,, f_j) = \lambda_j(\omega_j f_j\,, f_j)$ leads to the algebraic system

$$\begin{pmatrix} (1+c)\alpha_j + k + \lambda_j & c\alpha_j + k \\ c\alpha_j^2 + c(3+\nu)\alpha_j + (1+\nu)\alpha_j + 2k & c\alpha_j^2 + c(3+\nu)\alpha_j + 2k + \lambda_j \end{pmatrix} \begin{pmatrix} \omega_j \\ 1 \end{pmatrix} = 0.$$

Proceeding as *e.g.* in [7] we have two solutions:

$$\lambda_j^{\pm} = \frac{1}{2}\left(-B_j \pm \sqrt{B_j^2 - 4C_j}\right)$$

with

$$B_j = c\alpha_j^2 + [(1+c) + c(3+\nu)]\alpha_j + 3(1+c)(1+\nu)\,,$$

$$C_j = c\alpha_j^3 + 2c\alpha_j^2 + (1+c)(1-\nu^2)\alpha_j$$

and

$$\omega_j^{\pm} = \frac{c\alpha_j + (1+c)(1+\nu)}{\lambda_j^{\pm} + (1+c)\alpha_j + (1+c)(1+\nu)}.$$

Moreover, we may assume that the numbers $\lambda_0^{\pm}$, $\lambda_1^{\pm}$, ... are pairwise distinct and different from zero (this holds for all but countably many exceptional values of c).

Since $\alpha_j \to \infty$, one obtains easily the asymptotic relations

$$\lambda_j^+ \sim -\alpha_j\,, \quad \lambda_j^- \sim -c\alpha_j^2 \tag{2.5}$$

and hence

$$\omega_j^+ \sim 1\,, \quad \omega_j^- \sim -1/\alpha_j. \tag{2.6}$$

Applying Proposition 2.1 from [11], we conclude that the vectors

$$(\omega_j^{\pm} f_j\,, f_j)\,, \quad j = 0\,, 1\,, \ldots$$

form a Riesz basis in $\mathcal{H}$ and that the solutions of (2.1), (1.2), (1.3) (with $u = s'$) are given by the series

$$\begin{aligned}(s\,, w)(t) = \sum_j \left(a_j e^{\sqrt{\lambda_j^+/dt}} + b_j e^{-\sqrt{\lambda_j^+/dt}}\right)(\omega_j^+ f_j\,, f_j) + \\ + \sum_j \left(c_j e^{\sqrt{\lambda_j^-/dt}} + d_j e^{-\sqrt{\lambda_j^-/dt}}\right)(\omega_j^- f_j\,, f_j)\end{aligned} \tag{2.7}$$

with suitable complex coefficients a_j, b_j, c_j and d_j, depending on the initial data.

3. Proof of the uniqueness theorem

We begin by formulating a special case of a generalization of a classical theorem due to Beurling [3], proved in [1] and [2]. Let $(\lambda_n)_{n=-\infty}^{\infty}$ be a strictly increasing sequence of real numbers. Assume that there exists a number $\gamma' > 0$ such that

$$\lambda_{n+2} - \lambda_n \geq 2\gamma'$$

for all n. Set

$$A_1 := \{n \in \mathbb{Z} \;:\; \lambda_n - \lambda_{n-1} \geq \gamma' \text{ and } \lambda_{n+1} - \lambda_n \geq \gamma'\}\,,$$

$$A_2 := \{n \in \mathbb{Z} \;:\; \lambda_n - \lambda_{n-1} \geq \gamma' \text{ and } \lambda_{n+1} - \lambda_n < \gamma'\}\,,$$

and consider the sums of the form

$$f(t) = \sum_n b_n e^{i\lambda_n t} \tag{3.1}$$

with complex coefficients b_n. We only consider «finite» sums, *i.e.*, we assume that only finitely many coefficients are different from zero. Put

$$E(f) := \sum_{n\in A_1} |b_n|^2 + \sum_{n\in A_2} \left[|b_n + b_{n+1}|^2 + (\lambda_{n+1} - \lambda_n)^2 \left(|b_n|^2 + |b_{n+1}|^2 \right) \right]$$

for brevity. Furthermore, set

$$D^+ := \lim_{r\to\infty} \frac{n^+(r)}{r}$$

where $n^+(r)$ denotes the largest number of terms of the sequence (λ_n) contained in an interval of length r.

The following result is a special case of a theorem proved in [2].

THEOREM 3.1. *For every bounded interval* I *of length* $|I| > 2\pi D^+$ *there exist two constants* C_1, $C_2 > 0$ *such that*

$$C_1 E(f) \le \int_I |f(t)|^2 \, dt \le C_2 E(f) \tag{3.2}$$

for all functions f *of the form* (3.1).

REMARKS.

- By a standard density argument, the estimates (3.2) remain valid also for all *infinite* sums such that $E(f) < \infty$.
- Using a theorem of [11], the above theorem remains valid if there is also a finite number of *nonreal* exponents λ_n.

Now we are ready to prove Theorem 1.1. Let $T > 2\sqrt{d}(\theta_0 - \theta_1)$ and assume that $w(\theta_0, t) = 0$ for all $0 < t < T$. Then, using the representation (2.7) we have

$$\sum_j a_j f_j(\theta_0) e^{\sqrt{\lambda_j^+/d}\,t} + b_j f_j(\theta_0) e^{-\sqrt{\lambda_j^+/d}\,t} + c_j f_j(\theta_0) e^{\sqrt{\lambda_j^-/d}\,t} + d_j f_j(\theta_0) e^{-\sqrt{\lambda_j^-/d}\,t} = 0$$

for all $0 < t < T$.

Let us apply Theorem 3.1 and the above remarks for the sequence (λ_n) is formed of the numbers $\pm\sqrt{\lambda_j^{\pm}}$. Thanks to the asymptotic relations (2.3) and (2.5) we have $D^+ = \sqrt{d}(\theta_0 - \theta_1)/\pi$. Since $T > 2\pi D^+$, we conclude that

$$a_j f_j(\theta_0) = b_j f_j(\theta_0) = c_j f_j(\theta_0) = d_j f_j(\theta_0) = 0$$

for every j. Since the variational problem (2.2) is regular, none of the numbers $f_j(\theta_0)$ is equal to zero. Hence all coefficients a_j, b_j, c_j and d_j vanish. Using again the representation (2.7) we conclude that the solution (s, w) and then also (u, w) vanishes identically.

REMARK. There exist effectively exceptional values of the parameters c. Indeed, one can find by direct computation two different indices $j < k$ and values c, ν such that $\lambda_j^+ = \lambda_k^-$. Denoting this common value by λ, the formula

$$(s, w)(t) = e^{\sqrt{\lambda/d}\,t}\big(f_k(\theta_0)(\omega_j^+ f_j, f_j) - f_j(\theta_0)(\omega_k^- f_k, f_k)\big)$$

defines a nontrivial solution of (1.1)-(1.3) for which $w(\theta_0, t) = 0$ for all real t.

REFERENCES

[1] C. BAIOCCHI - V. KOMORNIK - P. LORETI, *Ingham type theorems and applications to control theory*. Boll. Un. Mat. Ital., B (8), II - B, n. 1 febbraio 1999, 33-63.

[2] C. BAIOCCHI - V. KOMORNIK - P. LORETI, *Généralisation d'un théorème de Beurling et application à la théorie du contrôle*. C. R. Acad. Sci. Paris Sér. I Math., 330 (4), 2000, 281-286.

[3] L. CARLESON - P. MALLIAVIN - J. NEUBERGER - J. WERMER (eds.), *The Collected Works of Arne Beurling*. Vol. 2, Birkhäuser, Boston 1989.

[4] G. GEYMONAT - P. LORETI - V. VALENTE, *Introduzione alla controllabilità esatta per la calotta sferica*. Quaderno IAC 8/1989.

[5] G. GEYMONAT - P. LORETI - V. VALENTE, *Exact Controllability of a shallow shell model*. Int. Series of Num. Math., 107, 1992, 85-97.

[6] G. GEYMONAT - P. LORETI - V. VALENTE, *Exact controllability of a spherical shell via harmonic analysis*. In: J.-L. LIONS - C. BAIOCCHI (eds.), *Boundary Values Problems for Partial Differential Equations and Applications*. Masson, Paris 1993.

[7] G. GEYMONAT - P. LORETI - V. VALENTE, *Spectral problems for thin shells and exact controllability*. In: *Spectral Analysis of Complex Structures*. Travaux en cours, 49, Hermann, Paris 1995, 35-57.

[8] A. E. INGHAM, *Some trigonometrical inequalities with applications in the theory of series*. Math. Z., 41, 1936, 367-379.

[9] S. JAFFARD - M. TUCSNAK - E. ZUAZUA, *On a theorem of Ingham*. J. Fourier Anal. Appl., 3, 1997, n. 5, 577-582.

[10] V. KOMORNIK - P. LORETI, *Ingham type theorems for vector-valued functions and observability of coupled linear systems*. SIAM J. Control Optim., 37, 1998, 461-485.

[11] V. KOMORNIK - P. LORETI, *Observability of compactly perturbed systems*. J. Math. Anal. Appl., 243, 2000, 409-428.

[12] J. LAGNESE, *Boundary Stabilization of Thin Plates*. SIAM Studies in Appl. Math., Philadelphia 1989.

[13] J. LAGNESE - J.-L. LIONS, *Modelling, Analysis and Control of Thin Plates*. Masson, Paris 1988.

[14] J.-L. LIONS, *Exact controllability, stabilization and perturbations for distributed systems*. SIAM J. Control Optim., 30, 1988, 1-68.

[15] J.-L. LIONS, *Contrôlabilité exacte et stabilisation de systèmes distribués*. Voll. 1-2, Masson, Paris 1988.

[16] P. LORETI, *Application of a new Ingham type theorem to the control of spherical shells*. In: V. ZAKHAROV (ed.), Proceedings of the 11th IFAC International Workshop *Control Applications of Optimization* (St. Petersburg, Russia, July 3-6, 2000). Pergamon, 2000.

[17] A. E. H. LOVE, *A treatise on the mathematical theory of elasticity*. Dover, New York 1944.

[18] E. SANCHEZ-PALENCIA, *Asymptotic and spectral properties of a class of singular-stiff problems*. J. Math. Pures Appl., 71, 1992, 379-406.

[19] S. TIMOSHENKO, *Theory of elastic stability*. McGraw-Hill, New York 1936.

[20] E. C. TITCHMARSH, *Eigenfunction Expansions Associated with Second-Order Differential Equations*. Clarendon Press, Oxford 1962.

Pervenuta il 17 ottobre 2000,
in forma definitiva il 20 novembre 2000.

Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate
Università degli Studi di Roma «La Sapienza»
Via A. Scarpa, 16 - 00161 ROMA
loreti@dmmm.uniroma1.it