Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Fulvio Ricci, Jérémie Unterberger

Solvability of invariant sublaplacians on spheres and group contractions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **12** (2001), n.1, p. 27–42.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2001_9_12_1_27_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2001.

Rend. Mat. Acc. Lincei
s. 9, v. 12:27-42 (2001)

Equazioni a derivate parziali. — *Solvability of invariant sublaplacians on spheres and group contractions.* Nota di Fulvio Ricci e Jérémie Unterberger, presentata (*) dal Socio F. Ricci.

Abstract. — In the first part of this paper we study the local and global solvability and the hypoellipticity of a family of left-invariant sublaplacians $\mathcal{L}_\alpha$ on the spheres $S^{2n+1} \simeq U(n+1)/U(n)$. In the second part, we introduce a larger family of left-invariant sublaplacians $\mathcal{L}_{\alpha,\beta}$ on $S^3 \simeq SU(2)$ and study the corresponding properties by means of a Lie group contraction to the Heisenberg group.

Key words: Local solvability; Hypoellipticity; Invariant differential operators; Lie group contractions.

Riassunto. — *Risolubilità di sub-Laplaciani invarianti su sfere e contrazioni di gruppi.* Nella prima parte del lavoro si studiano risolubilità locale e globale e ipoellitticità di una famiglia di sub-Laplaciani invarianti $\mathcal{L}_\alpha$ sulle sfere $S^{2n+1} \simeq U(n+1)/U(n)$. Nella seconda parte si introduce una famiglia più ampia di sub-Laplaciani invarianti a sinistra $\mathcal{L}_{\alpha,\beta}$ su $S^3 \simeq SU(2)$ e se ne studiano le corrispondenti proprietà per mezzo di una contrazione, nel senso dei gruppi di Lie, sul gruppo di Heisenberg.

Introduction

For $n \geq 1$, let $U(n)$ (resp. $SU(n)$) be the group of (resp. determinant 1) n by n unitary matrices. Considering the action of $U(n+1)$ on the base point $z_0 = (1, 0, \ldots, 0) \in \mathbb{C}^{n+1}$, we may see the complex sphere $S^{2n+1} = \{(z^0, \ldots, z^n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} |z_i|^2 = 1\}$ as the quotient $U(n+1)/U(n)$, where $U(n)$ acts on the n last coordinates, or, also, as $SU(n+1)/SU(n)$. Let $\mathfrak{u}(n+1) = \{X \in \mathfrak{gl}(n+1, \mathbb{C}) \mid X = -{}^t\overline{X}\}$ be the Lie algebra of $U(n+1)$, and $\mathfrak{su}(n+1)$ the subalgebra of null trace matrices in $\mathfrak{u}(n+1)$.

We propose to study the solvability properties of certain partial differential operators on S^{2n+1} that commute with the action of $U(n+1)$ or $SU(n+1)$.

We first recall some basic facts about invariant differential operators on a homogeneous space G/K. If D is G-left-invariant and K-right-invariant on G, then D induces a G-invariant operator $D^\flat$ on G/K as follows:

$$Df^\sharp = (D^\flat f)^\sharp,$$

where $f^\sharp(g) = f(gK)$, $g \in G$. It turns out (see [2, Chap. 2]) that the correspondence $D \mapsto D^\flat$ is onto, but not $1-1$. More details on this correspondence will be given in the first Section.

As we shall prove below, the algebra of $U(n+1)$-invariant differential operators on S^{2n+1} is commutative and generated by two elements. Clearly two possible generators are $\Delta^\flat$, where Δ is the Casimir operator of $SU(n+1)$, and $Z^\flat$, where Z generates the

(*) Nella seduta del 15 dicembre 2000.

center of $\mathfrak{u}(n+1)$ (seen as left-invariant vector field).

However, it is more convenient for us to introduce two different generators. Let X_j, Y_j $(j = 1, \dots, n)$, $H \in \mathfrak{su}(n+1)$ defined as

$$X_j = \frac{1}{2}\begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & & & & & & \\ -1 & & & & & & \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix}, \quad Y_j = \frac{1}{2}\begin{pmatrix} 0 & \cdots & 0 & i & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & & & & & & \\ i & & & & & & \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix},$$

where the non-zero coefficients appear at line j or column j (the indices starting from 0),

$$H = \frac{1}{2}\begin{pmatrix} i & & & \\ & -i/n & & \\ & & \ddots & \\ & & & -i/n \end{pmatrix}$$

and let $\mathcal{L}_0 = \sum_{j=1}^{n}(X_j^2 + Y_j^2)$ in the enveloping algebra $\mathcal{U}(\mathfrak{su}(n+1))$, seen as a left-invariant differential operator on $U(n+1)$. Then $\mathcal{L}_0^\flat$ and $H^\flat$ constitue another set of generators, obtained from the previous ones by the simple relations $H^\flat = Z^\flat$ and $\mathcal{L}_0^\flat = \Delta^\flat - (Z^\flat)^2$. One reason for choosing these generators is that $\mathcal{L}_0^\flat \pm inH^\flat$ represent the boundary sublaplacian $\Box_b = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b$ of the sphere, acting on functions and on $(0, n)$-forms respectively. Therefore $\mathcal{L}_0^\flat$ is naturally related to the structure of CR-manifold on S^{2n+1} (see [1, Chap. 7 and Chap. 8]). From now on we shall drop the alteration signs, giving explanations only in case of ambiguity.

The operators we consider are the $U(n+1)$-invariant sublaplacians $\mathcal{L}_\alpha = \mathcal{L}_0 - i\alpha H$ acting on $L^2(S^{n+1})$. The left action on $U(n+1)$ on $L^2(S^{n+1})$ decomposes as $L^2(S^{n+1}) \simeq$ $\simeq \oplus_{l,l'\geq 0}\mathcal{H}^{n,l,l'}$, where $\mathcal{H}^{n,l,l'}$ is the vector space of harmonic polynomials in $(z, \overline{z})$ $(z \in \mathbb{C}^{n+1})$ of bidegree (l, l') with respect to z and $\overline{z}$ (see [10, Chap. 11]). The spaces $\mathcal{H}^{n,l,l'}$ are irreducible with respect to $U(n+1)$, so $\mathcal{L}_\alpha$ is scalar on each of them. By means of an explicit computation of its eigenvalues, we prove the following theorem:

THEOREM. *Let $n \geq 1$. Then $\mathcal{L}_\alpha$ is locally solvable and hypoelliptic for $\alpha \neq \pm n$; modulo a subspace of finite dimension, it is even globally solvable on S^{2n+1}. If $\alpha = \pm n$, then it is neither locally solvable nor hypoelliptic.*

In a second part, we restrict ourselves to the case $n = 1$. The reason why we single out this case is the following. Whereas for $n \geq 2$ the algebra of $SU(n+1)$-invariant differential operators on S^{2n+1} is equal to the aforementioned algebra of $U(n+1)$-invariant differential operators, in the case of $S^3 \simeq SU(2)$, the former one is isomorphic to the whole enveloping algebra $\mathcal{U}(\mathfrak{su}(2))$. Thus, if we impose only an $SU(2)$-invariance condition on our operators, a natural generalization of $\mathcal{L}_\alpha$ is the more general family of «sublaplacians» $A^2 + B^2 - i\alpha[A, B]$, where $A, B \in \mathfrak{su}(2)$ are linearly independent, so that A, B and $[A, B]$ form a basis of $\mathfrak{su}(2)$. It is easy to see that these operators are conjugate in $\mathfrak{su}(2)$ (up to a constant) to the operators

$$\mathcal{L}_{\alpha,\beta} = X^2 + \beta^2 Y^2 - i\alpha\beta H \ ,$$

where $\beta \in \mathbb{R}$, and X, Y, H is the following standard basis of $\mathfrak{su}(2)$:

$$H = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Y = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

They satisfy the commutation relations $[H, X] = Y,\ [X, Y] = H,\ [Y, H] = X$. The operator $\mathcal{L}_{\alpha,\beta}$ is $U(1)$-invariant if and only if $\beta = 1$. Hence we may assume that $\beta \neq 1$.

We need to consider this time the decomposition of $L^2(S^3) \simeq L^2(SU(2))$ with respect to the left regular action of G (we write G for $SU(2)$ and $\mathfrak{g}$ for $\mathfrak{su}(2)$). Let V_n be the $n+1$-dimensional space of homogeneous polynomials of degree n on $\mathbb{C}^2$ with the following action of G:

$$\pi_n(g)p(x, y) = p(\alpha x + \beta y, -\overline{\beta}x + \overline{\alpha}y),\ \ g^{-1} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}.$$

Then the (V_n, π_n), $n \geq 0$, are representatives of all classes of unitary irreducible representations of G. Equivalently, by setting $y = 1$, V_n can be replaced by the space of polynomials on $\mathbb{C}$ of degree less than or equal to n, with a fractional linear action.

Let $L^2_0(G)$ be the space of L^2-functions f on $SU(2)$ such that $\int_G f = 0$; equivalently, $L^2_0(G)$ may be defined as the closure of the space spanned by the matrix coefficients of the representations V_j for $j \geq 1$. For $\beta \neq 1$, the analysis of $d\pi(\mathcal{L}_{\alpha,\beta})$ is made different by the fact that this operator does not appear to be diagonal in any natural basis. Let us say that α is a singular value (associated with π_n) if $d\pi_n(\mathcal{L}_{\alpha,\beta})$ is not invertible on L^2_0 for a certain $n \geq 1$ (note that $\mathcal{L}_{\alpha,\beta}$ always annihilates constant functions). Then define the cluster set of $\mathcal{L}_{\alpha,\beta}$ to be the set of values α for which there is a sequence of singular values α_j associated to π_{n_j}, with $\alpha_j \to \alpha$ and $n_j \to \infty$. This set is the union of the accumulation points of the set of singular values and of the values α for which $\mathcal{L}_\alpha$ has an infinite dimensional kernel (in particular, ± 1, as we shall see). It is easy to study local and global solvability if α is not in the cluster set. So the important matter is the study of the accumulation points of the set of singular values. This was also the underlying scheme of the proof of the above Theorem, but we shall focus on this notion only in this part, where it will receive an interpretation in terms of a Lie group contraction as follows.

Let G' be the three-dimensional Heisenberg group, and $\mathfrak{g}'$ its Lie algebra, generated by the standard basis (X', Y', H') with the single non trivial relation $[X', Y'] = H'$. Denote by (ζ, t), with $\zeta = x + iy, x, y, t \in \mathbb{R}$, the element $\exp(xX' + yY' + tH')$ of G'. A family of unequivalent unitary irreducible representations (called Bargmann-

Fock representations) $(\sigma_\lambda, \mathcal{H}_\lambda)$ $(\lambda \in \mathbb{R}^*)$ of G' is given by:

$$[\sigma_\lambda(\zeta, t)f](w) = e^{-\lambda(it+\overline{\zeta}w+\frac{1}{2}|\zeta|^2)} f(w+\zeta), \quad \lambda > 0,$$

$$[\sigma_\lambda(\zeta, t)f](w) = e^{\lambda(-it-\zeta w+\frac{1}{2}|\zeta|^2)} f(w-\overline{\zeta}), \quad \lambda < 0.$$

Let φ be any local diffeomorphism from a neighbourhood of the identity $e_{G'}$ of G' to a neighbourhood of the identity e_G of G such that $\varphi(e_{G'}) = e_G$ and $d\varphi(e_{G'})$ sends the triple (X', Y', H') into the triple (X, Y, H), and put

$$\varphi_\varepsilon(\zeta, t) = \varphi(\varepsilon^{\frac{1}{2}}\zeta, \varepsilon t), \quad \varepsilon \in]0, 1].$$

So $\Phi_\varepsilon := d\varphi_\varepsilon(e_{G'})$ is the invertible linear map given by $\Phi_\varepsilon(X') = \varepsilon^{\frac{1}{2}} X, \Phi_\varepsilon(Y') = = \varepsilon^{\frac{1}{2}} Y, \Phi_\varepsilon(H') = \varepsilon H$. Then φ_ε is a contraction of G onto G' in the following sense:

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon^{-1}[\Phi_\varepsilon V, \Phi_\varepsilon W] = [V, W], \quad V, W \in \mathfrak{g}'.$$

The first author proved that this contraction allows one to retrieve the Bargman-Fock representations as an asymptotic limit of the representations π_n of $SU(2)$. Namely, for all $V \in \mathfrak{g}$, P, Q polynomials in one variable,

$$\langle d\pi_n(V)P(\sqrt{n}.), Q(\sqrt{n}.)\rangle \sim_{n\to\infty} \langle d\sigma_1(\Phi_{1/n}^{-1} V)P, Q\rangle$$

(see [7, Theorem 2]). The left hand side is defined only if the degrees of P and Q are smaller than n, but the limit makes sense for all P and Q.

This formula can easily be generalized to V in the envelopping algebra $\mathcal{U}(\mathfrak{g})$. We obtain thus, for any P, Q:

$$\langle d\pi_n(\mathcal{L}_{\alpha,\beta})P(\sqrt{n}.), Q(\sqrt{n}.)\rangle \sim_{n\to\infty} n\langle d\sigma_1(\mathcal{L}'_{\alpha,\beta})P, Q\rangle,$$

where $\mathcal{L}'_{\alpha,\beta} = X'^2 + \beta^2 Y'^2 + i\alpha\beta H'$. On the Heisenberg group, $d\sigma_1(\mathcal{L}'_{\alpha,\beta})$ is equal to a conjugate of the harmonic oscillator, whose eigenvalues are the odd integers. The set of odd integers is also exactly the set of asymptotic singular values obtained in the first part, in the particular case of the operators $\mathcal{L}_{\alpha,1}$ on $SU(2)$. It is tempting to relate the cluster set to the eigenvalues of the limit operator $d\sigma_1(\mathcal{L}'_{0,\beta})$ on $\mathcal{H}_1$ obtained by the contraction.

Using the group contraction, we prove the following result:

THEOREM. *For any β, the set of asymptotic singular values is actually equal to the set $\pm(2n+1)$ $(n \in \mathbb{N})$ of eigenvalues of the modified harmonic oscillators $\frac{1}{\beta} d\sigma_{\pm 1}(\mathcal{L}'_{0,\beta})$.*

Using then a general theorem of Hörmander (see [3]) on the hypoellipticity of doubly characteristic operators, we deduce the following partial results for the operators $\mathcal{L}_{\alpha,\beta}$:

THEOREM.

(1) *If $\alpha \neq \pm 1, \pm 3, \ldots$, then $\mathcal{L}_{\alpha,\beta}$ is hypoelliptic, locally solvable, and globally solvable modulo a finite dimensional subspace.*

(2) *If $\alpha = \pm 1$, then $\mathcal{L}_{\alpha,\beta}$ is neither hypoelliptic nor locally solvable.*

For $\alpha = \pm 3, \pm 5, \ldots$, the analysis of $\mathcal{L}_{\alpha,\beta}$ (dimension of its kernel, construction of relative fundamental solution, etc.) depends on the arithmetic properties of its eigenvalues, which seem difficult to understand for $\beta \neq 1$ because of the non-diagonal character of $d\pi_n(\mathcal{L}_{\alpha,\beta})$.

Study of the operators $\mathcal{L}_\alpha = \mathcal{L}_0 - i\alpha H$ on the spheres

Let $n \geq 1$. We define the left-invariant vector fields X_j, Y_j, H in $\mathfrak{su}(n+1)$ as in the Introduction. Denote also by I the identity matrix in $\mathfrak{gl}(n+1, \mathbb{C})$. We first prove the following Lemma:

Lemma 1. *The operators $\mathcal{L}_0 = \sum_j X_j^2 + Y_j^2$ and H commute with each other and with $\mathfrak{u}(n)$.*

Proof. Let us first prove that $\mathcal{L}_0$ and H commute with each other. For $j \geq 1$, X_j, Y_j, H satisfy the commutation relations

$$[X_j, H] = -\frac{1}{2}\left(1 + \frac{1}{n}\right) Y_j, \quad [Y_j, H] = \frac{1}{2}\left(1 + \frac{1}{n}\right) X_j$$

(note that we get the canonical commutation relations of $\mathfrak{su}(2)$ if $n = 1$). Combining them with the formal relations

$$[X_j^2, H] = X_j[X_j, H] + [X_j, H]X_j, \quad [Y_j^2, H] = Y_j[Y_j, H] + [Y_j, H]Y_j$$

gives $[X_j^2 + Y_j^2, H] = 0$. So $[\mathcal{L}_0, H] = 0$.

Let now $M_V = \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix}$ ($V \in \mathfrak{u}(n)$) denote an element of $\mathfrak{u}(n) \subset \mathfrak{u}(n+1)$, and, for $z \in \mathbb{C}^n$, denote by Z_z the element $\sum_{j=1}^n \mathrm{Re} z_j X_j + \mathrm{Im} z_j Y_j$. We get easily $[M_V, Z_z] = Z_{Vz}$. If $z, z' \in \mathbb{C}^n$, we then get (using analogous formal relations as above)

$$[M_V, Z_z Z_{z'}] = Z_{Vz} Z_{z'} + Z_z Z_{VZ'} = \left.\frac{d}{dt}\right|_{t=0} (Z_{e^{tV}z}.Z_{e^{tV}z'}).$$

In particular, denoting by $(e_j, f_j)_{j=1,\ldots,n}$ the canonical orthonormal basis of $\mathbb{C}^n$ (with $f_j = ie_j$),

$$\left[M_V, \sum_j X_j^2 + Y_j^2\right] = \left.\frac{d}{dt}\right|_{t=0} \sum_j \left\{(Z_{e^{tV}e_j}.Z_{e^{tV}e_j}) + (Z_{e^{tV}f_j}.Z_{e^{tV}f_j})\right\} = 0\,.$$

Finally, that H should commute with $\mathfrak{u}(n)$ is straightforward. □

So $\mathcal{L}_0$ and H can be considered as left-invariant operators on $S^{2n+1} \simeq U(n+1)/U(n)$. Note that $\mathcal{L}_0$ comes naturally from the Casimir operator on $SU(n+1)$ as we already said in the Introduction. It remains to prove that they generate the algebra of left-invariant operators on the sphere. In the following theorem, if K is a closed subgroup of the compact Lie group K, we denote by $\mathbf{D}(G/K)$ the algebra of G-invariant operators on G/K. As we said in the Introduction, the «flat» operation from $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{k}_{\mathbb{C}}}$ (which can be identified with the algebra of G-left-invariant and K-right-invariant differential operators on G) into $\mathbf{D}(G/K)$ is onto, but not $1-1$. More precisely, by [2, Chap. II, Theorem 4.6], the algebra $\mathbf{D}(G/K)$ is isomorphic to $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{k}_{\mathbb{C}}}/\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{k}_{\mathbb{C}}} \cap \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}$.

THEOREM 1. *The algebras* $\mathbf{D}(U(n+1)/U(n))$ $(n \geq 1)$ *and* $\mathbf{D}(SU(n+1)/SU(n))$ $(n \geq 2)$ *of left-invariant operators on the sphere* S^{2n+1} *are both isomorphic to* $\mathbb{C}[\mathcal{L}_0, Z]$.

PROOF. Recall the aforementioned result from [2]. If in particular (G, K) is a rank 1 semi-simple riemannian symmetric pair, then $\mathbf{D}(G/K)$ is commutative and generated by the Casimir operator of G. This applies to the pair $(SU(n+1), S(U(1) \times U(n)))$. Viewed as an element of $\mathcal{U}(\mathfrak{su}(n+1))$, $\mathcal{L}_0$ is congruent with the Casimir operator of $SU(n+1)$ modulo $\mathcal{U}(s(\mathfrak{u}(1) \times \mathfrak{u}(n)))$, as we already noticed in the Introduction, and even modulo $\mathcal{U}(s(\mathfrak{u}(1) \times \mathfrak{u}(n)))^{s(\mathfrak{u}(1)\times\mathfrak{u}(n))}$. So we get $\mathbf{D}(SU(n+1)/S(U(1) \times U(n))) \simeq \mathbb{C}[\mathcal{L}_0]$ (see [2, Chap. II, Theorem 5.18]). As the identity matrix is central in $\mathcal{U}(\mathfrak{u}(n+1))$, we also get in a trivial way that $\mathbf{D}(U(n+1)/U(1) \times U(n)) \simeq \mathbb{C}[\mathcal{L}_0]$.

Consider now the (non symmetric) pair $(U(n+1), U(n))$. Let $D \in \mathcal{U}(\mathfrak{u}(n+1))^{\mathfrak{u}(n)}$. There is a unique way to write D as $D = \sum_{k\geq 0} Z^k D_k$ where $D_k \in \mathcal{U}(\mathfrak{su}(n+1))$ and $Z \in \mathfrak{u}(n+1)_{\mathbb{C}}$ is the identity matrix. As iZ is central in $\mathfrak{u}(n+1)$, the operators D_k also commute with $\mathfrak{u}(n)$, hence with $\mathfrak{u}(1) \times \mathfrak{u}(n) = i\mathbb{R}Z \oplus \mathfrak{u}(n)$. Let $k \geq 0$. It follows from the above remarks that there exists a polynomial P_k such that $D_k = P_k(\mathcal{L}_0) + D_k'$, where

$$D_k' \in \mathcal{U}(\mathfrak{u}(n+1))^{\mathfrak{u}(1)\times\mathfrak{u}(n)} \cap \mathcal{U}(\mathfrak{u}(n+1))(\mathfrak{u}(1) \times \mathfrak{u}(n)).$$

By Poincaré-Birkhoff-Witt's theorem, given a basis (A_j) of $\mathfrak{su}(n)$, (B_j) of the Killing orthogonal of $\mathfrak{su}(n)$ in $\mathfrak{su}(n+1)$, then the polynomials $B^\mu A^\nu Z^j$ form a basis of $\mathcal{U}(\mathfrak{u}(n+$ $+1))$. By hypothesis, only terms in $B^\mu A^\nu$ can show up in the decomposition of D_k'. So $D_k' \in \mathcal{U}(\mathfrak{u}(n+1))^{\mathfrak{u}(n)} \cap \mathcal{U}(\mathfrak{u}(n+1))\mathfrak{u}(n)$, which shows that, as an element of $\mathbf{D}(U(n+1)/U(n))$, D_k' is zero. As Z can be identified with H as an operator in $\mathbf{D}(U(n+1)/U(n))$, we get

$$D \equiv \sum_{k\geq 0} P_k(\mathcal{L}_0) H^k \quad \mod \ \mathcal{U}(\mathfrak{u}(n+1))^{\mathfrak{u}(n)} \cap \mathcal{U}(\mathfrak{u}(n+1))\mathfrak{u}(n).$$

Let $n \geq 2$ and $D \in \mathbf{D}(SU(n+1)/SU(n))$. Let D_1 be a representative of $D \in$ $\in \mathbf{D}(SU(n+1)/SU(n)) \simeq \mathcal{U}(\mathfrak{su}(n+1))^{\mathfrak{su}(n)} \cap \mathcal{U}(\mathfrak{su}(n+1))\mathfrak{su}(n)$ in $\mathcal{U}(\mathfrak{su}(n+1))^{\mathfrak{su}(n)}$. The spaces $\mathcal{H}^{n,l,l'}$ are also irreducible with respect to the action of $SU(n+1)$, so D_1 is scalar on each $\mathcal{H}^{n,l,l'}$. Hence $D_1 = P(\mathcal{L}_0, H) + D_3$ where $P(\mathcal{L}_0, H)$ is a polynomial in $\mathcal{L}_0$ and H, viewed as an element of $\mathcal{U}(\mathfrak{su}(n+1))^{\mathfrak{u}(n)}$, and $D_3 \in \mathcal{U}(\mathfrak{u}(n+$ $+1))^{\mathfrak{u}(n)} \cap \mathcal{U}(\mathfrak{u}(n+1))\mathfrak{u}(n)$. Note that both D_1 and $P(\mathcal{L}_0, H)$ are in $\mathcal{U}(\mathfrak{su}(n+1))$, so D_3 is too. Now, applying once more Poincaré-Birkhoff-Witt's theorem to $\mathcal{U}(\mathfrak{u}(n+1))$ with the basis (A_j), (B_j) and

$$E = \begin{pmatrix} 0 & \\ & iId \end{pmatrix} \in \mathfrak{u}(n),$$

we see that $\mathcal{U}(\mathfrak{u}(n+1))$ can be written in a unique way as a sum of terms of the form $B^\mu A^\nu E^k$, and $\mathcal{U}(\mathfrak{su}(n+1)) \subset \mathcal{U}(\mathfrak{u}(n+1))$ contains exactly the terms with $k=0$. So, decomposing in this way D_3, one gets only terms in $B^\mu A^\nu$ with $|\nu| > 0$, which shows that $D_3 \in \mathcal{U}(\mathfrak{su}(n+1))^{\mathfrak{su}(n)} \cap \mathcal{U}(\mathfrak{su}(n+1))\mathfrak{su}(n)$. □

We shall now study the operators $\mathcal{L}_\alpha = \mathcal{L}_0 - i\alpha H$. The irreducible representation spaces $\mathcal{H}^{n,l,l'}$ of $U(n+1)$ have been defined in the Introduction. Let us call $\pi_{n,l,l'}$ (or $\pi_{l,l'}$ for short) the left regular representation of $U(n+1)$ on $\mathcal{H}^{n,l,l'}$.

Lemma 2. *The operator $d\pi_{n,l,l'}(\mathcal{L}_\alpha)$ on $\mathcal{H}^{n,l,l'}$ is equal to*

$$\left[-\left(ll' + \frac{n}{2}(l+l')\right) + \frac{\alpha}{2}(l-l')\right] Id.$$

Proof. Recall the operator $d\pi_{n,l,l'}(\mathcal{L}_\alpha)$ is scalar because it commutes with $d\pi(\mathfrak{u}(n+1))$. Let P be the polynomial on $\mathbb{C}^{n+1}$ defined by $P(z,\overline{z}) = z_0^l \overline{z}_n^{l'}$ $(z \in \mathbb{C}^{n+1})$. It is obviously harmonic, so $P \in \mathcal{H}_n^{l,l'}$ and it suffices to compute the action of $\mathcal{L}_\alpha$ on $z_0^l \overline{z}_n^{l'}$.

As the left-invariant vector fields X_j and Y_j do not act on $L^2(S^{n+1})$ but on $L^2(SU(n+1))$, we shall need to consider P as the function $P^\sharp$ on $SU(n+1)$ defined by $P^\sharp(u) = P(u.z_0) = u_{00}^l \overline{u}_{n0}^{l'}$, $u = (u_{ij}) \in SU(n+1)$. By a straightforward computation, we get

$$2X_j P^\sharp(u) = 2\frac{d}{dt}\Big|_{t=0} P^\sharp(u \exp -tX_j) = l u_{0j} u_{00}^{l-1} \overline{u}_{n0}^{l'} + l' \overline{u}_{nj} u_{00}^l \overline{u}_{n0}^{l'-1}$$

and, similarly,

$$2Y_j P^\sharp(u) = il u_{0j} u_{00}^{l-1} \overline{u}_{n0}^{l'} - il' \overline{u}_{nj} u_{00}^l \overline{u}_{n0}^{l'-1},$$

hence

$$\begin{aligned} 4X_j^2 P^\sharp(u) = -(l+l') u_{00}^l \overline{u}_{n0}^{l'} + l u_{0j} \left[(l-1) u_{0j} u_{00}^{l-2} \overline{u}_{n0}^{l'} + l' \overline{u}_{nj} u_{00}^{l-1} \overline{u}_{n0}^{l'-1}\right] + \\ + l' \overline{u}_{nj} \left[l u_{0j} u_{00}^{l-1} \overline{u}_{n0}^{l'-1} + (l'-1) \overline{u}_{nj} u_{00}^l \overline{u}_{n0}^{l'-2}\right] \end{aligned}$$

and

$$\begin{aligned} 4Y_j^2 P^\sharp(u) = -(l+l') u_{00}^l \overline{u}_{n0}^{l'} - l u_{0j} \left[(l-1) u_{0j} u_{00}^{l-2} \overline{u}_{n0}^{l'} - l' \overline{u}_{nj} u_{00}^{l-1} \overline{u}_{n0}^{l'-1}\right] + \\ + l' \overline{u}_{nj} \left[l u_{0j} u_{00}^{l-1} \overline{u}_{n0}^{l'-1} - (l'-1) \overline{u}_{nj} u_{00}^l \overline{u}_{n0}^{l'-2}\right]. \end{aligned}$$

Putting together all terms, and using the fact that $u \in SU(n+1)$, one gets

$$2\sum_j (X_j^2 + Y_j^2) P(u) = -\left[n(l+l') + 2ll'\right] P.$$

Finally, it is easy to verify that $2HP = i(l-l')P$. □

Note that the action of $\mathcal{L}_\alpha$ is always trivial on $\mathcal{H}^{n,0,0}$. Let

$$L_0^2(S^{2n+1}) = \oplus_{l+l'>0} \mathcal{H}^{n,l,l'}$$

denote the completion in $L^2(S^{2n+1})$ of the space of harmonic polynomials with null constant term (or, in other words, the space of L^2-functions f on the sphere such that

$\int_{S^{2n+1}} f = 0$). The preceding lemma proves that $\mathcal{L}_\alpha$ is invertible on $L^2_0(S^{2n+1})$ when the values $-(ll' + \frac{n}{2}(l + l')) + \frac{\alpha}{2}(l - l')$, with $l + l' > 0$, are bounded away from zero. In particular, it is locally solvable in that case (since any L^2-function f on the sphere can be modified out of a small neighbourhood such that its integral be 0).

We shall say that α is a regular value if $\mathcal{L}_\alpha : L^2_0(S^{2n+1}) \to L^2_0(S^{2n+1})$ has a trivial kernel; otherwise it is said to be singular. We shall also be interested in the following notion: we shall say that α is an asymptotic singular value (or is in the cluster set) if there exists sequences α_j, l_j and l'_j with $\alpha_j \to \alpha$ and $l_j + l'_j \to \infty$ such that $d\pi_{n,l_j,l'_j}(\mathcal{L}_{\alpha_j})$ is zero. This definition is particularly motivated, as we explained in the Introduction, by the use of the Lie group contraction in the second part, but it will be useful also in the study of the local solvability for our family of operators.

LEMMA 3.

(1) *The cluster set consists of the integers* $\pm(n + 2j)$, $j \in \mathbb{N}$.

(2) *Let* $\alpha \neq \pm n$. *Then the eigenvalues* $C_{\alpha,l,l'}$ *of* $\mathcal{L}_\alpha$ *on* $\mathcal{H}^{n,l,l'}$ *are all non-zero, except for a finite number of pairs* (l, l'), *and there exists a constant* $C > 0$ *such that* $|C_{\alpha,l,l'}| \geq C$ *for all* (l, l') *such that* $C_{\alpha,l,l'} \neq 0$.

PROOF.

(1) The numbers of the form $\pm(n + 2j)$, $j \in \mathbb{N}$ are asymptotic singular values since, for l' fixed (see Lemma 2),

$$\frac{n(l + l') + 2ll'}{l - l'} \to_{l\to\infty} n + 2l',$$

and, for l fixed,

$$\frac{n(l + l') + 2ll'}{l - l'} \to_{l'\to\infty} -(n + 2l).$$

To show that there are no other (positive, for example) asymptotic singular values, it is enough to notice that, for fixed l', the map

$$l \mapsto \alpha_{l,l'} := \frac{n(l + l') + 2ll'}{l - l'} \quad (l \neq l')$$

is a decreasing function. So, if $\alpha \in [2j + n - 1, 2j + n + 1[$ is an asymptotic singular value, then $|\alpha_{l,l'} - \alpha| \geq 1$ for all $l' \geq j + 1$. But, for all $l' = 0, \ldots, j$, $\alpha_{l,l'} \to_{l\to\infty} 2l' + n$. So $\alpha = 2j + n$.

(2) If we write the eigenvalue of $\mathcal{L}_\alpha$ on $\mathcal{H}^{n,l,l'}$ as $\frac{\alpha-n}{2} l - \frac{\alpha+n}{2} l' - ll'$, it becomes at once clear that $d\pi_{l,l'}(\mathcal{L}_\alpha)$ is invertible for $l + l'$ large enough if $\alpha \neq \pm n$.

Set first $\alpha = 2j + n$ $(j = 1, 2, \ldots)$. Then

$$d\pi_{l,l'}(\mathcal{L}_\alpha) = d\pi_{l,l'}(\mathcal{L}_{\alpha_{l,l'}}) + \frac{l - l'}{2}(2j + n - \alpha_{l,l'}) = \frac{l - l'}{2}(2j + n - \alpha_{l,l'})$$

$(l \neq l')$ and

$$d\pi_{l,l}(\mathcal{L}_{\alpha,1}) = -(l^2 + nl).$$

For $l \geq N$, all $\alpha_{l,l'}$ with $l' \neq j$ are, say, less than $2j + n - 1$ or greater than $2j + n + 1$. So, for $l > N$, the operator $d\pi_{l,l'}(\mathcal{L}_\alpha)$ is invertible and its eigenvalues are (in absolute value) greater than

$$\min\left(\frac{1}{2}, \frac{1}{2}\left|(l-j)\left(2j+n-\frac{l(2j+n)+nj}{l-2j}\right)\right|\right) \sim_{l\to\infty} 1.$$

Using the antisymmetry in l and l', an analogous result is valid for l' large enough.

Assume now that $\alpha \geq 0$ and α is not in the cluster set, so $\alpha \in]2j+n-2, 2j+n[$ for a certain $j \in \mathbb{N}$. Then, for $l > N$, all $\alpha_{l,l'}$ for $l' < j$ are very close to their asymptotic value $2l' + n < \alpha$, and $\alpha_{l,l'} \geq 2j + n$ for larger positive values of l', so, for all l', $|\alpha_{l,l'} - \alpha| \geq C > 0$. Using the method above, we get also in this case a bound from below of the eigenvalues of $(d\pi_{l,l'}(\mathcal{L}_\alpha))^{-1}$ for $l + l'$ large enough. □

THEOREM 2.

(1) *If α is not a singular value, then $\mathcal{L}_\alpha$ is globally solvable on $L^2_0(S^{2n+1})$. More precisely, if $f \in L^2_0(S^{2n+1})$, then there is a (unique) function $u \in L^2_0(S^{2n+1})$ such that $\mathcal{L}_\alpha u = f$.*
(2) *If $\alpha \neq \pm n$, then $\mathcal{L}_\alpha$ is locally solvable on $L^2(S^{2n+1})$.*

PROOF.

(1) Let α be a regular value. Then, whether α be in the cluster set or not, the inverse of $\mathcal{L}_\alpha$ is bounded on $L^2_0(S^{2n+1})$ by Lemma 1.
(2) Consider the equation $\mathcal{L}_\alpha u = f$ in a neighbourhood of $x_0 \in S^{2n+1}$. Then, by the preceding point, and modifying f outside of a small neighbourhood on x_0 as before so that they have a null integral, it appears clearly that $\mathcal{L}_\alpha$ is locally solvable if α is not a singular value.

Let α be a singular value, $\alpha \neq \pm n$. Then the kernel $\mathcal{J}$ of $\mathcal{L}_\alpha$ consists of the linear span of a finite set of matrix coefficients. Let $\mathcal{J}^\perp$ denote its orthogonal in $L^2(S^{2n+1})$, and $\mathcal{R}$ be the cokernel of $\mathcal{L}_\alpha$ in $L^2(S^{2n+1})$. Then $\mathcal{L}_\alpha$ is an isomorphism of $\mathcal{J}^\perp$ onto $\mathcal{R}$ and its inverse is bounded, as one sees by an easy generalization of Lemma 1.

Let now $x \in S^{2n+1}$, U a small neighbourhood of x and $f \in L^2(S^{2n+1})$ with compact support on U. Then, by modifying f outside of U, one may suppose that $f \in \mathcal{R}$. So there exists $u \in \mathcal{J}^\perp$ such that $\mathcal{L}_{\alpha,1} u = f$ on U. □

Note that it is not clear which odd integers are singular values. For $n = 1$, it is easy to verify that 5 is one of them but 3 is not.

Consider now $\alpha = \pm n$. In the case of $S^3 \simeq SU(2)$, *i.e.* $n = 1$, then $\mathcal{L}_{\pm 1} = (X \pm iY)(X \mp iY)$ is not locally solvable: if it were so, then also $X \pm iY$ would be locally solvable, in contrast with Nirenberg's and Treves' criterion for operators of principal type (see [6]).

In general dimensions, the L^2-kernel of $\mathcal{L}_n$ is $\oplus_{l\geq 0}\mathcal{H}^{n,l,0}$, *i.e.*, the subspace of boundary values of functions in the Hardy space H^2 of the unit ball (see [4, formula (4.7.2)] and [5, Proposition 2.5]). The orthogonal projection of L^2 onto this space in given by the integral formula

$$P_n f(z) = \lim_{r\to 1^-} c_n \int_{S^{2n+1}} \frac{f(w)}{(1 - r\langle z, w\rangle)^{n+1}}\, dw,$$

where c_n is a positive constant, $\langle\ ,\ \rangle$ is the Hermitian inner product on $\mathbb{C}^{n+1}$, dw is the unit surface measure on S^{2n+1}, and the limit is in the L^2-sense. Similarly, the L^2-kernel of $\mathcal{L}_{-n}$ is $\oplus_{l'\geq 0}\mathcal{H}^{n,0,l'}$ and the corresponding orthogonal projection is $P_{-n}f = \overline{P_n(\overline{f})}$.

The following theorem is an analogue of a classical result on the Heisenberg group (see [8, Chap. 13, Section 4]).

Theorem 3. *Let f be a smooth function on S^{2n+1}. Then the equation $\mathcal{L}_n u = f$ (resp. $\mathcal{L}_{-n}u = f$) has a distributional solution in a neighbourhood of $z_0 \in S^{2n+1}$ if and only if $P_n f$ (resp. $P_{-n}f$) is real-analytic on a neighbourhood of z_0 in S^{2n+1}.*

Proof. Let us first assume the equation $\mathcal{L}_n u = f$ has a distributional solution u in a neighbourhood Ω of $z_0 \in S^{2n+1}$. So $\mathcal{L}_n u = f + g$ for a certain distribution g which is zero in Ω. Since $\mathcal{L}_n$ is self-adjoint,

$$P_n f = P_n(\mathcal{L}_n u) - P_n g = -P_n g.$$

Since the integral that gives $P_n g$ is not singular in Ω, the function $P_n f$ is real-analytic in Ω.

Now suppose $P_n f$ is real-analytic in a neighbourhood Ω of z_0. The operator $\sum_{j=1}^n (X_j^2 + Y_j^2) - i\alpha H$, viewed as acting on $U(n)$-invariant functions on $U(n+1)$, has the following relative fundamental solution on the L^2-orthogonal of its kernel $\oplus_{l\geq 0}\mathcal{H}^{n,l,0}$:

$$S(g) = \sum_{l\geq 0,\, l'\geq 1} \frac{d_{n,l,l'}}{C_{n,l,l'}}\varphi_{n,l,l'}, \qquad g \in U(n+1)$$

where $d_{n,l,l'}$ is the dimension of the representation $\pi_{n,l,l'}$, $C_{n,l,l'}$ is defined as in Lemma 3, and $\varphi_{n,l,l'}$ is the spherical function associated with $\pi_{n,l,l'}$ (see [2, Chap. 4, Theorem 4.2 and Chap. 5, Theorem 3.5]). Since the $d_{n,l,l'}$ have polynomial growth and the $\left|\frac{1}{C_{n,l,l'}}\right| = \frac{1}{(l+n)l'}$ have a common bound for $l' \geq 1$, the relative fundamental solution S is well-defined as a distribution. Hence a solution of the equation $\mathcal{L}_n u = f - P_n f$ is the function u on S^{2n+1} such that $u^\sharp = f^\sharp \star S$. Since the equation $\mathcal{L}_n v = P_n f$ has an analytic solution because of Cauchy-Kowalewska's theorem, the proof is now complete for $\mathcal{L}_n$.

The argument is the same for $\mathcal{L}_{-n}$. □

Study of the operators $\mathcal{L}_{\alpha,\beta} = X^2 + \beta^2 Y^2 - i\beta\alpha H$ on $SU(2)$

Let $A, B, [A, B]$ constitute a basis of $\mathfrak{su}(2)$. We first prove that the operator $\mathcal{D} = A^2 + B^2 - i\alpha[A, B]$ is conjugate (up to a constant) to $\mathcal{L}_{\alpha,\beta}$ for a certain value of β (see Introduction for the definition). The adjoint action of $SU(2)$ on $\mathfrak{su}(2)$ gives all

3-dimensional rotations. So we may assume that A, B are in the hyperplane $\mathbb{R}X \oplus \mathbb{R}Y$. Put $A = a_1 X + a_2 Y$, $B = b_1 X + b_2 Y$, with $\delta = (a_1 b_2 - a_2 b_1)^2$. Then

$$\mathcal{D} = (a_1^2 + b_1^2)X^2 + (a_2^2 + b_2^2)Y^2 + (a_1 a_2 + b_1 b_2)(XY + YX) - i\alpha\delta^{\frac{1}{2}}[X, Y] = \mathcal{D}_0 - i\alpha\delta^{\frac{1}{2}}[X, Y].$$

The operator $\mathcal{D}_0$ can be formally written as

$$\mathcal{D}_0 = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} a_1^2 + b^2 & a_1 a_2 + b_1 b_2 \\ a_1 a_2 + b_1 b_2 & a_2^2 + b_2^2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X & Y \end{pmatrix}^t U \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} U \begin{pmatrix} X \\ Y \end{pmatrix}$$

for a certain orthogonal matrix $U \in SO(2)$, with $\lambda_1 \lambda_2 = \delta$. Set $\begin{pmatrix} X' \\ Y' \end{pmatrix} = U \begin{pmatrix} X \\ Y \end{pmatrix}$. Then $[X', Y'] = [X, Y]$, so that $\mathcal{D} = \lambda_1 X'^2 + \lambda_2 Y'^2 - i\alpha\sqrt{\lambda_1\lambda_2}[X', Y']$. So, if $\beta = \sqrt{\frac{\lambda_2}{\lambda_1}}$, then $\mathcal{D}$ is conjugate to $\lambda_1 \mathcal{L}_{\alpha,\beta}$.

We first apply a theorem of Hörmander (see [3]) on the hypoellipticity of doubly characteristic operators to our problem.

Theorem 4. *If* $\alpha \neq \pm 1, \pm 3, \ldots$, *then* $\mathcal{L}_{\alpha,\beta}$ *is hypoelliptic and locally solvable.*

Proof. First remark that $\mathcal{L}_{\alpha,\beta}$ is self-adjoint, so hypoellipticity implies local solvability. We verify that $\mathcal{L}_{\alpha,\beta}$ is hypoelliptic at the origin, using Hörmander's criterion.

Let $g = \begin{pmatrix} t_1 + it_2 & u_1 + iu_2 \\ -u_1 + iu_2 & t_1 - it_2 \end{pmatrix} \in SU(2)$ (with $t_1 = \sqrt{1 - t_2^2 - u_1^2 - u_2^2}$). The parameters (t_2, u_1, u_2) give local coordinates near the origin. We get

$$2X = t_1\partial_{t_2} + u_1\partial_{u_2} - u_2\partial_{u_1}, \quad 2Y = t_1\partial_{u_2} + t_2\partial_{u_1} - u_1\partial_{t_2}, \quad 2Z = t_1\partial_{u_1} + u_2\partial_{t_2} - t_2\partial_{u_2}.$$

Let $(t_2, u_1, u_2, \tau_2, \mu_1, \mu_2)$ be coordinates in the cotangent space (τ_2, μ_1, μ_2 being the dual coordinates of t_2, u_1, u_2), and let P_2 be the principal symbol of $4\mathcal{L}_{\alpha,\beta}$.

At the origin $(t_2, u_1, u_2) = (0, 0, 0)$, we have $-P_2(0, 0, 0, \tau_2, \mu_1, \mu_2) = \tau_2^2 + \beta^2\mu_2^2 \geq 0$. So the fiber of the characteristic variety over the origin is of dimension one, given by $\tau_2 = \mu_2 = 0$.

Let us now compute the quadratic form Q giving the Taylor expansion to second order of P_2 near each of the characteristic points $(0, 0, 0, 0, \pm 1, 0)$. Taking, *e.g.*, the positive sign, we get

$$Q(t_2, u_1, u_2, \tau_2, 1 + \delta\mu_1, \mu_2) = \tau_2^2 + u_2^2 - 2u_2\tau_2 + \beta^2(t_2^2 + \mu_2^2 + 2t_2\mu_2).$$

Its kernel is the two-dimensional vector space $\mathbb{R}u_1 \oplus \mathbb{R}\mu_1$. Identify Q with its matrix in the coordinates $(t_2, u_2, \tau_2, \mu_2)$, and put $F = QJ$, where J is the matrix of the canonical symplectic form (see [3, p. 166]). The eigenvalues of F can be easily computed to be 0 and $\pm 2\beta$. The generalized null-space of F is two-dimensional and consists of the vectors such that $\mu_2 = -t_2$ and $\tau_2 = u_2$. It is quickly verified that $Q(v, v) = 0$ whenever $v = (t_2, u_1, u_2, u_2, \mu_1, -t_2)$, $t_2, u_1, u_2, \mu_1 \in \mathbb{R}$, that is, when v lies in the whole generalized null-space. So Hörmander's condition (*i.e.* (1.3) in [3]) is exactly that $\alpha \neq \pm 1, \pm 3, \ldots$. □

We shall now study the set of singular values.

For all n, $(V_n, \widetilde{\pi}_n)$ will denote the representation of G, equivalent to π_n, given by

$$\widetilde{\pi}_n(g)P(u) = \pi_n(g)(P(\sqrt{n}.))\left(\frac{u}{\sqrt{n}}\right)$$

and $\|\ \|_{\widetilde{\pi}_n}$ the norm on $\widetilde{V}_n$ transfered from the norm on V_n by the intertwining operator. So, if $P(z) = \sum a_k z^k \in V_n$,

$$\|P\|^2_{\pi_n} = \sum_{j=0}^{n} C_{n,j}^{-1}|a_j|^2, \quad \|P\|^2_{\widetilde{\pi}_n} = \sum C_{n,j}^{-1} n^j |a_j|^2.$$

Let $\gamma = \frac{1-\beta}{1+\beta}$.

LEMMA 4.

(1) *For all* n, α, β,

$$\frac{1}{n} d\widetilde{\pi}_n(\mathcal{L}_{\alpha,\beta}) = \left(\frac{1}{\gamma+1}\right)^2 \left((1+\gamma u^2/n)\frac{d}{du} - \gamma u\right)\left((\gamma + u^2/n)\frac{d}{du} - u\right) - \\ - \beta(\alpha+1)\left(\frac{u}{n}\frac{d}{du} - \frac{1}{2}\right).$$

(2) *The kernel of* $\mathcal{L}_{\pm 1,\beta}$ *is infinite-dimensional: the operator* $d\widetilde{\pi}_n(\mathcal{L}_{\pm 1,\beta})$ $(n \geq 1)$ *has a non trivial kernel of dimension 1 for all even n, generated by the polynomial* $P_{n,\pm 1} = = \left(1 + \frac{\gamma^{\pm 1} u^2}{n}\right)^{\frac{n}{2}}$.

Observe that, when $n \to \infty$, it tends formally to the operator

$$\left(\frac{1}{\gamma+1}\right)^2 \left(\frac{d}{du} - \gamma u\right)\left(\gamma\frac{d}{du} - u\right) + \frac{1}{2}\beta(\alpha+1),$$

which is equal to the operator $d\sigma_1(\mathcal{L}'_{\alpha,\beta})$. Also, for $\alpha = 1$, $P_{n,1}(u) \to_{n\to\infty} \varphi_0(u) = e^{\frac{\gamma u^2}{2}}$ and $\varphi_0 \in \mathrm{Ker}\ d\sigma_1(\mathcal{L}'_{1,\beta})$.

PROOF.

(1) It is easily checked that

$$d\pi_n(X) = \frac{1}{2}\left((w^2+1)\frac{d}{dw} - nw\right),\ d\pi_n(Y) = \frac{i}{2}\left((1-w^2)\frac{d}{dw} + nw\right),\ d\pi_n(iH) = w\frac{d}{dw} - \frac{n}{2}.$$

So (letting $u = \sqrt{n}w$)

$$d\widetilde{\pi}_n(X - i\beta Y) = \sqrt{n}\left(\frac{1}{\gamma+1}\right)\left((1+\gamma u^2/n)\frac{d}{du} - \gamma u\right),$$

$$d\widetilde{\pi}_n(X + i\beta Y) = \sqrt{n}\left(\frac{1}{\gamma+1}\right)\left((\gamma + u^2/n)\frac{d}{du} - u\right),$$

$$id\widetilde{\pi}_n(H) = u\frac{d}{du} - \frac{n}{2}$$

with $\gamma = \frac{1-\beta}{1+\beta}$. Now just remark that

$$\mathcal{L}_{\alpha,\beta} = (X - i\beta Y)(X + i\beta Y) - i\beta(\alpha+1)H = (X + i\beta Y)(X - i\beta Y) - i\beta(\alpha-1)H.$$

(2) Let $f \in L^2$ such that $\mathcal{L}_{1,\beta} f = 0$. Put $f = \sum_n f_n$, with $f_n \in V_n$. Then, for all l, l', $\mathcal{L}_{1,\beta} f_{l,l'} = 0$, so

$$0 = \langle (X - i\beta Y)(X + i\beta Y)f, f\rangle = ||(X + i\beta Y)f||^2.$$

It suffices then to verify by means of the above explicit expressions of $d\pi_n(X)$ and $d\pi_n(Y)$ that the polynomials $P_{n,1}$ generate the kernel of $X + i\beta Y$ on $\mathcal{H}^{n,l,l'}$. The case $\alpha = -1$ is completely analogue. □

We can now prove:

THEOREM 5. *The cluster set is equal to the set* $\{\pm 1, \pm 3, \dots\}$ *of eigenvalues of the operator* $\frac{1}{\beta} d\sigma_{\pm 1}(X'^2 + \beta^2 Y'^2)$.

PROOF.

(1) Let us first prove by contradiction that the cluster set contains the values $\{\pm 1, \pm 3, \dots\}$. Let φ_j be a non-zero element in the kernel of $d\sigma_1(\mathcal{L}'_{-(2j+1),\beta})$. We may choose for instance $\varphi_j(u) = (d\sigma_1(X' - i\beta Y'))^j e^{\frac{\gamma u^2}{2}}$ since

$$\left((X' + i\beta Y')(X' - i\beta Y') + i\beta\alpha H'\right)(X' - i\beta Y') = \\ = (X' - i\beta Y')\left((X' + i\beta Y')(X' - i\beta Y') + i\beta(\alpha - 2)H'\right).$$

So $\varphi_j(u) = Q_j(u) e^{\frac{\gamma u^2}{2}}$ where Q_j is a polynomial of degree j.

Fix $j = 0, 1, \dots$. Let $0 < \sigma < 1 - \gamma$ and, for all n, $\varphi_{j,n} = \sum_{k=0}^{[n\sigma]} a_k u^k$, with $a_k = \frac{\varphi_j^{(k)}(0)}{k!}$. Note that, for k large enough,

$$|a_k|^2 \le CP_j(k) \left(\frac{\gamma}{2}\right)^k \left(\left(1 + \left[\frac{k}{2}\right]\right)!\right)^{-2} \le CP_j(k) \frac{\gamma^k}{k!}$$

by Stirling's formula, where P_j is a certain polynomial.

Let $\mathcal{D} = u^b (\frac{\partial}{\partial u})^c$ $(b, c = 0, 1, \dots)$ and $\varepsilon = b - c$. Then

$$\mathcal{D}\varphi_{j,n} = \sum_{k=0}^{[\sigma n]} k(k-1)\cdots(k-c+1) a_k u^{k+\varepsilon}$$

so

$$||\mathcal{D}\varphi_{j,n}||^2_{\tilde{\pi}_n} \le \sum_{k=0}^{[\sigma n]} P_j(k) k^{2c} \frac{\gamma^k}{k!} \frac{n^{k+\varepsilon}}{C_{n,k+\varepsilon}} \le C_\varepsilon P_j(k) \sum_{k=0}^{[\sigma n]} k^{2c+\varepsilon} \gamma^k \frac{n^{k+\varepsilon}}{n(n-1)\cdots(n-k-\varepsilon+1)}.$$

Now, for $k \le \sigma n$,

$$\gamma^k \frac{n^{k+\varepsilon}}{n(n-1)\cdots(n-k-\varepsilon+1)} \le \left(\frac{\gamma}{1-\sigma}\right)^{k+\varepsilon}$$

so, by the hypothesis on σ, $||\mathcal{D}\varphi_{j,n}||_{\tilde{\pi}_n}$ is bounded for any n. By dominated convergence, $||\varphi_{j,n}||_{\tilde{\pi}_n} \to_{n\to\infty} ||\varphi_j||_{\sigma_1}$.

Observe that $d\sigma_1(\mathcal{L}'_{\alpha,\beta})(\varphi_{j,n} - \varphi_j) \in V_n$ since $d\sigma_1(\mathcal{L}'_{\alpha,\beta})(\varphi_j) = 0$. By the same method, we also get

$$||d\sigma_1(\mathcal{L}'_{\alpha,\beta})(\varphi_{j,n} - \varphi_j)||^2_{\widetilde{\pi}_n} = \sum_{k=[\sigma n]-2}^{[\sigma n]+2} |b_k|^2$$

(all other terms cancelling) where

$$|b_k| \leq CP_j(k)k^4\gamma^k \frac{n^{k+2}}{n(n-1)\cdots(n-k-1)} \leq C'P_j(k)k^4\left(\frac{\gamma}{1-\sigma}\right)^k$$

for $k = [\sigma n] - 2, \ldots, [\sigma n] + 2$, so $||d\sigma_1(\mathcal{L}'_{\alpha,\beta})(\varphi_{j,n} - \varphi_j)||^2_{\widetilde{\pi}_n}$ tends exponentially to 0 as n goes to infinity.

By Lemma 4, we can write $\frac{1}{n} d\widetilde{\pi}_n(\mathcal{L}_{\alpha,\beta}) = d\sigma_1(\mathcal{L}'_{\alpha,\beta}) + \frac{1}{n}\mathcal{D}_1 + \frac{1}{n^2}\mathcal{D}_2$ (where $\mathcal{D}_1$ and $\mathcal{D}_2$ do not depend on n), so we finally get

$$\left\|\frac{1}{n} d\widetilde{\pi}_n(\mathcal{L}_{-(2j+1),\beta})\varphi_{n,j}\right\|_{\widetilde{\pi}_n} \leq$$

$$\leq \frac{1}{n}\|\mathcal{D}_1\varphi_{j,n}\|_{\widetilde{\pi}_n} + \frac{1}{n^2}\|\mathcal{D}_2\varphi_{j,n}\|_{\widetilde{\pi}_n} + \|d\sigma_1(\mathcal{L}'_{-(2j+1)\beta,\beta})(\varphi_{j,n} - \varphi_j)\|_{\widetilde{\pi}_n} \leq \frac{C}{n}\|\varphi_{n,j}\|_{\widetilde{\pi}_n}.$$

So now suppose α is not a cluster point, and write

$$d\widetilde{\pi}_n(\mathcal{L}_{\alpha,\beta}) = d\widetilde{\pi}_n(X^2 + \beta^2 Y^2) - \alpha\beta d\widetilde{\pi}_n(iH) = A_n + \alpha B_n;$$

A_n and B_n are hermitian matrices and A_n is negative definite. More precisely, $A_n \leq \leq \min(1, \beta^2) d\widetilde{\pi}_n(X^2 + Y^2)$, and, by Section 1,

$$d\widetilde{\pi}_n(X^2 + Y^2) = \operatorname{diag}\left(-\frac{n}{2}(1 + 2k) + k^2\right) \leq -\frac{n}{2} Id.$$

So

$$\left|\left|\left|d\widetilde{\pi}_n(\mathcal{L}_{\alpha,\beta})^{-1}\right|\right|\right| = \alpha^{-1}\left|\left|\left|A_n^{-\frac{1}{2}}\left(\frac{1}{\alpha}Id + A_n^{-\frac{1}{2}}B_nA_n^{-\frac{1}{2}}\right)^{-1}A_n^{-\frac{1}{2}}\right|\right|\right| \leq$$

$$\leq \frac{2}{\alpha n}\left|\left|\left|\left(\frac{1}{\alpha}Id + A_n^{-\frac{1}{2}}B_nA_n^{-\frac{1}{2}}\right)^{-1}\right|\right|\right|.$$

Since the eigenvalues of $A_n^{-\frac{1}{2}}B_nA_n^{-\frac{1}{2}}$ are the $-\frac{1}{\alpha'}$ where α' is a singular value associated to $\widetilde{\pi}_n$, we get by hypothesis

$$\left|\left|\left|\left(\frac{1}{\alpha}Id + A_n^{-\frac{1}{2}}B_nA_n^{-\frac{1}{2}}\right)^{-1}\right|\right|\right| \leq C,$$

so

$$\left|\left|\left|\left(\frac{1}{n}d\widetilde{\pi}_n(\mathcal{L}_{\alpha,\beta})\right)^{-1}\right|\right|\right| \leq C'$$

whence the contradiction.

(2) Let $\alpha \neq \pm 1, \pm 3, \ldots$. Then, by Hörmander's theorem, we get the following subellipticity estimate for any $u \in C^\infty(G)$ (since G is compact):

$$||u||_1^2 \leq C(||\mathcal{L}_{\alpha,\beta}u||_0^2 + ||u||_0^2),$$

which, by Fourier inversion, gives

$$(n+1)^2||A||_{HS}^2 \le C(||\pi_n(\mathcal{L}_{\alpha,\beta})A||_{HS}^2 + ||A||_{HS}^2)$$

for all matrix $A \in$ End V_n. Hence, if n is large enough,

$$||A||_{HS} \le \frac{C}{n+1}||\pi_n(\mathcal{L}_{\alpha,\beta})A||_{HS}$$

for all A, and

$$\left|\left|\left| \left(\frac{1}{n}\pi_n(\mathcal{L}_{\alpha,\beta})\right)^{-1} \right|\right|\right| \le C.$$

Suppose now (by contradiction) that α is a cluster point, so there exists a sequence of non-zero $P_j \in V_{n_j}$ such that $d\pi_{n_j}(\mathcal{L}_{\alpha_j,\beta})P_j = 0$ with $\alpha_j \to \alpha$. Hence

$$\left\|\frac{1}{n_j}d\pi_{n_j}(\mathcal{L}_{\alpha,\beta})P_j\right\|_{\widetilde{\pi}_{n_j}} \le \frac{\alpha-\alpha_j}{n_j}\left\|u\frac{d}{du}P_j\right\|_{\widetilde{\pi}_{n_j}} \le \frac{\alpha-\alpha_j}{n_j}\left|\left|\left|u\frac{d}{du}\right|\right|\right| \, \|P_j\|_{\widetilde{\pi}_{n_j}}.$$

It is easy to verify (using the monomial basis of V_n) that $|||u\frac{d}{du}||| = n_j$, so

$$\left\|\frac{1}{n_j}d\pi_{n_j}(\mathcal{L}_{\alpha,\beta})P_j\right\|_{\widetilde{\pi}_{n_j}} \le (\alpha-\alpha_j)\|P_j\|_{\widetilde{\pi}_{n_j}},$$

whence we get a contradiction. □

Note finally that the operator $\mathcal{L}_{\alpha,\beta}$ is globally solvable on L_0^2 for all non real α. Namely, writing $d\widetilde{\pi}_n(\mathcal{L}_{\alpha,\beta}) = A_n + \alpha B_n$ as before, we get

$$\begin{aligned}\left|\left|\left|(A_n+\alpha B_n)^{-1}\right|\right|\right| &= |\alpha^{-1}| \; |||A_n^{-\frac{1}{2}}|||^2 \; \left|\left|\left|\left(A_n^{-\frac{1}{2}}B_nA_n^{-\frac{1}{2}} + \alpha^{-1}I\right)^{-1}\right|\right|\right| \\ &\le \frac{C}{n}\;|\alpha^{-1}| \; |\mathrm{Im}\alpha^{-1}|^{-1}.\end{aligned}$$

Acknowledgements

The second author is supported by a postdoc fellowship from the European Commission within the European TMR Network «Harmonic Analysis» 1998-2001 (Contract ERBFMRX-CT97-0159).

References

[1] A. Boggess, *CR Manifolds and the Tangential Cauchy-Riemann Complex*. Studies in Advanced Mathematics, CRC Press, 1991.
[2] S. Helgason, *Groups and geometric analysis*. Academic Press, 1984.
[3] L. Hörmander, *A Class of Hypoelliptic Pseudodifferential Operators with Double Characteristics*. Math. Ann., vol. 217, 1975, 165-188.
[4] L. K. Hua, *Harmonic analysis of Functions of Several complex variables in the classical domains*. Transl. Math. Monog., vol. 6, A.M.S., Providence RI 1963.
[5] A. Korányi, *The Poisson integral for generalized half-planes and bounded symmetric domains*. Ann. Math., 82, 1965.

[6] L. Nirenberg - F. Trèves, *On Local Solvability of Linear Partial Differential Equations. Part I: Necessary Conditions* Vol. 23, 1970, 1-38.
[7] F. Ricci, *A Contraction of SU(2) to the Heisenberg group*. Mh. Mat., 101, 1986, 211-225.
[8] E.M. Stein, *Harmonic analysis*. Princeton University Press, 1983.
[9] M.E. Taylor, *Noncommutative harmonic analysis*. AMS, Mathematical surveys and monographs, 1986.
[10] N.Ja. Vilenkin - A. U. Klimyk, *Representation of Lie Groups and Special Functions*. Vol. 2, Kluwer Academic Publishers, 1993.

Pervenuta il 10 ottobre 2000,
in forma definitiva il 24 ottobre 2000.

F. Ricci:
Scuola Normale Superiore di Pisa
Piazza dei Cavalieri, 7 - 56126 Pisa
fricci@sns.it

J. Unterberger:
Université Henri Poincaré (Nancy I)
Institut de Mathématiques Élie Cartan (UMR CNRS 9973)
54506 Vandoeuvre-lès-nancy, Cedex (Francia)
jeremie.unterberger@iecn.u–nancy.fr