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General and physically privileged solutions to certain symmetric systems of linear P.D.E.s with tensor functionals as unknowns

Memoria (*) di Adriano Montanaro e Diego Pigozzi

Abstract. — We characterize the general solutions to certain symmetric systems of linear partial differential equations with tensor functionals as unknowns. Then we determine the solutions that are physically meaningful in suitable senses related with the constitutive functionals of two simple thermodynamic bodies with fading memory that are globally equivalent, *i.e.* roughly speaking that behave in the same way along processes not involving cuts. The domains of the constitutive functionals are nowhere dense subsets of a suitable infinite-dimensional Hilbert space. By using the condition of material frame-indifference on the constitutive functionals and the theory [1] of differential calculus on convex sets (that may be nowhere dense), we give a rigorous meaning from a general point of view to the derivatives of these functionals, without assuming the possibility of extending them to an open set. Such results appear necessary for characterizing the couples of thermodynamic bodies with memory that are globally equivalent but are physically different; and such bodies exist.

Key words: Linear partial differential equations; Tensor functionals; Symmetric linear systems.

Riassunto. — *Soluzione generale e soluzioni fisicamente privilegiate di sistemi simmetrici di equazioni lineari alle derivate parziali aventi come incognite funzionali tensoriali.* Si caratterizza la soluzione generale di certi sistemi simmetrici di equazioni lineari alle derivate parziali aventi funzionali tensoriali come incognite. Quindi si determinano le soluzioni fisicamente significative in certi sensi collegati con i funzionali costitutivi di due corpi termodinamici con memoria evanescente che sono globalmente equivalenti, ossia che si comportano allo stesso modo nei processi non involgenti tagli. I domini dei funzionali costitutivi sono sottoinsiemi ovunque non densi di un opportuno spazio di Hilbert a infinite dimensioni. Usando la condizione di indifferenza materiale per i funzionali costitutivi e la teoria [1] di calcolo differenziale su un insieme convesso (che può essere ovunque non denso), viene dato un significato rigoroso, da un punto di vista generale, alle derivate di questi funzionali, senza assumere la possibilità di estenderli a qualche insieme aperto. Tali risultati sembrano necessari per caratterizzare le coppie di corpi termodinamici con memoria che sono globalmente equivalenti ma fisicamente diversi; e tali corpi esistono.

1. Introduction

Let Lin ($\simeq \mathbb{R}^{3\times 3}$) be the vector space of all second-order tensors on the real vector space $\mathcal{V} = \mathbb{R}^3$ equipped with the usual inner product, that in Cartesian co-ordinates is defined by

$$x_1 \cdot x_2 = \operatorname{tr}(x_1 x_2^T) = x_1^{ij} x_2^{ij} \quad \text{for} \quad x_1, x_2 \in Lin. \tag{1}$$

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be an *influence function*, that is a positive continuous function which

(*) Pervenuta in forma definitiva all'Accademia il 25 luglio 2000.

is bounded, monotone-decreasing and square-integrable. In continuum mechanics, with regard to simple materials with memory, an influence function is used to characterize the rate at which the memory of the material fades (cf. [2]). For each normed vector space $\mathcal{S} \in \{Lin, \mathcal{V}, \mathbb{R}\}$ let us consider the inner product

$$\langle \gamma_1, \gamma_2 \rangle = \left[\int_0^\infty \gamma_1(s) \cdot \gamma_2(s)\, h^2(s) ds \right]^{\frac{1}{2}} \tag{2}$$

for functions $\gamma_i : (0, \infty) \to \mathcal{S}$. The set $\mathcal{S}_\infty$ of all Lebesgue-measurable functions $\gamma : (0, \infty) \to \mathcal{S}$ such that $\langle \gamma, \gamma \rangle < \infty$ is a Hilbert space with the norm

$$||\gamma|| = \left[\int_0^\infty |\gamma(s)|^2 h^2(s) ds \right]^{\frac{1}{2}} < \infty, \qquad |\gamma(s)|^2 = \gamma(s) \cdot \gamma(s) \quad (^1) \tag{3}$$

induced by (2).

Consider the (smooth) functional

$$\widehat{\mathcal{F}} : \mathcal{A} \times \mathcal{U} \to Lin, \quad \mathcal{F} = \widehat{\mathcal{F}}(x, z, \xi, \zeta), \quad (x, z) \in \mathcal{A}, \quad (\xi, \zeta) \in \mathcal{U}, \tag{4}$$

where $\mathcal{A}$ and $\mathcal{U}$ are open connected subsets of $Lin \times \mathcal{V}$ and $Lin_\infty \times \mathcal{V}_\infty$, respectively. We characterize the solutions to the symmetric system of linear partial differential equations

$$\begin{aligned} &\frac{\partial \mathcal{F}^{aA}}{\partial x^b_B} + \frac{\partial \mathcal{F}^{aB}}{\partial x^b_A} = 0, \qquad && \frac{\partial \mathcal{F}^{aA}}{\partial z_B} + \frac{\partial \mathcal{F}^{aB}}{\partial z_A} = 0, \\ &\frac{\partial \mathcal{F}^{aA}}{\partial \xi^b_B} + \frac{\partial \mathcal{F}^{aB}}{\partial \xi^b_A} = 0, \qquad && \frac{\partial \mathcal{F}^{aA}}{\partial \zeta_B} + \frac{\partial \mathcal{F}^{aB}}{\partial \zeta_A} = 0, \end{aligned} \tag{5}$$

where the indices a, b, A and B run over $\{1, 2, 3\}$, in the unknown functional (4), which we assume to be of class C^1.

We also study the symmetric system of linear partial differential equations

$$\begin{aligned} &\frac{\partial \mathcal{Q}^{A}}{\partial x^b_B} + \frac{\partial \mathcal{Q}^{B}}{\partial x^b_A} = 0, \qquad && \frac{\partial \mathcal{Q}^{A}}{\partial z_B} + \frac{\partial \mathcal{Q}^{B}}{\partial z_A} = 0, \\ &\frac{\partial \mathcal{Q}^{A}}{\partial \xi^b_B} + \frac{\partial \mathcal{Q}^{B}}{\partial \xi^b_A} = 0, \qquad && \frac{\partial \mathcal{Q}^{A}}{\partial \zeta_B} + \frac{\partial \mathcal{Q}^{B}}{\partial \zeta_A} = 0, \end{aligned} \tag{6}$$

where b, A and B run over $\{1, 2, 3\}$, in the unknown functional

$$\widehat{\mathcal{Q}} : \mathcal{A} \times \mathcal{U} \to \mathcal{V}, \qquad \mathcal{Q} = \widehat{\mathcal{Q}}(x, z, \xi, \zeta), \quad (x, z) \in \mathcal{A}, \quad (\xi, \zeta) \in \mathcal{U}, \tag{7}$$

which we assume to be of class C^1.

In addition to the general solution we also characterize the classes of solutions that are physically privileged in certain senses specified below.

(1) Note that $\mathcal{S}_\infty$ is the «weighted» L^2 space, $L^2_h(\mathbb{R}^+, \mathcal{S})$, relative to the weighted element of measure $h^2(s) ds$.

The symmetric systems (5) and (6) arise in continuum thermodynamics: the difference $\mathcal{F}$ [$\mathcal{Q}$], between corresponding constitutive functionals of any two globally equivalent simple bodies with fading memory, satisfies (5) [(6)] as a consequence of the local balance laws.

In more detail, in connection with any given material point X of a continuous simple body $\mathcal{B}$ with fading memory, let x be the deformation gradient, z the temperature gradient and ξ [ζ] the past time-history of x [z]; *i.e.*, at the present time t we have

$$\xi(s) := x^t(s) := x(t-s)\,, \qquad \zeta(s) := z^t(s) := z(t-s)\,, \quad s \in (0\,,\infty). \tag{8}$$

The constitutive relations for the stress and heat flux in $\mathcal{B}$ at X are expressed in terms of functionals having the forms (4) and (7), respectively, where the dependence upon temperature has been neglected only for simplicity of notation. Furthermore, the domain of the response functionals for the stress, heat flux, internal energy and entropy in $\mathcal{B}$ at X is of the type $\mathcal{A} \times \mathcal{U}$ for suitable choices of the open connected sets $\mathcal{A} \subset Lin \times \mathcal{V}$ and $\mathcal{U} \subset Lin_\infty \times \mathcal{V}_\infty$.

In [3] the notion of *global physical equivalence* for simple bodies is introduced. Roughly, let k be a bijection between the material points of the bodies $\mathcal{B}$ and $\mathcal{B}'$. We say that in the time interval $I = [t_0\,, t_1]$ the bodies $\mathcal{B}$ and $\mathcal{B}'$ are subjected to the same external actions if in I they are subjected to k-corresponding fields for the body force and the heat supply and to k-corresponding boundary conditions. Assume that in the time interval I the bodies $\mathcal{B}$ and $\mathcal{B}'$ undergo k-corresponding (thermokinetic) processes if and only if they are subjected to the same external actions and to k-corresponding initial conditons (at time t_0 for position, velocity and temperature). In this case we say that $\mathcal{B}$ and $\mathcal{B}'$ are globally k-equivalent (from the physical point of view, see [3, Definition 2.2]). Of course, the aforementioned processes constitute the solutions of the typical initial-boundary-value problem for the involved bodies.

Now, let $\mathcal{B}$ and $\mathcal{B}'$ be two simple bodies with fading memory that, with respect to certain configurations, are globally k-equivalent. If we interpret the functional $\widehat{\mathcal{F}}$ in (4) [$\widehat{\mathcal{Q}}$ in (7)] as the difference $\widehat{\mathcal{P}} - \widehat{\mathcal{P}}'$ [$\widehat{q} - \widehat{q}'$] between the response functionals for the first Piola stress tensors [heat flux vectors] in $\mathcal{B}$ and $\mathcal{B}'$, then it is easy to show [3] that $\widehat{\mathcal{F}}$ [$\widehat{\mathcal{Q}}$] must solve equations (5) [(6)] along any couple of k-corresponding processes of $\mathcal{B}$ and $\mathcal{B}'$. Hence, the frame-indifferent solutions to (5) and (6) are useful in order to find relations between the corresponding response functionals $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{P}}'$ [$\widehat{q}$ and $\widehat{q}'$] of the two globally equivalent bodies $\mathcal{B}$ and $\mathcal{B}'$.

We say that the functional (4) is a *physically privileged solution* to (the system of) equations (5) if it satisfies the condition

$$\mathcal{F}^{aA} x^b{}_A = \mathcal{F}^{bA} x^a{}_A\,, \tag{9}$$

which is related to the symmetry of the stress and furthermore satisfies the (material) frame-indifference property with respect to the groups of Galileian or Euclidean coordinate transformations of space-time (see Section 8).

Similarly, we say that the functional (7) is a *physically privileged solution* to equations (6) if it satisfies the property of frame-indifference with respect to the group of Galileian or Euclidean co-ordinate transformations of space-time (see Section 8).

In the present paper we characterize the classes of physically privileged solutions to (5) and (6) in each one of the two aforementioned senses.

The results of this paper extend to the infinite-dimensional case certain theorems of the paper [4] for tensor functions defined in finite-dimensional domains, which are related with the response functions of a thermoelastic body. The results of [4] are used in [3] to characterize the class of the thermoelastic bodies that are globally equivalent to a given thermoelastic body referred to a given configuration; parallel to the results of [4], the results of the present paper constitute the essential tool in order to study the anologous class of global equivalence in the case of simple bodies with fading memory.

Any result of Sections 1 to 8 refers to a functional (4) or (7), which is defined on an open subset of a suitable Hilbert space. But the true physical domain $\mathcal{D}$ of the constitutive functionals of a continuous simple body with fading memory is a nowhere dense subset of a suitable infinite-dimensional Hilbert space (cf. [5]). As a consequence the derivatives of the functionals (4) and (7) can be considered only if such functionals are extended to an open set containing $\mathcal{D}$.

Note that $\mathcal{D}$ is nowhere dense and nonconvex; however the restrictions of the constitutive functionals to pure-stretch histories are defined on a convex subset of $\mathcal{D}$.

By invoking the theory of differential calculus [1] on convex sets, in Section 9 we show that a rigorous meaning can be given to the derivatives of the constitutive functionals which are defined on the nowhere dense and nonconvex set $\mathcal{D}$.

The property of frame-indifference, in the stronger form of Euclidean invariance, is essentially used to reach the objective. As a consequence all the results of the present paper remain valid without extending the constitutive functionals (4) and (7) to an open set.

2. General solution to equations $(5)_4$ and $(6)_4$

Now we consider the functional

$$\widehat{\mathcal{Q}} : \mathcal{U}_1 \to \mathcal{V}\,, \qquad \zeta \mapsto \mathcal{Q} = \widehat{\mathcal{Q}}(\zeta)\,, \tag{10}$$

with $\mathcal{U}_1$ open connected subset of $\mathcal{V}_\infty$, and the symmetric system of equations

$$\frac{\partial \mathcal{Q}^A}{\partial \zeta_B} + \frac{\partial \mathcal{Q}^B}{\partial \zeta_A} = 0 \qquad (A\,, B = 1\,, 2\,, 3) \tag{11}$$

in the unknown functional (10); note that $\langle \partial \mathcal{Q}^A / \partial \zeta_B \mid \gamma_B \rangle$, with $\gamma_B \in \mathbb{R}_\infty$, rewritten in Coleman's notations [2] is $\delta \mathcal{Q}^A(\zeta_B | \gamma_B)$.

Equations (11) extend to the infinite-dimensional case equations

$$\frac{\partial Q^A}{\partial G_B} + \frac{\partial Q^B}{\partial G_A} = 0\,, \qquad (A\,, B = 1\,, 2\,, 3) \tag{12}$$

in the unknown function

$$\widehat{Q} : \mathcal{U}_0 \to \mathcal{V}, \qquad G \mapsto Q = \widehat{Q}(G), \tag{13}$$

with $\mathcal{U}_0$ open connected subset of $\mathcal{V}$. The general solution to equations (12), due to Euler, is

$$Q = V + MG, \qquad i.e. \qquad Q^A = V^A + M^{AB} G_B, \tag{14}$$

where V is any vector and M is any second-order skew tensor. Therefore, we have $M^{AB} = \varepsilon^{ABC} W_C$ with $W_C = \varepsilon_{CAB} M^{BA}/2$. Hence $(14)_2$ becomes

$$Q^A = V^A + \varepsilon^{ABC} W_C G_B. \tag{15}$$

A proof of (14) which uses the assumption $Q \in C^1$ is due to Gurtin and Williams (see [6, pp. 98, 258]).

We point out that the proof of Gurtin and Williams remains valid by replacing $\mathcal{V}$ with $\mathcal{V}_\infty$ and thus (10) and (11) with (13) and (12), respectively. To allow this generalization we preliminarly need some definitions.

Definition 2.1. Let $n \geq 1$, $D := (0, +\infty)^n \subset \mathbb{R}^n$ and let $\mathcal{S}$ be a normed space. Then $L^2_h(D, \mathcal{S})$ denotes the (weighted) L^2 space of all functions from D to $\mathcal{S}$, with (weighted) element of measure

$$h^2(s_1) \dots h^2(s_n) ds_1 \dots ds_n,$$

whose inner product and norm are given by

$$\langle a | b \rangle = \iint \dots \int_D a(s_1, \dots, s_n) \cdot b(s_1, \dots, s_n) h^2(s_1) \dots h^2(s_n) ds_1 \dots ds_n \tag{16}$$

and $\|a\| = \langle a, a \rangle^{\frac{1}{2}}$, respectively; in (16) the dot « · » denotes the inner product on $\mathcal{S}$.

Remark 2.1. Let each function $g_i(s_i)$, $i = 1, \dots, n$, assume tensor values of order $o(g_i) = 1$ or 2 and let the function $f(s_1, \dots, s_n)$ assume tensor values whose order is greater than $\sum_{i=1}^n o(g_i)$. By the Riesz representation theorem any multilinear function from Lin_∞ to $\mathbb{R}$ has the form

$$\begin{aligned}\langle f | g_1, g_2, \dots, g_n \rangle &= \int_0^\infty \left\{ \dots \left[\int_0^\infty f(s_1, \dots, s_n) \cdot g_1(s_1) h^2(s_1) ds_1 \right] \dots \right\} \cdot g_n(s_n) h^2(s_n) ds_n = \\ &= \iint \dots \int_D f(s_1, \dots, s_n) \cdot g_1(s_1) \dots \cdot g_n(s_n) h^2(s_1) \dots h^2(s_n) ds_1 \dots ds_n.\end{aligned} \tag{17}$$

As pointed out above Definition 2.1, by (14) the general solution (10) to equations (11) is given by

$$\mathcal{Q}^A = V^A + \langle M^{AB}(s) \mid \zeta_B(s) \rangle, \qquad (A, B = 1, 2, 3) \tag{18}$$

or, equivalently, by

$$\mathcal{Q}^A = V^A + \varepsilon^{ABC} \langle W_C(s) \mid \zeta_B(s) \rangle, \qquad W_C(s) = \frac{1}{2} \varepsilon_{CAB} M^{BA}(s), \tag{19}$$

for any choice of the constants $V^A \in \mathbb{R}$ and of the tensor functions $M^{AB}(\cdot) \in L^2_h[\mathbb{R}^+ , \mathbb{R}]$ such that $M^{AB} = -M^{BA}$.

This result holds also for any functional $\widehat{\mathcal{F}} = [\widehat{\mathcal{F}}^{a_1 \ldots a_n A}]$, which solves equations $(5)_4$. To simplify notations we put

$$a := a_1 a_2 \ldots a_n \quad \text{for} \quad n > 0 \quad \text{and} \quad a := \emptyset \quad \text{for} \quad n = 0 .$$

For instance, when $n = 0$ we have $\mathcal{S} = \mathcal{V}$, $\mathcal{F} = [\widehat{\mathcal{F}}^A]$ and when $n = 1$ we have $\mathcal{S} = Lin$, $\mathcal{F} = [\widehat{\mathcal{F}}^{aA}]$.

The above considerations in the case $n = 1$, where

$$\widehat{\mathcal{F}} = [\widehat{\mathcal{F}}^{aA}] : \mathcal{U}_1 \to Lin , \qquad \zeta \mapsto \mathcal{F} = \widehat{\mathcal{F}}(\zeta) \ , \tag{20}$$

with $\mathcal{U}_1$ open connected subset of $\mathcal{V}_\infty$, yield the following

LEMMA 2.1. *The functional* (20) *is a solution on* $\mathcal{U}_1$ *to equations* $(5)_4$ *if and only if*

$$\mathcal{F}^{aA} = \widehat{\mathcal{F}}^{aA}(\zeta) = V^{aA} + \langle M^{aAB}(s) \mid \zeta_B(s) \rangle \qquad (a , A = 1 , 2 , 3) \ , \tag{21}$$

for any choice of the constants $V^{aA} \in \mathbb{R}$ *and of the functions* $M^{aAB}(\cdot) \in L^2_h[\mathbb{R}^+ , \mathbb{R}]$ *such that the tensor* M^{aAB} *is skew in the indices* A , B.

Note that by putting $W^a_C(s) = \varepsilon^{ABC} M^{aBA}/2$ the equalities (21) become

$$\mathcal{F}^{aA} = V^{aA} + \varepsilon^{ABC} \langle W^a_C(s) \mid \zeta_B(s) \rangle \ , \tag{22}$$

with $W^a_C(\cdot) \in L^2_h[\mathbb{R}^+ , \mathbb{R}]$.

3. General solution to equations $(5)_3$

Now we apply the results of the previous section in order to characterize the solutions to equations $(5)_3$ in the unknown functional

$$\widehat{\mathcal{F}} = [\widehat{\mathcal{F}}^{aA}] : \mathcal{U}_2 \to \mathcal{S} , \qquad \xi \mapsto \mathcal{F} = \widehat{\mathcal{F}}(\xi) , \tag{23}$$

where $\mathcal{U}_2$ is an open connected subset of Lin_∞.

LEMMA 3.1. *The functional* (23) *is a solution on* $\mathcal{U}_2$ *to equations* $(5)_3$ *if and only if* [2]

$$\begin{aligned} \mathcal{F}^{aA} = \widehat{\mathcal{F}}^{aA}(\xi) = \langle L^{aABCD}(s_1 , s_2 , s_3) \mid \xi^1_{\ B}(s_1) , \xi^2_{\ C}(s_2) , \xi^3_{\ D}(s_3) \rangle + \\ + \sum_{b=1}^{3} \langle M^{abABC}(s_{b+1} , s_{b+2}) \mid \xi^{b+1}_{\ \ B}(s_{b+1}) , \xi^{b+2}_{\ \ C}(s_{b+2}) \rangle + \\ + \langle N^{a\ AB}_{\ b}(s) \mid \xi^b_{\ B}(s) \rangle + U^{aA} \qquad (a , A = 1 , 2 , 3) \end{aligned} \tag{24}$$

[2] In the equality below, $b + i$ is replaced by its remainder when divided by 3, for $b + i > 3$.

for any choice of the constants $U^{aA} \in \mathbb{R}$ and of the functions

$$
\begin{aligned}
L^{aABCD} &= L^{aABCD}(s_1, s_2, s_3), & L^{aABCD}(\cdot) &\in L_b^2[\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}],\\
M^{abABC} &= M^{abABC}(s_1, s_2), & M^{abABC}(\cdot) &\in L_b^2[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}],\\
N^{a\ AB}_{\ b} &= N^{a\ AB}_{\ b}(s), & N^{a\ AB}_{\ b}(\cdot) &\in L_b^2[\mathbb{R}^+, \mathbb{R}],
\end{aligned}
$$

such that the tensors L^{aABCD}, M^{abABC} and $N^{a\ AB}_{\ b}$ are skew in their capital indices.

PROOF. Let (23) be solution of $(5)_3$. Fix $B \in \{1, 2, 3\}$; by Lemma 2.1 the function $\mathcal{F}^{aA}$ is linear in ξ^b_B for $b = 1, 2, 3$; thus $\mathcal{F}^{aA}$ is multilinear in ξ^b_B for $b, B = 1, 2, 3$. Now by Remark 2.1 $\widehat{\mathcal{F}}^{aA}$ is a sum of monomials of the types

$$
\langle L^{aABCD}(s_1, s_2, s_3) \mid \xi^1_{\ B}(s_1), \xi^2_{\ C}(s_2), \xi^3_{\ D}(s_3)\rangle,
$$

$$
\langle M^{abABC}(s_{b+1}, s_{b+2}) \mid \xi^{b+1}_{\ B}(s_{b+1}), \xi^{b+2}_{\ C}(s_{b+2})\rangle, \qquad \langle N^{a\ AB}_{\ b}(s_1) \mid \xi^b_{\ B}(s_1)\rangle.
$$

By Lemma 2.1 the tensors L^{aABCD}, M^{abABC} and $N^{a\ AB}_{\ b}$ are skew in the pairs of indices (A, B), (A, C) and (A, D), respectively. Hence these tensors are (totally) skew (-symmetric) in their capital indices (3) and (24) holds. Conversely, by substitution one checks that the functional (24) solves $(5)_3$. □

REMARK 3.1. Note that (*a*) a totally skew tensor of order greater than 3 vanishes; (*b*) up to the multiplication by a real number, the Ricci tensor is the unique third-order skew tensor; (*c*) any second-order skew tensor W has components $W^{AB} = \varepsilon^{ABC} w_C$ for some vector w.

By this Remark the functions $L^{aABC}(\cdot)$ vanish whereas the functions $M^{abABC}(\cdot)$ and $N^{a\ AB}_{\ b}(\cdot)$ have the form

$$
M^{abABC}(\cdot) = \varepsilon^{ABC} V^{ab}(\cdot), \qquad N^{a\ AB}_{\ b}(\cdot) = \varepsilon^{ABC} W^a_{\ \ bC}(\cdot).
$$

Consequently, the general solution (24) to equations $(5)_3$ becomes

$$
\mathcal{F}^{aA} = \varepsilon^{ABC} \sum_{b=1}^{3} \langle V^{ab}(s_{b+1}, s_{b+2}) \mid \xi^{b+1}_{\ B}(s_{b+1}), \xi^{b+2}_{\ C}(s_{b+2})\rangle + \varepsilon^{ABC}\langle W^a_{\ \ bC}(s) \mid \xi^b_{\ B}(s)\rangle + U^{aA}, \tag{25}
$$

that is,

$$
\mathcal{F}^{aA} = \frac{1}{2}\varepsilon_{bcd}\, \varepsilon^{ACD}\langle V^{ab}(s_{b+1}, s_{b+2}) \mid \xi^c_{\ C}(s_c), \xi^d_{\ D}(s_d)\rangle + \varepsilon^{ABC}\langle W^a_{\ \ bC}(s) \mid \xi^b_{\ B}(s)\rangle + U^{aA}. \tag{26}
$$

We have proved the following

LEMMA 3.2. *The functional* (23) *is a solution on* $\mathcal{U}_1$ *to equations* $(5)_3$ *if and only if the components of* (23) *are given by* (25) *or* (26) *for any choice of the functions* $V^{ab}(\cdot) \in L_b^2[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}]$ *and* $W^a_{\ \ bC}(\cdot) \in L_b^2[\mathbb{R}^+, \mathbb{R}]$.

(3) Let $\mathcal{A}$ and $\mathcal{B}$ be non-disjoint sets of indices of a given tensor L; if L is skew in both $\mathcal{A}$ and $\mathcal{B}$, then L is skew in $\mathcal{A} \cup \mathcal{B}$.

Incidentally, note that in the proof of Theorem 4.1 below, equation (26) is used in the equivalent form

$$(27)\qquad \mathcal{F}^{aA} = \frac{1}{2}\varepsilon_{bcd}\langle M^{abACD}(s_{b+1}\,,\,s_{b+2}) \mid \xi^c{}_C(s_c)\,,\,\xi^d{}_D(s_d)\rangle + \langle N^a{}_b{}^{AB} \mid \xi^b{}_B\rangle + U^{aA}\,,$$

where M^{abACD} and $N^a{}_b{}^{AB}$ are skew in the capital indices.

4. General solution to equations $(5)_{3\text{-}4}$

Now we consider the functional

$$(28)\qquad \widehat{\mathcal{F}} : \mathcal{U}_2 \times \mathcal{U}_1 \to Lin\,, \qquad (\xi\,,\zeta) \mapsto \mathcal{F} = \widehat{\mathcal{F}}(\xi\,,\zeta)\,,$$

with $\mathcal{U}_1 \times \mathcal{U}_2$ open connected subset of $Lin_\infty \times \mathcal{V}_\infty$, and the coupled equations

$$(29)\qquad \frac{\partial \mathcal{F}^{aA}}{\partial \xi^b{}_B} + \frac{\partial \mathcal{F}^{aB}}{\partial \xi^b{}_A} = 0\,, \qquad \frac{\partial \mathcal{F}^{aA}}{\partial \zeta_B} + \frac{\partial \mathcal{F}^{aB}}{\partial \zeta_A} = 0$$

in the unknown functional (28). The next theorem characterizes the general solution to equations (29).

Theorem 4.1. *The functional* (28) *is a solution on* $\mathcal{U}_2 \times \mathcal{U}_1$ *to equations* (29) *if and only if*

$$(30)\qquad \begin{aligned} \mathcal{F}^{aA} = {}& \overset{[00]}{\tau}{}^{aA} + \varepsilon^{ABC}\langle \overset{[10]}{\tau}{}^a{}_{bC} \mid \xi^b{}_B\rangle + \varepsilon^{ACE}\langle \overset{[01]}{\tau}{}^a{}_C \mid \zeta_E\rangle + \\ & + \varepsilon_{bcd}\varepsilon^{ACD}\langle \overset{[20]}{\tau}{}^{ab}(s_{b+1}\,,\,s_{b+2}) \mid \xi^c{}_C(s_c)\,,\,\xi^d{}_D(s_d)\rangle + \varepsilon^{ABE}\langle \overset{[11]}{\tau}{}^a{}_b \mid \xi^b{}_B\,,\,\zeta_E\rangle \end{aligned}$$

for any choice of the constants $\overset{[00]}{\tau}{}^{aA} \in \mathbb{R}$ *and of the functions*

$$\overset{[01]}{\tau}{}^a{}_C(\cdot)\,,\ \overset{[10]}{\tau}{}^a{}_{bC}(\cdot) \in L^2_h[\mathbb{R}^+\,,\mathbb{R}]\ , \qquad \overset{[11]}{\tau}{}^a{}_b(\cdot)\,,\ \overset{[20]}{\tau}{}^{ab}(\cdot) \in L^2_h[\mathbb{R}^+ \times \mathbb{R}^+\,,\mathbb{R}].$$

Proof. By Lemmas 3.2 and 2.1 each solution to equations (29) can be written in both the forms (27) and (21); hence we obtain

$$\begin{aligned} \mathcal{F}^{aA} = {}& \overset{[00]}{\psi}{}^{aA} + \langle \overset{[10]}{\psi}{}^a{}_b{}^{AB}(s) \mid \xi^b{}_B(s)\rangle + \langle \overset{[01]}{\psi}{}^{aAE}(s) \mid \zeta_E(s)\rangle + \\ & + \varepsilon_{bcd}\langle \overset{[20]}{\psi}{}^{abACD}(s_{b+1}, s_{b+2}) \mid \xi^c{}_C(s_c)\,,\,\xi^d{}_D(s_d)\rangle + \langle \overset{[11]}{\psi}{}^a{}_b{}^{ABE}(s_1, s_2) \mid \xi^b{}_B(s_1)\,,\,\zeta_E(s_2)\rangle + \\ & + \varepsilon_{bcd}\langle \overset{[21]}{\psi}{}^{abACDE}(s_{b+1}\,,\,s_{b+2}\,,\,s_b) \mid \xi^c{}_C(s_c)\,,\,\xi^d{}_D(s_d)\,,\,\zeta_E(s_b)\rangle\,, \end{aligned}$$

with $\overset{[00]}{\psi}{}^{aA} \in \mathbb{R}$, $\overset{[01]}{\psi}{}^{aAE}(\cdot)\,,\ \overset{[10]}{\psi}{}^a{}_b{}^{AB}(\cdot) \in L^2_h[\mathbb{R}^+\,,\mathbb{R}]$, $\overset{[11]}{\psi}{}^a{}_b{}^{ABD}(\cdot)\,,\ \overset{[20]}{\psi}{}^{abACD}(\cdot) \in L^2_h[\mathbb{R}^+ \times \mathbb{R}^+\,,\mathbb{R}]$ and $\overset{[21]}{\psi}{}^{abACDE}(\cdot) \in L^2_h[\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+\,,\mathbb{R}]$.

By Lemmas 3.2 and 2.1 again, the tensor functions $\overset{[..]}{\psi}{}^{\cdots}(\cdot)$ are skew in their capital indices. Hence Remark 3.1 yields $\overset{[21]}{\psi}(\cdot) \equiv 0$ and

$$\overset{[10]}{\psi}{}^{a\,AB}_{\ b}(\cdot) \equiv \varepsilon^{ABC}\overset{[10]}{\tau}{}^{a}_{\ bC}(\cdot)\,, \qquad \overset{[01]}{\psi}{}^{aAE}(\cdot) \equiv \varepsilon^{ACE}\overset{[01]}{\tau}{}^{a}_{\ C}(\cdot)\,,$$

$$\overset{[20]}{\psi}{}^{abACD}(\cdot) \equiv \varepsilon^{ACD}\overset{[20]}{\tau}{}^{ab}(\cdot)\,, \qquad \overset{[11]}{\psi}{}^{a\,ABE}_{\ b}(\cdot) \equiv \varepsilon^{ABE}\overset{[11]}{\tau}{}^{a}_{\ b}(\cdot)$$

for some functions $\overset{[01]}{\tau}{}^{a}_{\ C}(\cdot)$, $\overset{[10]}{\tau}{}^{a}_{\ bC}(\cdot)$, $\overset{[20]}{\tau}{}^{ab}(\cdot)$ and $\overset{[11]}{\tau}{}^{a}_{\ b}(\cdot)$; hence (30) holds. □

5. General solution to equations (5)

Now we characterize the general solution to equations (5) in the unknown functional (4).

Theorem 5.1. *The functional* (4) *is a solution on* $\mathcal{A}\times\mathcal{U}$ *to equations* (5) *if and only if*

$$\begin{aligned}
\mathcal{F}^{aA} = {} & \overset{[0000]}{\tau}{}^{aA} + \varepsilon^{ABC}\overset{[1000]}{\tau}{}^{a}_{\ bC}\,x^{b}_{\ B} + \varepsilon^{ALC}\overset{[0100]}{\tau}{}^{a}_{\ C}\,z_L + \\
& + \varepsilon^{ADC}\langle \overset{[0010]}{\tau}{}^{a}_{\ dC}(s) \mid \xi^{d}_{\ D}(s)\rangle + \varepsilon^{AMC}\langle \overset{[0001]}{\tau}{}^{a}_{\ C}(s) \mid \zeta_M(s)\rangle + \\
& + \varepsilon^{ABL}\overset{[1100]}{\tau}{}^{a}_{\ b}\,x^{b}_{\ B}z_L + \varepsilon^{ABD}\langle \overset{[1010]}{\tau}{}^{a}_{\ bd}(s) \mid \xi^{d}_{\ D}(s)\rangle x^{b}_{\ B} + \\
& + \varepsilon^{ABM}\langle \overset{[1001]}{\tau}{}^{a}_{\ b}(s) \mid \zeta_M(s)\rangle\, x^{b}_{\ B} + \varepsilon^{ADL}\langle \overset{[0110]}{\tau}{}^{a}_{\ d}(s) \mid \xi^{d}_{\ D}(s)\rangle z_L + \\
& + \varepsilon^{AML}\langle \overset{[0101]}{\tau}{}^{a}(s) \mid \zeta_M(s)\rangle z_L + \varepsilon^{ADM}\langle \overset{[0011]}{\tau}{}^{a}_{\ d}(s_1\,,s_2) \mid \xi^{d}_{\ D}(s_1)\,,\zeta_M(s_2)\rangle + \\
& + \varepsilon_{hbc}\varepsilon^{ABC}\,\overset{[2000]}{\tau}{}^{ah}\,x^{b}_{\ B}x^{c}_{\ C} + \varepsilon_{hde}\varepsilon^{ADE}\langle \overset{[0020]}{\tau}{}^{ah}(s_1\,,s_2) \mid \xi^{d}_{\ D}(s_1)\,,\,\xi^{e}_{\ E}(s_2)\rangle\,,
\end{aligned} \tag{31}$$

$a\,,A = 1\,,2\,,3\,,$ *for any choice of the constants*

$$\overset{[0000]}{\tau}{}^{aA}\,,\quad \overset{[1000]}{\tau}{}^{a}_{\ bC}\,,\quad \overset{[0100]}{\tau}{}^{a}_{\ C}\,,\quad \overset{[1100]}{\tau}{}^{a}_{\ b}\,,\quad \overset{[2000]}{\tau}{}^{ah} \in \mathbb{R}\,,$$

and of the functions

$$\overset{[0010]}{\tau}{}^{a}_{\ dC}(\cdot)\,,\ \overset{[0001]}{\tau}{}^{a}_{\ C}(\cdot)\,,\ \overset{[1010]}{\tau}{}^{a}_{\ bd}(\cdot)\,,\ \overset{[1001]}{\tau}{}^{a}_{\ b}(\cdot)\,,\ \overset{[0110]}{\tau}{}^{a}_{\ d}(\cdot)\,,\ \overset{[0101]}{\tau}{}^{a}(\cdot) \in L^2_h[\mathbb{R}^+\,,\mathbb{R}]\,,$$

$$\overset{[0101]}{\tau}{}^{a}_{\ d}(\cdot)\,,\ \overset{[0020]}{\tau}{}^{ah}(\cdot) \in L^2_h[\mathbb{R}^+\times\mathbb{R}^+\,,\mathbb{R}].$$

Proof. From [1], firstly, we note that when the functional in (4) does not depend on ξ and ζ, *i.e.* $\mathcal{F} = \widehat{\mathcal{F}}(x, z)$, the general solution to equations $(5)_{1,2}$ is given by

$$\mathcal{F}^{aA} = \overset{[00]}{\varphi}{}^{aA} + \overset{[10]}{\varphi}{}^{a\,AB}_{\ b}x^{b}_{\ B} + \overset{[01]}{\varphi}{}^{aAB}z_B + \varepsilon_{bcd}\overset{[20]}{\varphi}{}^{abACD}x^{c}_{\ C}x^{d}_{\ D} + \overset{[11]}{\varphi}{}^{a\,ABD}_{\ b}x^{b}_{\ B}z_D\,, \tag{32}$$

where the tensors $\overset{[..]}{\varphi}$ are skew in their capital indices.

Now let (4) solve (5). By Remark 2.1 and Lemmas 2.1, 3.2 it follows that $\mathcal{F}^{aA}$ can be simultaneously written in each of the forms (32), (21) and (27):

$$
\begin{aligned}
\mathcal{F}^{aA} = {} & \overset{[0000]}{\psi}{}^{aA} + \overset{[1000]}{\psi}{}^{a}{}_{b}{}^{AB} x^{b}{}_{B} + \overset{[0100]}{\psi}{}^{aAL} z_{L} + \langle \overset{[0010]}{\psi}{}^{a}{}_{d}{}^{AD}(s) \mid \xi^{d}{}_{D}(s) \rangle + \\
& + \langle \overset{[0001]}{\psi}{}^{aAM}(\cdot) \mid \zeta_{M}(s) \rangle + \overset{[1100]}{\psi}{}^{a}{}_{b}{}^{ABL} x^{b}{}_{B} z_{L} + \langle \overset{[1010]}{\psi}{}^{a}{}_{bd}{}^{ABD}(s) \mid \xi^{d}{}_{D}(s) \rangle x^{b}{}_{B} + \\
& + \langle \overset{[1001]}{\psi}{}^{a}{}_{b}{}^{ABM}(s) \mid \zeta_{M}(s) \rangle x^{b}_{B} + \langle \overset{[0110]}{\psi}{}^{a}{}_{d}{}^{ADL}(s) \mid \xi^{d}{}_{D}(s) \rangle z_{L} + \\
& + \langle \overset{[0101]}{\psi}{}^{aAML}(s) \mid \zeta_{M}(s) \rangle z_{L} + \langle \overset{[0011]}{\psi}{}^{a}{}_{d}{}^{ADM}(s_{1}, s_{2}) \mid \xi^{d}{}_{D}(s_{1}), \zeta_{M}(s_{2}) \rangle + \\
& + \overset{[2000]}{\psi}{}^{a}{}_{bc} x^{b}{}_{B} x^{c}{}_{C} + \langle \overset{[0020]}{\psi}{}^{a}{}_{de}{}^{ADE}(s_{1}, s_{2}) \mid \xi^{d}{}_{D}(s_{1}), \xi^{e}{}_{E}(s_{2}) \rangle + \\
& + \langle \overset{[1110]}{\psi}{}^{a}{}_{bd}{}^{ABDL}(s) \mid \xi^{d}{}_{D}(s) \rangle x^{b}{}_{B} z_{L} + (\ldots),
\end{aligned} \tag{33}
$$

where $(\ldots)$ denotes the sum of all monomials whose components $\overset{[\ldots]}{\psi}{}^{\cdots}$ have more than 3 capital indices and the tensors $\overset{[\ldots]}{\psi}{}^{\cdots}$ are totally skew in their capital indices.

Thus, *e.g.*, $\overset{[1110]}{\psi}{}^{a}{}_{bd}{}^{ABDL}$ is skew in both (A, B) and (A, L) because (33) solves $(5)_{1,2}$; furthermore it is skew in (A, D) because (33) solves $(5)_3$; consequently $\overset{[1110]}{\psi}{}^{a}{}_{bd}{}^{ABDL}$ is totally skew in (A, B, D, L). By Remark 3.1 equations (33) and (31) are equivalent. Conversely, by direct substitution one cheks that the functional (4) solves equations (5). □

6. Frame-indifferent solutions to equations (5)

Next we characterize the solutions (4) to equations (5) which are frame-indifferent in each one of the two senses (*GI*) and (*EI*) below.

Remind that the functional $\widehat{\mathcal{F}}$ in (4) may be interpreted as the difference between the constitutive functionals $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{P}}'$ for the Piola stress-tensor in certain two globally equivalent bodies. The principle of material frame-indifference requires that the response of a material do not depend on the motion of the observer frame (*e.g.*, see [6]).

A Galilean frame of reference represents an observer that moves with a constant translatory rigid motion w.r.t. an inertial frame, whereas an Euclidean frame of reference represents an observer that moves with an arbitrarily given rigid motion w.r.t. an inertial frame. Here we characterize the solutions of equations (5) that are frame-indifferent in the sense of Galilean invariance and then in the stronger sense of Euclidean invariance. These solutions can be used to investigate how the class of the bodies (with fading memory), that are globally equivalent to a given body, depends on the observer motion.

In the two conditions of frame-indifference below, R and ρ respectively represent the constant value of the rotation tensor and the total history in the observer motion.

As is customary, let

$$I = \text{identity tensor}\,, \qquad Lin^{+} := \{L \in Lin \mid \det L > 0\}\,, \tag{34}$$

$$Orth := \{Q \in Lin \mid QQ^{T} = I\}\,, \qquad Orth^{+} := Orth \cap Lin^{+}\,. \tag{35}$$

(GI) - (Galilean invariance condition)

$$\widehat{\mathcal{F}}(Rx\,,\, z\,,\, R\xi\,,\, \zeta) = R\widehat{\mathcal{F}}(x\,,\, z\,,\, \xi\,,\, \zeta) \tag{36}$$

for each $R \in 0rth^{+}$ *and* $(x\,,\, z\,,\, \xi\,,\, \zeta) \in \mathcal{A} \times \mathcal{U}$.

(EI) - (Euclidean invariance condition)

$$\widehat{\mathcal{F}}(Rx\,,\, z\,,\, \rho\,\xi\,,\, \zeta) = R\widehat{\mathcal{F}}(x\,,\, z\,,\, \xi\,,\, \zeta)\ , \quad R := \rho(0)\ , \tag{37}$$

for each history $\rho : [0\,,\, \infty) \to 0rth^{+}$ *and* $(x\,,\, z\,,\, \xi\,,\, \zeta) \in \mathcal{A} \times \mathcal{U}$.

REMARK 6.1. A tensor T is said to be *weakly isotropic* if it is $Orth^{+}$-invariant, *i.e.*

$$T^{h_1 h_2 \dots h_n} R_{h_1 k_1} R_{h_2 k_2} \dots R_{h_n k_n} = T^{k_1 k_2 \dots k_n} \qquad \forall\, R \in Orth^{+}.$$

As a consequence, any weakly isotropic tensor of order 1 vanishes and any weakly isotropic tensor of order 2 or 3 has the respective form

$$T^{ij} = d\, \delta^{ij} \qquad \text{or} \qquad T^{ijk} = d\, \varepsilon^{ijk}\,, \qquad d \in \mathbb{R}. \tag{38}$$

The next two theorems characterize the solutions to equations (5) which satisfy (GI) or (EI). To prove them we use the general solution (31) to equations (5) given by Theorem 5.1. In the proofs below we shall use the following

REMARK 6.2. In view of (17), if

$$\langle\, T^{\cdots}(s_1\,,\, \dots\,,\, s_n) \mid \eta_1(s_1)\,,\, ..\,,\, \eta_n(s_n)\,\rangle = 0 \quad \forall\, \eta_1(s_1)\,,\, ..\,,\, \eta_n(s_n) \in \mathbb{R}_{\infty}\ ,$$

then $T^{\cdots}(s_1\,,\, \dots\,,\, s_n)$ identically vanishes almost everywhere on its domain. For functions belonging to a L^2 space the symbol « $\equiv$ » will be used to mean equality almost everywhere. For instance $T^{\cdots}(\cdot) \equiv U^{\cdots}(\cdot)$ means that $T^{\cdots}(\cdot)$ and $U^{\cdots}(\cdot)$ differ in values at most on a set of points of measure zero.

THEOREM 6.1. *The solution* (31) *to equations* (5) *satisfies the invariance condition* (36) *if and only if*

$$\begin{gathered}
\overset{[0000]}{\tau}{}^{aA} = 0\,, \quad \overset{[0100]}{\tau}{}^{a}{}_{C} = 0\,, \quad \overset{[0001]}{\tau}{}^{a}{}_{C} \equiv 0\,, \quad \overset{[0101]}{\tau}{}^{a} \equiv 0\,, \\
\overset{[1000]}{\tau}{}^{a}{}_{bC} = \overset{[1000]}{d}{}_{C}\delta^{a}{}_{b}\,, \quad \overset{[0010]}{\tau}{}^{a}{}_{dC} \equiv \overset{[0010]}{d}{}_{C}\delta^{a}{}_{d}\,, \quad \overset{[1100]}{\tau}{}^{a}{}_{b} = \overset{[1100]}{d}\,\delta^{a}{}_{b}\,, \\
\overset{[1010]}{\tau}{}^{a}{}_{bd} \equiv \overset{[1010]}{d}\,\varepsilon^{a}{}_{bd}\,, \quad \overset{[1001]}{\tau}{}^{a}{}_{b} \equiv \overset{[1001]}{d}\,\delta^{a}{}_{b}\,, \quad \overset{[0110]}{\tau}{}^{a}{}_{d} \equiv \overset{[0110]}{d}\,\delta^{a}{}_{d}\,, \\
\overset{[0011]}{\tau}{}^{a}{}_{d} \equiv \overset{[0011]}{d}\,\delta^{a}{}_{d}\,, \quad \overset{[2000]}{\tau}{}^{ah} = \overset{[2000]}{d}\,\delta^{ah}\,, \quad \overset{[0020]}{\tau}{}^{ah} \equiv \overset{[0020]}{d}\,\delta^{ah}\,;
\end{gathered} \tag{39}$$

hence, if and only if

$$
\begin{aligned}
\mathcal{F}^{aA} = {} & \varepsilon^{ABC} \overset{[1000]}{d}{}_C\, x^a{}_B + \varepsilon^{ADC} \langle \overset{[0010]}{d}{}_C(s) \mid \xi^a{}_D(s) \rangle + \varepsilon^{ABL} \overset{[1100]}{d}\, x^a{}_B z_L + \\
& + \varepsilon^{ABD} \varepsilon^a{}_{bd} \langle \overset{[1010]}{d}(s) \mid \xi^d{}_D(s) \rangle x^b{}_B + \varepsilon^{ABM} \langle \overset{[1001]}{d}(s) \mid \zeta_M(s) \rangle x^a{}_B + \\
& + \varepsilon^{ADL} \langle \overset{[0110]}{d}(s) \mid \xi^a{}_D(s) \rangle z_L + \varepsilon^{ADM} \langle \overset{[0011]}{d}(s_1 , s_2) \mid \xi^a{}_D(s_1) , \zeta_M(s_2) \rangle + \\
& + \varepsilon^{ABC}\, \varepsilon^a{}_{bc} \overset{[2000]}{d}\, x^b{}_B x^c{}_C + \varepsilon^{ADE}\, \varepsilon^a{}_{de} \langle \overset{[0020]}{d}(s_1 , s_2) \mid \xi^d{}_D(s_1) , \xi^e{}_E(s_2) \rangle
\end{aligned}
\tag{40}
$$

for any choice of the constants

$$\overset{[1000]}{d}{}_C , \quad \overset{[1100]}{d} , \quad \overset{[2000]}{d} \in \mathbb{R}$$

and of the functions

$$\overset{[0010]}{d}{}_C , \quad \overset{[1010]}{d} , \quad \overset{[1001]}{d} , \quad \overset{[0110]}{d} \in L^2_h(\mathbb{R}^+ , \mathbb{R}) , \qquad \overset{[0011]}{d} , \quad \overset{[0020]}{d} \in L^2_h[\mathbb{R}^+ \times \mathbb{R}^+ , \mathbb{R}].$$

THEOREM 6.2. *The solution* (31) *to equations* (5) *satisfies the invariance condition* (37) *if and only if equations* (39) *hold and furthermore*

$$\overset{[0010]}{d}{}_C \equiv 0 , \quad \overset{[1010]}{d} \equiv 0 , \quad \overset{[0110]}{d} \equiv 0 , \quad \overset{[0011]}{d} \equiv 0 , \quad \overset{[0020]}{d} \equiv 0 ; \tag{41}$$

hence, if and only if

$$
\begin{aligned}
\mathcal{F}^{aA} = {} & \varepsilon^{ABC} \overset{[1000]}{d}{}_C\, x^a{}_B + \varepsilon^{ABL} \overset{[1100]}{d}\, x^a{}_B z_L + \\
& + \varepsilon^{ABM} \langle \overset{[1001]}{d}(s) \mid \zeta_M(s) \rangle x^a{}_B + \varepsilon^{ABC} \varepsilon^a{}_{bc} \overset{[2000]}{d}\, x^b{}_B x^c{}_C
\end{aligned}
\tag{42}
$$

for any choice of the constants

$$\overset{[1000]}{d}{}_C , \quad \overset{[1100]}{d} , \quad \overset{[2000]}{d} \in \mathbb{R}$$

and of the function

$$\overset{[1001]}{d} \in L^2_h(\mathbb{R}^+ , \mathbb{R}).$$

REMARK 6.3. The invariance condition (*EI*) is stronger than (*GI*); consequently, we can prove Theorems 6.1 and 6.2 in a unified manner by applying to each monomial of the general solution (31) to (5) the steps (*i*) to (*iii*) below.

(*i*) To replace the general solution (31) to (5) in (37).

(*ii*) To find the restrictions that the invariance condition (*GI*) implies on $\overset{[\dots]}{\tau}$ by choosing a constant rotation history

$$\rho : [0 , \infty) \to Orth^+ , \qquad \rho(s) \equiv R := \rho(0) \qquad \forall\, s > 0 , \tag{43}$$

where $\rho(0) \in Orth^+$ is pre-fixed *ad arbitrium*.

(*iii*) To find the restrictions that (*EI*) imposes on $\overset{[\dots]}{\tau}$ by choosing *ad arbitrium* some non-constant history $\rho : [0\,,\infty) \to Orth^+$ which satisfies condition (A) or condition (B) below.

(A) For all $\bar{s}_1\,, \bar{s}_2 > 0$ there is ρ such that

$$\bar{\rho} := \rho(\bar{s}_1) = \rho(\bar{s}_2) \neq R := \rho(0). \tag{44}$$

(B) For all $\bar{s} > 0$ there is ρ such that

$$\bar{\rho} := \rho(\bar{s}) \neq R := \rho(0). \tag{45}$$

In the proofs below the well known identities

$$\varepsilon^a{}_{bd} R_a{}^m R^b{}_i = \varepsilon^m{}_{is} R_d{}^s\,, \quad \varepsilon^a{}_{de} R_r{}^d R_s{}^e = \varepsilon^i{}_{rs} R^a{}_i\,, \quad \varepsilon_{hde} R^d{}_r R^e{}_s = \varepsilon_{irs} R_h{}^i\,, \tag{46}$$

where $R \in Orth^+$, will be used.

Proof of both Theorems 6.1, 6.2.

(*Step* (*i*) *in Remark* 6.3). The solution (31) to equations (5) satisfies (37) of (*EI*) if and only if for each $\rho : [0\,,\infty) \to Orth^+$ we have

$$\begin{aligned}
&\overset{[0000]}{\tau}{}^{aA} + \varepsilon^{ABC} \overset{[1000]}{\tau}{}^{a}{}_{bC}\, R^b{}_j x^j{}_B + \varepsilon^{ALC} \overset{[0100]}{\tau}{}^{a}{}_{C} z_L + \\
&+ \varepsilon^{ADC} \langle \overset{[0010]}{\tau}{}^{a}{}_{dC}(s) \mid \rho^d{}_j(s) \xi^j{}_D(s) \rangle + \varepsilon^{AMC} \langle \overset{[0001]}{\tau}{}^{a}{}_{C}(s) \mid \zeta_M(s) \rangle + \\
&+ \varepsilon^{ABL} \overset{[1100]}{\tau}{}^{ab}\, R^b{}_i x^i{}_B z_L + \varepsilon^{ABD} \langle \overset{[1010]}{\tau}{}^{a}{}_{bd}(s) \mid \rho^d{}_j(s) \xi^j{}_D(s) \rangle\, R^b{}_i x^i{}_B + \\
&+ \varepsilon^{ABM} \langle \overset{[1001]}{\tau}{}^{a}{}_{b}(s) \mid \zeta_M(s) \rangle\, R^b{}_i x^i{}_B + \varepsilon^{ADL} \langle \overset{[0110]}{\tau}{}^{a}{}_{d}(s) \mid \rho^d{}_j(s) \xi^j{}_D(s) \rangle z_L + \\
&+ \varepsilon^{AML} \langle \overset{[0101]}{\tau}{}^{a}(s) \mid \zeta_M(s) \rangle z_L + \varepsilon^{ADM} \langle \overset{[0011]}{\tau}{}^{a}{}_{d}(s_1\,, s_2) \mid \rho^d{}_j(s_1) \xi^j{}_D(s_1)\,, \zeta_M(s_2) \rangle + \\
&+ \varepsilon^{ABC} \varepsilon_{hbc} \overset{[2000]}{\tau}{}^{ah}\, R^b{}_i x^i{}_B\, R^c{}_j x^j{}_C + \\
&+ \varepsilon^{ADE} \varepsilon_{hde} \langle \overset{[0020]}{\tau}{}^{ah}(s_1\,, s_2) \mid \rho^d{}_i(s_1) \xi^i{}_D(s_1)\,,\ \rho^e{}_j(s_2) \xi^j{}_E(s_2) \rangle = \\
&= R^a{}_\ell \overset{[0000]}{\tau}{}^{\ell A} + R^a{}_\ell\, \varepsilon^{ABC} \overset{[1000]}{\tau}{}^{\ell}{}_{bC} x^b{}_B + R^a{}_\ell\, \varepsilon^{ALC} \overset{[0100]}{\tau}{}^{\ell}{}_{C} z_L + \\
&+ R^a{}_\ell\, \varepsilon^{ADC} \langle \overset{[0010]}{\tau}{}^{\ell}{}_{dC}(s) \mid \xi^d{}_D(s) \rangle + \varepsilon^{AMC} R^a{}_\ell \langle \overset{[0001]}{\tau}{}^{\ell}{}_{C}(s) \mid \zeta_M(s) \rangle + \\
&+ R^a{}_\ell\, \varepsilon^{ABL} \overset{[1100]}{\tau}{}^{\ell}{}_{b}\, x^b{}_B z_L + R^a_\ell\, \varepsilon^{ABD} \langle \overset{[1010]}{\tau}{}^{\ell}{}_{bd}(s) \mid \xi^d{}_D(s) \rangle x^b{}_B + \\
&+ \varepsilon^{ABM}\, R^a{}_\ell \langle \overset{[1001]}{\tau}{}^{\ell}{}_{b}(s) \mid \zeta_M(s) \rangle x^b{}_B + \\
&+ R^a{}_\ell\, \varepsilon^{ADL} \langle \overset{[0110]}{\tau}{}^{\ell}{}_{d}(s) \mid \xi^d{}_D(s) \rangle z_L + R^a{}_\ell\, \varepsilon^{AML} \langle \overset{[0101]}{\tau}{}^{\ell}(s) \mid \zeta_M(s) \rangle z_L + \\
&+ R^a{}_\ell\, \varepsilon^{ADM} \langle \overset{[0011]}{\tau}{}^{\ell}{}_{d}(s_1\,, s_2) \mid \xi^d{}_D(s_1)\,, \zeta_M(s_2) \rangle + \\
&+ R^a{}_\ell\, \varepsilon^{ABC} \varepsilon_{hbc} \overset{[2000]}{\tau}{}^{\ell h} x^b{}_B x^c{}_C + \varepsilon^{ADE}\, \varepsilon_{hde}\, R^a{}_\ell \langle \overset{[0020]}{\tau}{}^{\ell h}(s_1\,, s_2) \mid \xi^d{}_D(s_1)\,, \xi^e{}_E(s_2) \rangle.
\end{aligned} \tag{47}$$

Next we isolate the terms involving $\overset{[0020]}{\tau}$ in (47) by taking the derivative $\partial^2/\partial\xi^r{}_R\partial\xi^s{}_S$ of both its sides; using (17) we obtain

$$(48)\qquad \begin{aligned} &\varepsilon^{ARS}\varepsilon_{hde}\int_0^{+\infty}\int_0^{+\infty}\overset{[0020]}{\tau}{}^{ah}(s_1\,,\,s_2)[\,\rho^d{}_r(s_1)\;\rho^e{}_s(s_2)+\\ &\quad+\;\rho^e{}_s(s_1)\;\rho^d{}_r(s_2)]f(s_1)g(s_2)h^2(s_1)ds_1h^2(s_2)ds_2=\\ &\qquad=2\varepsilon^{ARS}\varepsilon_{hrs}\,R^a{}_\ell\int_0^{+\infty}\int_0\overset{[0020]}{\tau}{}^{\ell h}(s_1\,,\,s_2)f(s_1)g(s_2)h^2(s_1)ds_1h^2(s_2)ds_2\end{aligned}$$

for all increments $f, g \in \mathbb{R}_\infty$.

(*Step* (*ii*) *in Remark* 6.3). Choose ρ satisfying (43); then (48) yields

$$(49)\qquad \begin{aligned} &\varepsilon_{hde}R^d{}_rR^e{}_s\int_0^{+\infty}\int_0^{+\infty}\overset{[0020]}{\tau}{}^{ah}(s_1\,,\,s_2)f(s_1)g(s_2)h^2(s_1)ds_1h^2(s_2)ds_2=\\ &\qquad=\varepsilon_{hrs}R^a{}_\ell\int_0^{+\infty}\int_0^{+\infty}\overset{[0020]}{\tau}{}^{\ell h}(s_1\,,\,s_2)f(s_1)g(s_2)h^2(s_1)ds_1h^2(s_2)ds_2\;,\end{aligned}$$

which by the arbitrariness of the increments $f,\ g\in\mathbb{R}_\infty$ yields

$$(50)\qquad \varepsilon_{hde}R^d{}_rR^e{}_s\overset{[0020]}{\tau}{}^{ah}(s_1\,,\,s_2)\equiv\varepsilon_{hrs}R^a{}_\ell\overset{[0020]}{\tau}{}^{\ell h}(s_1\,,\,s_2).$$

By $(46)_3$, equation (50) becomes

$$(51)\qquad \varepsilon_{irs}R_h{}^i\overset{[0020]}{\tau}{}^{ah}(s_1\,,\,s_2)\equiv\varepsilon_{hrs}R^a{}_\ell\overset{[0020]}{\tau}{}^{\ell h}(s_1\,,\,s_2).$$

The multiplication of both the sides of (51) by $\varepsilon^{mrs}R_a{}^k$ yields

$$(52)\qquad R_h{}^mR_a{}^k\overset{[0020]}{\tau}{}^{ah}=\overset{[0020]}{\tau}{}^{km};$$

thus the tensor $\overset{[0020]}{\tau}$ is weakly isotropic (see Remark 6.1) and $(39)_{13}$ holds.

(*Step* (*iii*) *in Remark* 6.3). By replacing $(39)_{13}$ in (48) we obtain

$$(53)\qquad \begin{aligned} &\delta^{ah}\varepsilon_{hde}\int_0^{+\infty}\int_0^{+\infty}\overset{[0020]}{d}(s_1\,,\,s_2)[\rho^d{}_r(s_1)\rho^e{}_s(s_2)+\\ &\quad+\rho^e{}_s(s_1)\rho^d{}_r(s_2)]f(s_1)g(s_2)h^2(s_1)ds_1h^2(s_2)ds_2=\\ &\qquad=2\delta^{\ell h}\varepsilon_{hrs}\int_0^{+\infty}\int_0^{+\infty}\overset{[0020]}{d}(s_1\,,\,s_2)f(s_1)g(s_2)R^a{}_\ell h^2(s_1)ds_1h^2(s_2)ds_2\end{aligned}$$

and thus by the arbitrariness of $f, g\in\mathbb{R}_\infty$ we have

$$(54)\quad \delta^{ah}\varepsilon_{hde}\overset{[0020]}{d}(s_1,s_2)[\rho^d{}_r(s_1)\rho^e{}_s(s_2)+\rho^e{}_s(s_1)\rho^d{}_r(s_2)]\equiv2\delta^{\ell h}\varepsilon_{hrs}\overset{[0020]}{d}(s_1,s_2)R^a_j\quad\forall\,\rho\in Orth^+_\infty.$$

By (A) in Remark 6.3 the last equation yields

$$(55)\qquad \delta^{ah}\varepsilon_{hde}\overset{[0020]}{d}(\bar s_1\,,\,\bar s_2)\bar\rho^d{}_r\bar\rho^e{}_s\equiv\delta^{\ell h}\varepsilon_{hrs}\overset{[0020]}{d}(\bar s_1\,,\,\bar s_2)R^a{}_\ell\;,$$

which by $(46)_3$ becomes

$$\delta^{ah}\varepsilon_{irs}\overset{[0020]}{d}(\bar{s}_1, \bar{s}_2)\,\bar{\rho}_h{}^i \equiv \delta^{\ell h}\varepsilon_{hrs}\overset{[0020]}{d}(\bar{s}_1, \bar{s}_2)R^a{}_\ell\,, \tag{56}$$

i.e.

$$\overset{[0020]}{d}(\bar{s}_1, \bar{s}_2)[\bar{\rho}^{ah} - R^{ah}] \equiv 0. \tag{57}$$

By $(44)_3$ we have $\bar{\rho}^{ah} \neq R^{ah}$ for some $a, h \in \{1, 2, 3\}$; hence (57) yields $(39)_5$.

Next we isolate the terms involving $\overset{[2000]}{\tau}$ in (47) by taking the derivative $\partial^2/\partial x^r{}_R \partial x^s{}_S$ of both its sides; we obtain

$$\varepsilon_{hbc}\overset{[2000]}{\tau}{}^{ah}R^b{}_r R^c{}_s = R^a{}_\ell\,\varepsilon_{hrs}\overset{[2000]}{\tau}{}^{\ell h}\,, \tag{58}$$

which by $(46)_3$ is equivalent to

$$\varepsilon_{irs}R_h{}^i\overset{[2000]}{\tau}{}^{ah} = R^a{}_\ell\,\varepsilon_{hrs}\overset{[2000]}{\tau}{}^{\ell h}. \tag{59}$$

By multiplying both the sides of equation (59) by $\varepsilon^{krs}R_a{}^j$ we obtain

$$R_a{}^j R_h{}^i\overset{[2000]}{\tau}{}^{ah} = \overset{[2000]}{\tau}{}^{ji}\,,$$

so that the tensor $\overset{[2000]}{\tau}$ is weakly isotropic and $(39)_{12}$ holds.

Next we study the terms of (47) involving $\overset{[0011]}{\tau}$. By taking the derivative $\partial^2/\partial\xi^s{}_S\partial\zeta_R$ of both the sides of equation (47) we obtain

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\overset{[0011]}{\tau}{}^a{}_d(s_1, s_2)\rho^d{}_s(s_1)f(s_1)g(s_2)h^2(s_1)ds_1h^2(s_2)ds_2 = \\ &\quad = R^a{}_\ell\int_0^{+\infty}\int_0^{+\infty}\overset{[0011]}{\tau}{}^\ell{}_s(s_1, s_2)f(s_1)g(s_2)h^2(s_1)ds_1h^2(s_2)ds_2\,.\end{aligned} \tag{60}$$

(*Step* (*ii*) *in Remark* 6.3). The arbitrariness of $f, g \in \mathbb{R}_\infty$ in (60) yields (see (43))

$$R^d{}_s\overset{[0011]}{\tau}{}^a{}_d(s_1, s_2) \equiv R^a{}_\ell\overset{[0011]}{\tau}{}^\ell{}_s(s_1, s_2); \tag{61}$$

hence multiplying both the sides of equation (61) by $R_m{}^s$ yields

$$\overset{[0011]}{\tau}{}^a{}_m(s_1, s_2) \equiv R^a{}_\ell R_m{}^s\overset{[0011]}{\tau}{}^\ell{}_s(s_1, s_2); \tag{62}$$

namely, $\overset{[0011]}{\tau}$ is weakly isotropic and $(39)_{11}$ holds.

(*Step* (*iii*) *in Remark* 6.3). By replacing $(39)_{11}$ in (60) we obtain

$$\begin{aligned}&\int_0^{+\infty}\overset{[0011]}{d}(s_1, s_2)\rho^a{}_s(s_1)f(s_1)g(s_2)h^2(s_1)ds_1h^2(s_2)ds_2 = \\ &\quad = R^a{}_s\int_0^{+\infty}\overset{[0011]}{d}(s_1, s_2)f(s_1)g(s_2)h^2(s_1)ds_1h^2(s_2)\,ds_2\,,\end{aligned} \tag{63}$$

which by the arbitrariness of $f, g \in \mathbb{R}_\infty$ yields

$$\overset{[0011]}{d}(s_1, s_2)\rho^a{}_s(s_1) \equiv R^a{}_s \overset{[0011]}{d}(s_1, s_2) . \tag{64}$$

By (B) in Remark 6.3, the last equation yields $\overset{[0011]}{d}(\bar{s}_1, \bar{s}_2)(\bar{\rho}^a{}_s - R^a{}_s) \equiv 0$, which by $(44)_3$ yields $(41)_4$.

Next we study the terms of (47) involving $\overset{[0101]}{\tau}$. By taking the derivative $\partial^2/\partial z_R \partial \zeta_S$ of both the sides of equation (47) and using (17) we obtain

$$\int_0^{+\infty} \overset{[0101]}{\tau}{}^a(s) f(s) h^2(s) ds = R^a{}_\ell \int_0^{+\infty} \overset{[0101]}{\tau}{}^\ell(s) f(s) h^2(s) ds. \tag{65}$$

By the arbitrariness of $f \in \mathbb{R}_\infty$ the last equation yields

$$\overset{[0101]}{\tau}{}^a \equiv R^a{}_\ell \overset{[0101]}{\tau}{}^\ell , \tag{66}$$

so that $\overset{[0101]}{\tau}$ vanishes (see Remark 6.1) and $(39)_4$ holds.

Next we study the terms of (47) involving $\overset{[0110]}{\tau}$. By taking the derivative $\partial^2/\partial \xi^s{}_S \partial z_R$ of both the sides of equation (47) we obtain

$$\int_0^{+\infty} \overset{[0110]}{\tau}{}^a{}_d(s)\rho^d{}_s(s) f(s) ds = R^a{}_\ell \int_0^{+\infty} \overset{[0110]}{\tau}{}^\ell{}_s(s) f(s) ds , \tag{67}$$

which by the arbitrariness of $f \in \mathbb{R}_\infty$ yields

$$\overset{[0110]}{\tau}{}^a{}_d(s)\rho^d{}_s(s) \equiv \overset{[0110]}{\tau}{}^\ell{}_s(s) R^a{}_\ell . \tag{68}$$

(*Step* (*ii*) *in Remark* 6.3). Let ρ satisfy equation (43); multiplying both the sides of equation (68) by $R_m{}^s$ yields

$$\overset{[0110]}{\tau}{}^a{}_m(s) \equiv \overset{[0110]}{\tau}{}^\ell{}_s(s) R^a{}_\ell R_m{}^s , \tag{69}$$

so that $\overset{[0110]}{\tau}$ is weakly isotropic and $(39)_{10}$ holds.

(*Step* (*iii*) *in Remark* 6.3). By replacing $(39)_{10}$ in (67) we obtain

$$\overset{[0110]}{d}(s)\rho^a{}_s(s) \equiv \overset{[0110]}{d}(s)\, R^a{}_s , \tag{70}$$

which by (B) in Remark 6.3 implies $(41)_3$.

Next we study the terms of (47) involving $\overset{[1001]}{\tau}$. By taking the derivative $\partial^2/\partial x^i{}_B \partial \zeta_R$ of both the sides of equation (47) we obtain

$$R^b{}_i \int_0^{+\infty} \overset{[1001]}{\tau}{}^a{}_b(s) f(s) h^2(s) ds = R^a{}_\ell \int \overset{[1001]}{\tau}{}^\ell{}_i(s) f(s) h^2(s) ds. \tag{71}$$

The arbitrariness of $f \in \mathbb{R}_\infty$ in the last equation yields

$$\overset{[1001]}{\tau}{}^a{}_b \equiv \overset{[1001]}{\tau}{}^\ell{}_i\, R^a{}_\ell\, R_b{}^i , \tag{72}$$

so that the tensor $\overset{[1001]}{\tau}$ is weakly isotropic and $(39)_9$ holds. Note that in (71) the rotation history $\rho = \rho(s)$ appears just through its value $R := \rho(0)$; hence the stronger condition (*EI*) give no further restriction on $\overset{[1001]}{\tau}$.

Next we study the terms of (47) involving $\overset{[1010]}{\tau}$. By taking the derivative $\partial^2/\partial\xi^d{}_D\partial x^b{}_B$ of both the sides of equation (47) we obtain

$$\int_0^{+\infty} \overset{[1010]}{\tau}{}^a{}_{bd}(s)\rho^d{}_j(s)\, R^b{}_i f(s)h^2(s)ds = R^a{}_\ell \int_0^{+\infty} \overset{[1010]}{\tau}{}^\ell{}_{ij}(s)f(s)h^2(s)ds, \tag{73}$$

which by the arbitrariness of $f \in \mathbb{R}_\infty$ yields

$$\overset{[1010]}{\tau}{}^a{}_{bd}\,\rho^d{}_j R^b{}_i R_a{}^m \equiv \overset{[1010]}{\tau}{}^m{}_{ij}. \tag{74}$$

(*Step* (*ii*) *in Remark* 6.3). Let ρ satisfy equation (43); then, by (74), the tensor $\overset{[1010]}{\tau}$ is weakly isotropic and thus $(39)_8$ holds.

(*Step* (*iii*) *in Remark* 6.3). By replacing $(39)_8$ in (73) we obtain

$$\overset{[1010]}{d}\,\varepsilon^a{}_{bd}\,\rho^d{}_j\, R^b{}_i\, R_a{}^m \equiv \overset{[1010]}{d}\,\varepsilon^m{}_{ij}; \tag{75}$$

by $(46)_1$ this equality is equivalent to

$$\overset{[1010]}{d}\,\varepsilon^m{}_{is}\, R_d{}^s\,\rho^d{}_j \equiv \overset{[1010]}{d}\,\varepsilon^m{}_{ij}, \tag{76}$$

which by (B) in Remark 6.3 yields $(41)_2$.

Next we study the terms in (47) involving $\overset{[1100]}{\tau}$. By taking the derivative $\partial^2/\partial x^i{}_B\partial z_L$ of both the sides of equation (47) we obtain an integral equality which is equivalent to $\overset{[1100]}{\tau}{}^a{}_b R^b{}_i = R^a{}_\ell \overset{[1100]}{\tau}{}^\ell{}_i$; multiplying this by $R_m{}^i$ yields

$$\overset{[1100]}{\tau}{}^a{}_m = R_m{}^i R^a{}_\ell \overset{[1100]}{\tau}{}^\ell{}_i\,, \tag{77}$$

so that $\overset{[1100]}{\tau}$ is weakly isotropic and $(39)_7$ holds.

Note that in (77) the rotation history $\rho = \rho(s)$ appears just through its value $R :=$ $:= \rho(0)$; hence the stronger condition (*EI*) give no further restriction on $\overset{[1100]}{\tau}$.

Next we study the terms of (47) involving $\overset{[0001]}{\tau}$. By taking the derivative $\partial/\partial\zeta_M$ of both the sides of equation (47), in view of the already proved equalities $(39)_{2,4,5,7\text{-}13}$ and $(41)_{3\text{-}5}$, we obtain an integral equality which is equivalent to $\overset{[0001]}{\tau}{}^a{}_C = R^a{}_\ell \overset{[0001]}{\tau}{}^\ell{}_C$; hence for any fixed $C \in \{1, 2, 3\}$ the vector $\overset{[0001]}{\tau}{}^\ell{}_C$ is weakly isotropic and $(39)_3$ holds.

In an analogous way one shows that $(39)_2$ holds.

Next we study the terms of (47) containing $\overset{[0010]}{\tau}$. By taking the derivative $\partial/\partial\xi^s{}_S$ of both the sides of equation (47) and using the already proved equalities $(39)_{2\text{-}5,7\text{-}13}$, we obtain

$$\overset{[0010]}{\tau}{}^{a}{}_{dC}\rho^{d}{}_{s} \equiv R^{a}{}_{\ell}\overset{[0010]}{\tau}{}^{\ell}{}_{sC} \,. \tag{78}$$

(*Step* (*ii*) *in Remark* 6.3). Let ρ satisfy equation (43); the multiplication of (78) by $R_{m}{}^{s}$ yields

$$\overset{[0010]}{\tau}{}^{a}{}_{mC} \equiv R^{a}{}_{\ell}R_{m}{}^{s}\overset{[0010]}{\tau}{}^{\ell}{}_{sC} \,; \tag{79}$$

hence for any fixed $C \in \{1\,,2\,,3\}$ the second-order tensor $\overset{[0010]}{\tau}{}^{a}{}_{mC}$ is weakly isotropic and $(39)_6$ holds.

(*Step* (*iii*) *in Remark* 6.3). By replacing $(39)_6$ in (78) we obtain

$$\overset{[0010]}{d}{}_{C}\,\rho^{a}{}_{s} \equiv R^{a}{}_{s}\overset{[0010]}{d}{}_{C} \,, \tag{80}$$

which by (B) in Remark 6.3 yields $(41)_1$.

Lastly we study the terms of (47) involving $\overset{[0000]}{\tau}$; by replacing the already proved equalities $(39)_{2\text{-}13}$ we obtain

$$\overset{[0000]}{\tau}{}^{aA} = R^{a}{}_{\ell}\overset{[0000]}{\tau}{}^{\ell A}; \tag{81}$$

hence for any fixed $A \in \{1\,,2\,,3\}$ the vector $\overset{[0000]}{\tau}{}^{aA}$ is weakly isotropic and $(39)_1$ holds. $\square$

7. Frame-indifferent solutions to equations (5) satisfying a certain symmetry condition

For a simple body the local law of angular momentum is equivalent to

$$Px^{T} = xP^{T} \,, \tag{82}$$

where P is the first Piola-Kirchhoff stress tensor and x is the deformation gradient.

The next theorems characterize the frame-indifferent solutions (4) to equations (5) that are Galilean-invariant or Euclidean-invariant and furthermore satisfy the symmetry condition (9), which is equivalent to (82).

Theorem 7.1. *The Galilean invariant functional* $\widehat{\mathcal{F}}$, *with components* (40), *satisfies the symmetry condition* (9) *if and only if*

$$\begin{aligned} &\overset{[1000]}{d}{}_{C} = 0\,, \quad &&\overset{[0010]}{d} \equiv 0\,, \quad &&\overset{[1100]}{d} = 0\,, \quad &&\overset{[1010]}{d} \equiv 0\,, \\ &\overset{[1001]}{d} \equiv 0\,, \quad &&\overset{[0110]}{d} \equiv 0\,, \quad &&\overset{[0011]}{d} \equiv 0\,, \quad &&\overset{[0020]}{d} \equiv 0\,; \end{aligned} \tag{83}$$

hence, if and only if

$$\mathcal{F}^{aA} = \varepsilon^{a}{}_{bc}\varepsilon^{ABC}\overset{[2000]}{d}\,x^{b}{}_{B}x^{c}{}_{C} \tag{84}$$

for any choice of the constant $\overset{[2000]}{d} \in \mathbb{R}$.

PROOF. The solution (40) to (5) satisfies (9) if and only if

$$
\begin{aligned}
&\varepsilon^{ABC}\overset{[1000]}{d}{}_C x^a{}_B x^b{}_A + \\
&+ \varepsilon^{ADC}\langle \overset{[0010]}{d}{}_C(s) \mid \xi^a{}_D(s)\rangle\, x^b{}_A + \varepsilon^{ACD}\varepsilon^a{}_{cd}\langle \overset{[1010]}{d}(s) \mid \xi^d{}_D(s)\rangle x^c{}_C\, x^b{}_A + \\
&+ \varepsilon^{ABL}\overset{[1100]}{d}\, x^a{}_B x^b{}_A z_L + \varepsilon^{ABM}\langle \overset{[1001]}{d}(s) \mid \zeta_M(s)\rangle x^a{}_B x^b{}_A + \\
&+ \varepsilon^{ADL}\langle \overset{[0110]}{d}(s) \mid \xi^a{}_D(s)\rangle z_L\, x^b{}_A + \varepsilon^{ADM}\langle \overset{[0011]}{d}(s_1, s_2) \mid \xi^a{}_D(s_1), \zeta_M(s_2)\rangle\, x^b{}_A + \\
&+ \varepsilon^{ABC}\varepsilon^a{}_{sc}\overset{[2000]}{d}\, x^s{}_B x^c{}_C x^b{}_A + \varepsilon^{ADE}\varepsilon^a{}_{de}\langle \overset{[0020]}{d}(s_1, s_2) \mid \xi^d{}_D(s_1), \xi^e{}_E(s_2)\rangle x^b{}_A = \\
&= \varepsilon^{ABC}\overset{[1000]}{d}{}_C x^b{}_B x^a{}_A + \\
&+ \varepsilon^{ADC}\langle \overset{[0010]}{d}{}_C(s) \mid \xi^b{}_D(s)\rangle\, x^a{}_A + \varepsilon^{ACD}\varepsilon^b{}_{cd}\langle \overset{[1010]}{d}(s) \mid \xi^d{}_D(s)\rangle x^c{}_C\, x^a{}_A + \\
&+ \varepsilon^{ABL}\overset{[1100]}{d}\, x^b{}_B x^a{}_A z_L + \varepsilon^{ABM}\langle \overset{[1001]}{d}(s) \mid \zeta_M(s)\rangle x^b{}_B x^a{}_A + \\
&+ \varepsilon^{ADL}\langle \overset{[0110]}{d}(s) \mid \xi^b{}_D(s)\rangle z_L\, x^a{}_A + \varepsilon^{ADM}\langle \overset{[0011]}{d}(s_1, s_2) \mid \xi^b{}_D(s_1), \zeta_M(s_2)\rangle\, x^a{}_A + \\
&+ \varepsilon^{ABC}\varepsilon^b{}_{sc}\overset{[2000]}{d}\, x^a{}_A x^s{}_B x^c{}_C + \varepsilon^{ADE}\varepsilon^b{}_{de}\langle \overset{[0020]}{d}(s_1, s_2) \mid \xi^d{}_D(s_1), \xi^e{}_E(s_2)\rangle x^a{}_A .
\end{aligned}
\tag{85}
$$

Next, by using (85) we study the restrictions that (9) places on $\overset{[0020]}{d}$. By taking the derivative $\partial^3/\partial\xi^i{}_I\partial\xi^j{}_J\partial x^m_M$ in both the sides of equation (85) we obtain

$$
\varepsilon^{MIJ}\varepsilon^a{}_{ij}\delta^{bm}\langle \overset{[0020]}{d}(s_1, s_2) \mid f(s_1), g(s_2)\rangle = \varepsilon^{MIJ}\varepsilon^b{}_{ij}\delta^{am}\langle \overset{[0020]}{d}(s_1, s_2) \mid f(s_1), g(s_2)\rangle. \tag{86}
$$

By the arbitrariness of $f, g \in \mathbb{R}_\infty$, equation (86) yields

$$
\overset{[0020]}{d}(\varepsilon^a{}_{ij}\delta^{bm} - \varepsilon^b{}_{ij}\delta^{am}) \equiv 0\,, \tag{87}
$$

which implies $(83)_8$ because $\varepsilon^a{}_{ij}\delta^{bm} \neq \varepsilon^b{}_{ij}\delta^{am}$ for some a, i, j, b, m.

Next, by using (85) we find the restrictions that (9) places on $\overset{[2000]}{d}$; by taking the derivative $\partial^3/\partial x^i{}_I\,\partial x^j{}_J\,\partial x^m{}_M$ in both the sides of equation (85) we obtain

$$
\begin{aligned}
&\overset{[2000]}{d}(\varepsilon^{MIJ}\varepsilon^a{}_{ij}\delta^{mb} + \varepsilon^{JIM}\varepsilon^a{}_{im}\delta^{jb} + \varepsilon^{MJI}\varepsilon^a{}_{ji}\delta^{mb} + \varepsilon^{JMI}\varepsilon^a{}_{mi}\delta^{jb} + \varepsilon^{IJM}\varepsilon^a{}_{jm}\delta^{ib} + \varepsilon^{IMJ}\varepsilon^a{}_{mj}\delta^{ib}) = \\
&= \overset{[2000]}{d}(\varepsilon^{MIJ}\varepsilon^b{}_{ij}\delta^{ma} + \varepsilon^{JIM}\varepsilon^b{}_{im}\delta^{ja} + \varepsilon^{MJI}\varepsilon^b{}_{ji}\delta^{ma} + \varepsilon^{JMI}\varepsilon^b{}_{mi}\delta^{ja} + \varepsilon^{IJM}\varepsilon^b{}_{jm}\delta^{ia} + \varepsilon^{IMJ}\varepsilon^b{}_{mj}\delta^{ia})\,,
\end{aligned}
$$

that is,

$$
\overset{[2000]}{d}\,\varepsilon^{IJM}(\varepsilon^a{}_{ji}\delta^{mb} + \varepsilon^a{}_{mi}\delta^{jb} + \varepsilon^a{}_{jm}\delta^{ib} + \varepsilon^b{}_{ij}\delta^{ma} + \varepsilon^b{}_{mi}\delta^{ja} + \varepsilon^b{}_{jm}\delta^{ia}) = 0. \tag{88}
$$

Adding the vanishing quantity $\varepsilon^{i}{}_{mj}\delta^{ab} + \varepsilon^{i}{}_{jm}\delta^{ab}$ in the left side of (88) yields

$$\overset{[2000]}{d}\,\varepsilon^{IJM}(\varepsilon^{a}{}_{ji}\delta^{mb}+\varepsilon^{i}{}_{jm}\delta^{ab}+\varepsilon^{a}{}_{mi}\delta^{jb}+\varepsilon^{a}{}_{jm}\delta^{ib}+\varepsilon^{b}{}_{ij}\delta^{ma}+\varepsilon^{i}{}_{mj}\delta^{ab}+\varepsilon^{b}{}_{mi}\delta^{ja}+\varepsilon^{b}{}_{jm}\delta^{ia})=0. \tag{89}$$

By the identity [7, p. 843]

$$\varepsilon^{b}{}_{ij}\delta^{ma} + \varepsilon^{i}{}_{mj}\delta^{ab} + \varepsilon^{b}{}_{mi}\delta^{ja} + \varepsilon^{b}{}_{jm}\delta^{ia} = 0$$

equation (89) holds for any $\overset{[2000]}{d} \in \mathbb{R}$; *i.e.* (9) does not restrict the coefficient $\overset{[2000]}{d}$.

Next, by using (85) we study the restrictions that (9) places on $\overset{[0011]}{d}$; by taking the derivative $\partial^3/\partial\xi^{i}_{I}\partial x^{j}_{J}\partial\zeta_{L}$ in both the sides of equation (85) we obtain

$$\varepsilon^{JIL}\langle \overset{[0011]}{d}(s_1, s_2) \mid f(s_1)\delta^{ai}, g(s_2)\rangle\delta^{bj} = \varepsilon^{JIL}\langle \overset{[0011]}{d}(s_1, s_2) \mid f(s_1)\delta^{bi}, g(s_2)\rangle\delta^{aj}, \tag{90}$$

which by the arbitrariness of $f, g \in \mathbb{R}_\infty$ yields

$$\overset{[0011]}{d}\,(\delta^{ai}\delta^{bj} - \delta^{bi}\delta^{aj}) \equiv 0\,; \tag{91}$$

hence $(83)_7$ holds because

$$\delta^{ai}\delta^{bj} \neq \delta^{aj}\delta^{bi} \quad \text{for some} \quad a, b, i, j\,. \tag{92}$$

Next we study the restrictions that (9) places on $\overset{[0110]}{d}$; by taking the derivative $\partial^3/\partial\xi^{i}{}_{I}\partial x^{j}{}_{J}\partial z_{L}$ in both the sides of equation (85) we have

$$\varepsilon^{JIL}\langle \overset{[0110]}{d}(s) \mid f(s)\delta^{ai}\rangle\delta^{bj} = \varepsilon^{JIL}\langle \overset{[0110]}{d}(s) \mid f(s)\delta^{bi}\rangle\delta^{aj}. \tag{93}$$

By the arbitrariness of $f \in \mathbb{R}_\infty$ equation (93) yields

$$\overset{[0110]}{d}(s)\,(\delta^{ai}\delta^{bj} - \delta^{aj}\delta^{bi}) \equiv 0\,, \tag{94}$$

which in view of (92) implies $(83)_6$.

Next we find the restrictions that (9) places on $\overset{[1001]}{d}$; by taking the derivative $\partial^3/\partial x^{i}_{I}\,\partial x^{j}_{J}\,\partial\zeta_{N}$ in both the sides of equation (85) we obtain

$$\varepsilon^{JIN}\langle \overset{[1001]}{d}(s) \,|f(s)\rangle\delta^{ai}\delta^{bj} + \varepsilon^{IJN}\langle \overset{[1001]}{d}(s) \,|f(s)\rangle\delta^{aj}\delta^{bi} =$$

$$= \varepsilon^{JIN}\langle \overset{[1001]}{d}(s) \,|f(s)\rangle\delta^{aj}\delta^{bi} + \varepsilon^{IJN}\langle \overset{[1001]}{d}(s) \,|f(s)\rangle\delta^{ai}\delta^{bj}\,,$$

that is,

$$\langle \overset{[1001]}{d}(s) \,|f(s)\rangle(\delta^{ai}\delta^{bj} - \delta^{aj}\delta^{bi}) = 0\,;$$

this by (92) is equivalent to

$$\langle \overset{[1001]}{d}(s) \,|f(s)\rangle = 0; \tag{95}$$

which by the arbitrariness of $f \in \mathbb{R}_\infty$ yields $(83)_5$.

Next we find the restrictions that (9) places on $\overset{[1010]}{d}$; by taking the derivative $\partial^3/\partial\xi^i_I\partial x^j_J\partial x^m_M$ in both the sides of equation (85) we obtain

$$\text{(96)}\quad \langle \overset{[1010]}{d}(s) \mid f(s)\rangle(\varepsilon^{MJI}\varepsilon^a{}_{ji}\delta^{bm}+\varepsilon^{JMI}\varepsilon^a{}_{mi}\delta^{bj})=\langle \overset{[1010]}{d}(s) \mid f(s)\rangle(\varepsilon^{MJI}\varepsilon^b{}_{ji}\delta^{am}+\varepsilon^{JMI}\varepsilon^b{}_{mi}\delta^{aj}),$$

that is,

$$\text{(97)}\qquad \langle \overset{[1010]}{d}(s) \mid f(s)\rangle(\varepsilon^a{}_{ji}\delta^{bm}-\varepsilon^a{}_{mi}\delta^{bj}-\varepsilon^b{}_{ji}\delta^{am}+\varepsilon^b{}_{mi}\delta^{aj})=0.$$

Note that $\varepsilon^a{}_{ji}\delta^{bm}-\varepsilon^a{}_{mi}\delta^{bj}-\varepsilon^b{}_{ji}\delta^{am}+\varepsilon^b{}_{mi}\delta^{aj}\neq 0$ for some choice of (a, b, i, j, m), *e.g.* for $(a, j, i, b, m)=(1, 2, 3, 3, 3)$; hence (97) is equivalent to $\langle \overset{[1010]}{d}(s) \mid f(s)\rangle = 0$, which by the arbitrariness of $f\in\mathbb{R}_\infty$ yields $(83)_4$.

Next we find the restrictions that (9) places on $\overset{[1100]}{d}$; by taking the derivative $\partial^3/\partial x^i_I\partial x^j_J\partial z_N$ in both the sides of equation (85) we obtain

$$\text{(98)}\qquad \varepsilon^{JIN}\overset{[1100]}{d}\delta^{ai}\delta^{bj}+\varepsilon^{IJN}\overset{[1100]}{d}\delta^{aj}\delta^{bi}=\varepsilon^{JIN}\overset{[1100]}{d}\delta^{aj}\delta^{bi}+\varepsilon^{IJN}\overset{[1100]}{d}\delta^{ai}\delta^{bj},$$

that is,

$$\text{(99)}\qquad \overset{[1100]}{d}(\delta^{ai}\delta^{bj}-\delta^{aj}\delta^{bi})=0.$$

This equality and (92) yield $(83)_3$.

Next we find the restrictions that (9) places on $\overset{[0010]}{d}$; by taking the derivative $\partial^2/\partial\xi^i_I\partial x^j_J$ in both the sides of equation (85) and in view of the already proved equalities $(83)_{3\text{-}8}$ we obtain

$$\text{(100)}\qquad \varepsilon^{JIC}\langle \overset{[0010]}{d}_C(s) \mid f(s)\delta^{ai}\rangle\delta^{bj}=\varepsilon^{JIC}\langle \overset{[0010]}{d}_C(s) \mid f(s)\delta^{bi}\rangle\delta^{aj}.$$

By the arbitrariness of $f\in\mathbb{R}_\infty$ equation (100) yields

$$\text{(101)}\qquad \overset{[0010]}{d}_C(\delta^a_i\delta^b_j-\delta^b_i\delta^a_j)\equiv 0,$$

which by (92) yields $(83)_2$.

Lastly, by using (85) we study the restrictions that (9) places on $\overset{[1000]}{d}$; in view of $(83)_{2,3}$, by taking the derivative $\partial^2/\partial x^i_I\partial x^j_J$ in both the sides of equation (85) we have

$$\varepsilon^{JIC}\overset{[1000]}{d}_C\delta^{ai}\delta^{bj}+\varepsilon^{IJC}\overset{[1000]}{d}_C\delta^{aj}\delta^{bi}=\varepsilon^{JIC}\overset{[1000]}{d}_C\delta^{aj}\delta^{bi}+\varepsilon^{IJC}\overset{[1000]}{d}_C\delta^{ai}\delta^{bj},$$

that is,

$$\text{(102)}\qquad \overset{[1000]}{d}_C(\delta^{ai}\delta^{aj}-\delta^{aj}\delta^{bi})=0.$$

By (92) this equality is equivalent to $(83)_1$. $\square$

The proof of the next theorem is quite similar to the proof of Theorem 7.1. In the latter the Galilean invariant solutions (40) to equations (6) are required to satisfy the symmetry condition (9). The same steps constitute a proof for the next theorem because they can also be applied to the Euclidean invariant solutions (42) to equations (6). Hence, to prove the next theorem one only has to disregard the terms in (40) which do not appear in (42).

THEOREM 7.2. *The Euclidean invariant functional* $\widehat{\mathcal{F}}$, *with components* (42), *satisfies the symmetry condition* (9) *if and only if equations* $(83)_{1,3,5}$ *hold; hence, if and only if*

$$\mathcal{F}^{aA} = \varepsilon^{a}{}_{bc}\,\varepsilon^{ABC}\,\overset{[2000]}{d}\,x^{b}_{B}x^{c}_{C} \tag{103}$$

for any choice of $\overset{[2000]}{d} \in \mathbb{R}$.

8. Frame-indifferent solutions to equations (6)

Next we characterize the classes of solutions (7) to equations (6) which are frame-indifferent in each one of the two senses (*GI*) or (*EI*) in Section 6. We use the general solution (104) to equations (6) given by Theorem 8.1 below, whose proof is obtained simply by dropping the index «*a*» everywhere in (31) and in Theorem 5.1 (see Remark 2.1).

THEOREM 8.1. *The functional* (7) *is a solution on* $\mathcal{A}\times\mathcal{U}$ *to equations* (6) *if and only if*

$$\begin{aligned} Q^{A} = {}& \overset{[0000]}{\gamma}{}^{A} + \varepsilon^{ABC}\overset{[1000]}{\gamma}{}_{bC}\,x^{b}_{B} + \varepsilon^{CAL}\overset{[0100]}{\gamma}{}_{C}\,z_{L} + \varepsilon^{CAD}\langle\overset{[0010]}{\gamma}{}_{dC}\mid\xi^{d}_{D}\rangle + \\ & + \varepsilon^{AMC}\langle\overset{[0001]}{\gamma}{}_{C}\mid\zeta_{M}\rangle + \varepsilon^{ABL}\overset{[1100]}{\gamma}{}_{b}\,x^{b}_{B}z_{L} + \varepsilon^{ABD}\langle\overset{[1010]}{\gamma}{}_{bd}\mid\xi^{d}_{D}\rangle\,x^{b}_{B} + \\ & + \varepsilon^{ABM}\langle\overset{[1001]}{\gamma}{}_{b}\mid\zeta_{M}\rangle\,x^{b}_{B} + \varepsilon^{ADL}\langle\overset{[0110]}{\gamma}{}_{d}\mid\xi^{d}_{D}\rangle z_{L} + \varepsilon^{AML}\langle\overset{[0101]}{\gamma}\mid\zeta_{M}\rangle z_{L} + \\ & + \varepsilon^{ADM}\langle\overset{[0011]}{\gamma}{}_{d}\mid\xi^{d}_{D},\zeta_{M}\rangle + \varepsilon^{ABC}\varepsilon_{hbc}\overset{[2000]}{\gamma}{}^{h}\,x^{b}_{B}x^{c}_{C} + \varepsilon^{ADE}\varepsilon_{hde}\langle\overset{[0020]}{\gamma}{}^{h}\mid\xi^{d}_{D},\xi^{e}_{E}\rangle \end{aligned} \tag{104}$$

for any choice of the constants

$$\overset{[0000]}{\gamma}{}^{A},\quad \overset{[1000]}{\gamma}{}_{bC},\quad \overset{[0100]}{\gamma}{}_{C},\quad \overset{[1100]}{\gamma}{}_{b},\quad \overset{[2000]}{\gamma}{}^{h} \quad\in\mathbb{R}$$

and of the functions

$$\overset{[0010]}{\gamma}{}_{dC},\quad \overset{[0001]}{\gamma}{}_{C},\quad \overset{[1010]}{\gamma}{}_{bd},\quad \overset{[1001]}{\gamma}{}_{b},\quad \overset{[0110]}{\gamma}{}_{d},\quad \overset{[0101]}{\gamma} \quad\in L^{2}(\mathbb{R}^{+},\mathbb{R}),$$

$$\overset{[0011]}{\gamma}{}_{d},\quad \overset{[0020]}{\gamma}{}^{h} \quad\in L^{2}[\mathbb{R}^{+}\times\mathbb{R}^{+},\mathbb{R}].$$

Remind that the functional (7) may be interpreted as the difference between the constitutive functionals $\widehat{q}$ and $\widehat{q}'$ for the heat flux in two globally equivalent simple bodies with fading memory. The property of material frame-indifference, in the form of Galilean or Euclidean invariance, requires that the response of a material be independent of the observer motion in the Galilean or Euclidean class of motions, respectively. Hence we require that the constitutive functionals $\widehat{q}$ and $\widehat{q}'$, thus $\widehat{\mathcal{Q}} = \widehat{q} - \widehat{q}'$ too, satisfy the conditions (GI) or (EI) below. In these conditions R $[\rho]$ represents the constant [possibly non constant] history of the rotation tensor in the observer motion.

(GI) - (Galilean invariance condition)

$$\mathcal{Q}(Rx\,,\, z\,,\, R\xi\,,\, \zeta) = \mathcal{Q}(x\,,\, z\,,\, \xi\,,\, \zeta) \tag{105}$$

for each $R \in 0rth^{+}$ *at any* $(x\,,\, z\,,\, \xi\,,\, \zeta) \in \mathcal{A} \times \mathcal{U}$.

(EI) - (Euclidean invariance condition)

$$\mathcal{Q}(Rx\,,\, z\,,\, \rho\xi\,,\, \zeta) = \mathcal{Q}(x\,,\, z\,,\, \xi\,,\, \zeta) \tag{106}$$

for each history $\rho : [0\,,\,\infty) \to 0rth^{+}$, *where* $R := \rho(0)$, *at any* $(x\,,\, z\,,\, \xi\,,\, \zeta) \in \mathcal{A} \times \mathcal{U}$.

The Theorems 8.2 and 8.3 below characterize the solutions to equations (6) which satisfy the condition (GI) and (EI), respectively.

Theorem 8.2. *The solution* (104) *to equations* (6) *satisfies the invariance condition* (105) *if and only if*

$$\begin{gathered} \overset{[1000]}{\gamma}_{bC} = 0\,, \quad \overset{[0010]}{\gamma}_{dC} \equiv 0\,, \quad \overset{[1100]}{\gamma}_{b} = 0\,, \quad \overset{[1010]}{\gamma}_{bd} \equiv \overset{[1010]}{\gamma}\,\delta_{bd}\,, \\ \overset{[1001]}{\gamma}_{b} \equiv 0\,, \quad \overset{[0110]}{\gamma}_{d} \equiv 0\,, \quad \overset{[0011]}{\gamma}_{d} \equiv 0\,, \quad \overset{[2000]}{\gamma}{}^{b} = 0\,, \quad \overset{[0020]}{\gamma}{}^{b} \equiv 0\,; \end{gathered} \tag{107}$$

hence, if and only if

$$\begin{aligned} \mathcal{Q}^{A} = {} & \overset{[0000]}{\gamma}{}^{A} + \varepsilon^{ALC}\overset{[0100]}{\gamma}_{C}\, z_L + \varepsilon^{AMC}\langle \overset{[0001]}{\gamma}_{C} \mid \zeta_M \rangle + \\ & + \varepsilon^{ABD}\langle \overset{[1010]}{\gamma}_{bd} \mid \xi^{d}_{D} \rangle\, x^{b}_{B} + \varepsilon^{AML}\langle \overset{[0101]}{\gamma} \mid \zeta_M \rangle z_L \end{aligned} \tag{108}$$

for any choice of the constants $\overset{[0000]}{\gamma}{}^{A}$, $\overset{[0100]}{\gamma}_{C} \in \mathbb{R}$ *and of the functions* $\overset{[0001]}{\gamma}_{C}$, $\overset{[1010]}{\gamma}_{bd}$, $\overset{[0101]}{\gamma} \in L^{2}_{h}(\mathbb{R}^{+}\,,\,\mathbb{R})$.

Theorem 8.3. *The solution* (104) *to equations* (5) *satisfies the invariance condition* (106) *if and only if the equalities* (107) *hold and, in addition,*

$$\overset{[1010]}{\gamma}_{bd} \equiv 0\,; \tag{109}$$

hence, if and only if

$$\mathcal{Q}^{A} = \overset{[0000]}{\gamma}{}^{A} + \varepsilon^{ALC}\overset{[0100]}{\gamma}_{C}\, z_L + \varepsilon^{AMC}\langle \overset{[0001]}{\gamma}_{C} \mid \zeta_M \rangle + \varepsilon^{AML}\langle \overset{[0101]}{\gamma} \mid \zeta_M \rangle z_L \tag{110}$$

for any choice of the constants $\overset{[0000]}{\gamma}{}^{A}$, $\overset{[0100]}{\gamma}_{C} \in \mathbb{R}$ *and of the functions* $\overset{[0001]}{\gamma}_{C}$, $\overset{[0101]}{\gamma} \in L^{2}(\mathbb{R}^{+},\mathbb{R})$.

The proofs of the above theorems are given below in a unified fashion; their scheme is similar to the unified proof of Theorems 6.1, 6.2, which follows the steps written in Remark 6.3. Here we use the version of this remark which is obtained by replacing in it (5) with (6) on the third-fourth lines and (31) with (104) on the fourth line; when below we invoke Remark 6.3 we always refer to this last version of it.

Proof of both Theorems 8.2, 8.3.

(*Step* (*i*) *in Remark* 6.3). The solution (104) to equations (6) satisfies (106) if and only if for each $\rho : [0\,,\infty) \to Orth^{+}$ we have

$$
\begin{aligned}
&\overset{[0000]}{\gamma}{}^{A} + \varepsilon^{ABC}\overset{[1000]}{\gamma}{}_{bC}\, x^{b}{}_{B} + \varepsilon^{ADC}\langle \overset{[0010]}{\gamma}{}_{dC}(s) \mid \xi^{d}{}_{D}(s)\rangle + \\
&+ \varepsilon^{ALC}\overset{[0100]}{\gamma}{}_{C}\, z_{L} + \varepsilon^{AMC}\langle \overset{[0001]}{\gamma}{}_{C}(s) \mid \zeta_{M}(s)\rangle + \\
&+ \varepsilon^{ABL}\overset{[1100]}{\gamma}{}_{b}\, x^{b}{}_{B} z_{L} + \varepsilon^{ABD}\langle \overset{[1010]}{\gamma}{}_{bd}(s) \mid \xi^{d}{}_{D}(s)\rangle\, x^{b}{}_{B} + \\
&+ \varepsilon^{ABM}\langle \overset{[1001]}{\gamma}{}_{b}(s) \mid \zeta_{M}(s)\rangle\, x^{b}{}_{B} + \varepsilon^{ADL}\langle \overset{[0110]}{\gamma}{}_{d}(s) \mid \xi^{d}{}_{D}(s)\rangle z_{L} + \\
&+ \varepsilon^{AML}\langle \overset{[0101]}{\gamma}(s) \mid \zeta_{M}(s)\rangle z_{L} + \varepsilon^{ADM}\langle \overset{[0011]}{\gamma}{}_{d}(s_1\,,s_2) \mid \xi^{d}{}_{D}(s_1)\,, z_{M}(s_2)\rangle + \\
&+ \varepsilon^{ABC}\varepsilon_{hbc}\overset{[2000]}{\gamma}{}^{h}\, x^{b}{}_{B}x^{c}{}_{C} + \varepsilon^{ADE}\varepsilon_{hde}\langle \overset{[0020]}{\gamma}{}^{h}(s_1\,,s_2) \mid \xi^{d}{}_{D}(s_1)\,,\xi^{e}{}_{E}(s_2)\rangle = \\
&= \overset{[0000]}{\gamma}{}^{A} + \varepsilon^{ABC}\overset{[1000]}{\gamma}{}_{bC}\; R^{b}{}_{s}\, x^{s}{}_{B} + \varepsilon^{ADC}\langle \overset{[0010]}{\gamma}{}_{dC}(s) \mid\; \rho^{d}{}_{s}(s)\xi^{s}{}_{D}(s)\rangle + \\
&+ \varepsilon^{ALC}\overset{[0100]}{\gamma}{}_{C}\, z_{L} + \varepsilon^{AMC}\langle \overset{[0001]}{\gamma}{}_{C}(s) \mid \zeta_{M}(s)\rangle + \\
&+ \varepsilon^{ABL}\overset{[1100]}{\gamma}{}_{b}\; R^{b}{}_{s}\, x^{s}{}_{B} z_{L} + \varepsilon^{ABD}\langle \overset{[1010]}{\gamma}{}_{bd}(s) \mid\; \rho^{d}{}_{s}(s)\xi^{s}{}_{D}(s)\rangle\; R^{b}{}_{\ell}\, x^{\ell}{}_{B} + \\
&+ \varepsilon^{ABM}\langle \overset{[1001]}{\gamma}{}_{b}(s) \mid \zeta_{M}(s)\rangle\; R^{b}{}_{s}\, x^{s}{}_{B} + \varepsilon^{ADL}\langle \overset{[0110]}{\gamma}{}_{d}(s) \mid\; \rho^{d}{}_{s}(s)\xi^{s}{}_{D}(s)\rangle z_{L} + \\
&+ \varepsilon^{ADM}\langle \overset{[0011]}{\gamma}{}_{d}(s_1\,,s_2) \mid\; \rho^{d}{}_{s}(s_1)\xi^{s}{}_{D}(s_1)\,,\zeta_{M}(s_2)\rangle + \\
&+ \varepsilon^{AML}\langle \overset{[0101]}{\gamma}(s) \mid \zeta_{M}(s)\rangle z_{L} + \varepsilon^{ABC}\varepsilon_{hbc}\overset{[2000]}{\gamma}{}^{h}\; R^{b}{}_{s}\, x^{s}{}_{B}\, R^{c}{}_{\ell}\, x^{\ell}{}_{C} + \\
&\qquad + \varepsilon^{ADE}\varepsilon_{hde}\langle \overset{[0020]}{\gamma}{}^{h}(s_1\,,s_2) \mid\; \rho^{d}{}_{s}(s_1)\,\xi^{s}{}_{D}(s_1)\,,\; \rho^{e}{}_{\ell}(s_2)\,\xi^{\ell}{}_{E}(s_2)\rangle.
\end{aligned}
\tag{111}
$$

Next we study the terms of (111) involving $\overset{[0020]}{\gamma}$. By taking the derivative $\partial^2/\partial\xi^{m}_{M}\partial\xi^{s}_{S}$ of both the sides of (111) and using (17) we obtain

$$
\begin{aligned}
&2\varepsilon_{hsm}\int_0^{+\infty}\int_0^{+\infty}\overset{[0020]}{\gamma}{}_{h}(s_1\,,s_2)f(s_1)g(s_2)h^2(s_1)ds_1\,h^2(s_2)ds_2 + \\
&=\varepsilon_{hde}\int_0^{+\infty}\int_0\overset{[0020]}{\gamma}{}_{h}(s_1,s_2)[\rho^{d}{}_{s}(s_1)\,\rho^{e}{}_{m}(s_2)+\rho^{e}{}_{m}(s_1)\,\rho^{d}{}_{s}(s_2)]f(s_1)g(s_2)h^2(s_1)ds_1\,h^2(s_2)ds_2.
\end{aligned}
\tag{112}
$$

By the arbitrariness of $f, g \in \mathbb{R}_\infty$ equation (112) yields

$$
2\varepsilon_{hsm}\overset{[0020]}{\gamma}{}_{h} \equiv \varepsilon_{hde}\overset{[0020]}{\gamma}{}_{h}[\,\rho^{d}{}_{s}\;\rho^{e}{}_{m}\;+\;\rho^{e}{}_{m}\;\rho^{d}{}_{s}\,]. \tag{113}
$$

(*Step* (*ii*) *in Remark* 6.3). If ρ satisfies (43), then (113) becomes $\varepsilon^h{}_{sm} \overset{[0020]}{\gamma}{}_h \equiv$ $\equiv \varepsilon^h{}_{de} \overset{[0020]}{\gamma}{}_h R^d{}_s R^e{}_m$, which by $(46)_2$ is equivalent to

$$\overset{[0020]}{\gamma}{}_h \equiv R^i{}_h \overset{[0020]}{\gamma}{}_i \, ; \tag{114}$$

that is, the vector $\overset{[0020]}{\gamma}$ is weakly isotropic and $(107)_9$ holds.

Next we study the terms of (111) involving $\overset{[2000]}{\gamma}$. By taking the derivative $\partial^2/\partial x^s{}_S \partial x^m_M$ of both the sides of equation (111) we obtain

$$\varepsilon^h{}_{sm} \overset{[2000]}{\gamma}{}_h = \varepsilon^h{}_{bc} \overset{[2000]}{\gamma}{}_h \, R^b{}_s R^c{}_m \ , \tag{115}$$

which by $(46)_2$ becomes

$$\varepsilon^h{}_{sm} \overset{[2000]}{\gamma}{}_h = \varepsilon^i{}_{sm} R^h{}_i \overset{[2000]}{\gamma}{}_h \, ;$$

up to the multiplication with $\varepsilon^r{}_{sm}$ the last equality is equivalent to

$$\overset{[2000]}{\gamma}{}_r = R^h{}_r \overset{[2000]}{\gamma}{}_h . \tag{116}$$

That is, the vector $\overset{[2000]}{\gamma}$ is weakly isotropic (see Remark 6.1) and $(107)_8$ holds.

Next we study the terms of (111) involving $\overset{[0011]}{\gamma}$. By taking the derivative $\partial^2/\partial \xi^s{}_S \partial \zeta_N$ of both the sides of (111) and using (17) we obtain

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \overset{[0011]}{\gamma}{}_s(s_1 \, , \, s_2) f(s_1) g(s_2) h^2(s_1) ds_1 h^2(s_2) ds_2 = \\ &\qquad = \int_0^{+\infty} \int_0^{+\infty} \overset{[0011]}{\gamma}{}_d(s_1 \, , \, s_2) \ \rho^d{}_s \, (s_1) f(s_1) g(s_2) h^2(s_1) ds_1 h^2(s_2) ds_2 . \end{aligned} \tag{117}$$

(*Step* (*ii*) *in Remark* 6.3). If ρ satisfies (43), then by the arbitrariness of $f, g \in \mathbb{R}_\infty$ equation (117) yields

$$\overset{[0011]}{\gamma}{}_s \equiv R^d{}_s \overset{[0011]}{\gamma}{}_d \, . \tag{118}$$

That is, the vector $\overset{[0011]}{\gamma}$ is weakly isotropic and thus $(107)_7$ holds.

Next we study the terms of (111) involving $\overset{[0101]}{\gamma}$. By taking the derivative $\partial^2/\partial z_S \partial \zeta_N$ of both the sides of equation (111) we obtain an identity not involving ρ; thus no restriction on $\overset{[0011]}{\gamma}$ is imposed by conditions (*GI*) and (*EI*).

Next we study the terms of (111) involving $\overset{[0110]}{\gamma}$. By taking the derivative $\partial^2/\partial \xi^s{}_S \partial z_N$ of both the sides of equation (111) and using (17) we obtain

$$\int_0^{+\infty} \overset{[0110]}{\gamma}{}_s(s) f(s) h^2(s) ds = \int_0^{+\infty} \overset{[0110]}{\gamma}{}_d(s) \ \rho^d{}_s \, (s) f(s) h^2(s) ds. \tag{119}$$

By the arbitrariness of $f \in \mathbb{R}_\infty$ equation (119) yields

$$\overset{[0110]}{\gamma}{}_s \equiv R^d{}_s(s) \overset{[0110]}{\gamma}{}_d \ , \tag{120}$$

which by Remark 6.1 yields $(107)_6$.

Analogously, by taking the derivative $\partial^2/\partial x^s{}_S \partial \zeta_N$ [$\partial^2/\partial x^s{}_S \partial z_N$] of both the sides of equation (111) we see that the vector $\overset{[1001]}{\gamma}{}_s$ [$\overset{[1100]}{\gamma}{}_s$] is weakly isotropic and then by Remark 6.1 equation $(107)_5$ [$(107)_3$] holds.

Next we study the terms of (111) involving $\overset{[1010]}{\gamma}$. By taking the derivative $\partial^2/\partial \xi^s{}_S \partial x^m_M$ of both the sides of equation (111) and using (17) we obtain

$$\int_0^{+\infty} \overset{[1010]}{\gamma}{}_{ms}(s) f(s) h^2(s)\, ds = R^b{}_m \int_0^{+\infty} \overset{[1010]}{\gamma}{}_{bd}(s)\ \rho^d{}_s(s) f(s) h^2(s)\, ds. \tag{121}$$

The arbitrariness of $f \in \mathbb{R}_\infty$ yields

$$\overset{[1010]}{\gamma}{}_{ms} \equiv R^b{}_m \rho^d{}_s \overset{[1010]}{\gamma}{}_{bd}. \tag{122}$$

(*Step* (*ii*) *in Remark* 6.3). If ρ satisfies (43), then (122) becomes $\overset{[1010]}{\gamma}{}_{ms} \equiv R^b{}_m R^d{}_s \overset{[1010]}{\gamma}{}_{bd}$; hence the tensor $\overset{[1010]}{\gamma}{}_{bd}$ is weakly isotropic and $(107)_4$ holds.

(*Step* (*iii*) *in Remark* 6.3). In view of $(107)_4$ equation (122) becomes

$$\overset{[1010]}{d}\ (\delta_{ms} - R^b{}_m \overline{\rho}{}^b{}_s) \equiv 0 \ , \tag{123}$$

which by (B) in Remark 6.3 yields (109).

Next we study the terms of (111) involving $\overset{[0010]}{\gamma}$. By taking the derivative $\partial/\partial \xi^s{}_S$ of both the sides of equation (111) and using the already proved equalities (109), we have

$$\int_0^{+\infty} \overset{[0010]}{\gamma}{}_{hs}(s) f(s) h^2(s)\, ds = \int_0^{+\infty} \overset{[0010]}{\gamma}{}_{hd}(s)\ \ \rho^d{}_s(s) f(s) h^2(s)\, ds. \tag{124}$$

(*Step* (*ii*)-(*iii*) *in Remark* 6.3). The arbitrariness of $f \in \mathbb{R}_\infty$ in (124) yields

$$\overset{[0010]}{\gamma}{}_{hs} \equiv \overset{[0010]}{\gamma}{}_{hd} R^d{}_s \ . \tag{125}$$

Hence for each $h \in \{1\,,2\,,3\}$ the vector $\overset{[0010]}{\gamma}{}_{hs}$ is weakly isotropic and by Remark 6.1 equation $(107)_2$ holds.

The deduction of $(107)_1$ is quite similar to the above deduction of $(107)_2$.

Next we study the terms of (111) involving $\overset{[0001]}{\gamma}$ [$\overset{[0100]}{\gamma}$]. By taking the derivative $\partial/\partial \zeta_S$ [$\partial/\partial z_S$] of both the sides of equation (111) we obtain an identity not involving ρ; thus the conditions (*GI*) and (*EI*) do not restrict $\overset{[0001]}{\gamma}$ [$\overset{[0100]}{\gamma}$].

Lastly note that equations (107) reduce each equality (111) to an identity involving $\overset{[0000]}{\gamma}$. Thus the conditions (*GI*) and (*EI*) do not restrict $\overset{[0000]}{\gamma}$. □

9. Differentiability of constitutive functionals defined on physical domains that are nowhere dense and nonconvex

As is customary, let

$$Sym := \{S \in Lin \mid S = S^T\} \ ,$$

$$PSym := \{S \in Sym \mid S \quad is \ \ positive \ \ definite \ \} \, ; \tag{126}$$

furthermore let Lin_∞ $[\overline{Lin}_\infty]$ denote the Hilbert space of all Lebesgue-measurable functions $\gamma : (0\,,\infty) \to Lin$ $[\ \gamma : [0\,,\infty) \to Lin\ ]$ such that $\langle \gamma\,,\gamma\rangle < \infty$, equipped with the inner product (2) and norm (3); lastly, for each subset $\mathcal{S}$ of Lin let $\mathcal{S}_\infty$ $[\overline{\mathcal{S}}_\infty]$ denote the subset of Lin_∞ $[\overline{Lin}_\infty]$ formed by the $\mathcal{S}$-valued functions.

Mizel and Wang [5] pointed out that the natural domain of the constitutive maps of a continuous simple body with fading memory is the cone $\mathcal{N}$ formed by the (total) histories

$$(F^t(\cdot)\,,\theta^t(\cdot)\,,G^t(\cdot)) \in \overline{Lin}_\infty \times \overline{\mathbb{R}}_\infty \times \overline{\mathcal{V}}_\infty$$

such that

$$\det F^t(s) > 0 \quad \text{and} \quad \theta^t(s) > 0 \quad \forall\, s \geq 0.$$

They pointed out that $\mathcal{N}$ is nowhere dense in the Banach space $E := \overline{Lin}_\infty \times \overline{\mathbb{R}}_\infty \times \overline{\mathcal{V}}_\infty$ and thus the usual differential calculus, which is concerned with maps defined on open sets, cannot be used for these maps. Then Mizel and Wang noticed that, in order to use the standard notion of differentiability, one should assume that any constitutive map admits a smooth extension from the cone $\mathcal{N}$ to the whole space E.

Incidentally, note that this assumption should also be made in the present paper in order to render meaningful any result on the solutions to the symmetric systems of equations studied here. But, even if the assumption seems mathematically reasonable, it has no physical motivation.

In order to avoid the above requirements for smooth extendibility of constitutive maps, Mizel and Wang [5, pp. 126, 127] proposed a new definition of Fréchet differentiability for maps defined on nowhere dense sets. Precisely, they introduced the following definitions.

Definition 9.1. A function $\Lambda(\cdot) = (F^t(\cdot)\,,\theta^t(\cdot)\,,G^t(\cdot)) : (0\,,\infty) \to Lin \times \mathbb{R} \times \mathcal{V}$ is *admissible* if $\Lambda(\cdot) \in E$ and $\det F(s) > 0\,,\ \theta(s) > 0$ for almost all $s \geq 0$.

Definition 9.2. Let $\mathcal{W}$ be a vector space. The functional

$$f : \mathcal{N} \to \mathcal{W} \ , \quad f = f(\Lambda(\cdot))\,, \tag{127}$$

is *smooth* if, for each fixed admissible $\Lambda(\cdot)$, the first-order asymptotic expansion

$$f(\Lambda(\cdot) + \Gamma(\cdot)) = f(\Lambda(\cdot)) + \delta f(\Lambda(\cdot)) \cdot \Gamma(\cdot) + o(\|\Gamma(\cdot)\|) \tag{128}$$

holds for all $\Gamma(\cdot) \in E$ for which $\Lambda(\cdot) + \Gamma(\cdot)$ is admissible.

Here $\delta f(\Lambda(\cdot))$ denotes a continuous linear functional defined on the closed subspace of E spanned by the collection of all $\Gamma(\cdot)$ such that $\Lambda(\cdot) + \Gamma(\cdot)$ is admissible. It is assumed that the linear functional $\delta f(\Lambda(\cdot))$ is continuous in $\Lambda(\cdot)$.

Note that the aforementioned closed subspace, which is the domain of $\delta f(\Lambda(\cdot))$, is defined in correspondence with $\Lambda(\cdot)$, hence a priori it may depend on $\Lambda(\cdot)$.

In the paper [1] a differentiability notion is given for mappings f defined on any given convex subset of a Banach space that may be nowhere dense. Incidentally, they show that the afore-mentioned closed subspace, in which $\delta f(\Lambda(\cdot))$ is defined, does not depend on $\Lambda(\cdot)$. When the domain of f is open the frame of differential calculus in [1] coincides with the usual one.

Now let f be the constitutive functional for the first Piola stress-tensor P in a heat-conducting deformable body formed of a simple material with fading memory (4). That is, at any time t let

$$P = f(F^t(\cdot)\,,\theta^t(\cdot)\,,G^t(\cdot))\,. \tag{129}$$

The principle of material frame-indifference yields

$$f(Q^t(\cdot)F^t(\cdot)\,,\theta^t(\cdot)\,,G^t(\cdot)) = Q(t)f(F^t(\cdot)\,,\theta^t(\cdot)\,,G^t(\cdot)) \tag{130}$$

for each smooth function $Q^t(.) : [0\,,\infty) \to Orth^+$ at any admissible $(F^t(\cdot)\,,\theta^t(\cdot)\,,G^t(\cdot))$; as a consequence, at any time t the tensor P reads

$$P = R(t)f(U^t(\cdot)\,,\theta^t(\cdot)\,,G^t(\cdot))\,, \tag{131}$$

where $U^t : [0\,,\infty) \to PSym$ is the history of the right stretch tensor and $R(t)$ is the rotation tensor.

Note that $\overline{PSym}_\infty$ is a convex subset of $\overline{Lin}_\infty$. Hence a natural constitutive domain $\mathcal{U}$ for f can be chosen that is convex, for instance $\mathcal{U} := \overline{PSym}_\infty \times \overline{\mathbb{R}}^+_\infty \times \overline{\mathcal{V}}_\infty$. Thus the theory of differential calculus presented in [1] can be applied to the (reduced) constitutive functional $f(U^t(\cdot)\,,\theta^t(\cdot)\,,G^t(\cdot))$ in (131), or equivalently to the restriction of the functional (129) to pure-stretch histories.

Next we show that *rigorous meanings can be given to the derivatives of the unrestricted functional* (129) simply by employing the condition of material frame-indifference (130) and by applying the theory of differential calculus [1]. As a consequence *any result of the present paper, on the solutions to the symmetric systems of equations studied here, remains true for the constitutive functionals of any simple material with fading memory, without requiring their domains be extendible to an open set.*

(4) The considerations below can be adapted for the heat flux vector functional and the scalar functionals of internal energy and entropy. Indeed, by the condition of frame-indifference these functionals satisfy condition (135) below, which is the point of departure in order to render meaningful the derivative of the extension map $f(\xi)$ in its left side provided the map $f(v)$ in its right side be differentiable on its convex (and nowhere dense) domain.

As is well known, by the polar decomposition theorem any $x \in Lin^+$ can be uniquely written in the form

$$x = RU, \qquad \text{with} \quad R \in Orth^+ , \quad U \in PSym .$$

Consequently, any given $\xi \in Lin^+_\infty$ can be written as

$$\xi = \rho \upsilon , \qquad \text{with} \quad \rho \in Orth^+_\infty \ , \ \upsilon \in PSym_\infty .$$

At any fixed admissible $(\theta^t(\cdot)\,,\, G^t(\cdot))$ let us rewrite the functional (129) in the form

$$f = f(x\,,\xi)\,, \quad f : \omega \times \mathcal{W} \to Lin\,, \tag{132}$$

where ω is an open subset of Lin^+ and

$$\mathcal{W} := \bigcup_{\rho \in Orth^+_\infty} \rho\, \mathcal{U} = \{\rho\upsilon \mid \rho \in Orth^+_\infty \ , \ \ \upsilon \in \mathcal{U}\} \tag{133}$$

for any given convex subset $\mathcal{U}$ of $PSym_\infty$ (5). In (132) the variable ξ represents the past (*i.e.* only defined for $s > 0$) history of the position gradient and x is its value at time t, so that $(x\,,\xi) = F^t(.)$. The subset $\mathcal{W}$ of Lin^+_∞ is nowhere dense because it is formed by histories ξ such that $\det \xi(s) > 0$ almost everywhere on $(0\,,\infty)$ (see [1, Section 2]).

The Euclidean condition of material frame-indifference (130) yields

$$f = f(Qx\,, q\xi) = Qf(x\,,\xi) \quad \forall\, Q \in Orth^+ , \quad \forall q \in Orth^+_\infty\,, \tag{134}$$

at each $(x\,,\xi) \in \omega \times \mathcal{W}$.

For the sake of simplicity, from now onward we disregard in equations (132) to (134) the dipendence on the finite-dimensional variable x.

We note that for $Q = I$ and $q = \rho^T$ equation (134) yields

$$f(\xi) = f(\upsilon)\,, \quad \text{with} \quad \xi = \rho\upsilon\,, \tag{135}$$

$\forall \rho \in Orth^+_\infty$, $\forall \upsilon \in \mathcal{U}$, $\mathcal{U} \subseteq PSym_\infty$ convex. Moreover note that $\mathcal{W}$ is a nowhere dense subset of Lin_∞ which is nonconvex; hence we cannot use the theory of [1] to assert that $f(\xi)$ is differentiable, *i.e.*, that $\partial f/\partial\xi$ exists. However we can consider derivatives for the functional

$$f = f(\upsilon)\,, \quad f : \mathcal{U} \to Lin\,, \tag{136}$$

because $\mathcal{U}$ is convex in $PSym_\infty$, hence in Lin_∞. Note that from [1] we have $\partial f(\upsilon)/\partial\upsilon \in$ $\in \mathcal{L}(\mathcal{V}_{\mathcal{U}}\,, Lin)$, where

$$\mathcal{V}_{\mathcal{U}} = cl\langle\mathcal{G}_{\mathcal{U}}\rangle\,, \quad \text{with} \quad \mathcal{G}_{\mathcal{U}} = \{b \in Lin_\infty \mid a + b \in \mathcal{U}\,, \ \text{for some } a \in \mathcal{U}\}\,,$$

(5) Remind that $PSym$ is a convex subset of Lin; hence, *e.g.*, $\mathcal{U} := PSym_\infty$ is convex in Lin_∞ (see [1]).

is a closed subspace of Sym_∞. In view of this, next we show that

(A) *if the subset* $\mathcal{U}$ *of* $PSym_\infty$ *is convex, then the functional*

$$f = f(\xi)\,, \quad f : \mathcal{W} \to Lin\,, \quad \mathcal{W} = \bigcup_{\rho \in Orth_\infty^+} \rho\,\mathcal{U}\ , \tag{137}$$

with

$$\xi = \rho v\,, \quad \rho \in Orth_\infty^+\,, \quad v \in PSym_\infty\,,$$

is differentiable provided that its restriction (136) *to* $\mathcal{U}$ *is differentiable, also when* $\mathcal{W}$ *is a (nowhere dense and) nonconvex subset of* Lin_∞.

In fact, we show that

(B) *for the functional* (137) *the asymptotic expansion*

$$f(\xi + b) = f(\xi) + \left\langle \frac{\partial f}{\partial v}(v) \mid v' - v \right\rangle + o(|v' - v|)\,, \tag{138}$$

holds if $b = \rho' v' - \rho v$ *for some* $\rho' \in Orth_\infty^+$ *and* $v' \in \mathcal{U}$; i.e., $\forall\ b \in Lin_\infty$ *such that* $\xi + b \in \mathcal{W}$.

Following [1, Section 3] we put

$$\mathcal{V}_{\mathcal{W}} := cl < \mathcal{G}_{\mathcal{W}} > \ , \quad \mathcal{G}_{\mathcal{W}} = \{b \in Lin_\infty \ : \ \xi + b \in \mathcal{W} \quad \text{for some} \quad \xi \in \mathcal{W}\}. \tag{139}$$

In view of (B) we are induced to define the map $\langle \frac{\partial f}{\partial \xi}(\xi) \mid - \rangle\ :\ \mathcal{V}_{\mathcal{W}} \to Lin$ by

$$\left\langle \frac{\partial f}{\partial \xi}(\xi) \mid b \right\rangle := \left\langle \frac{\partial f}{\partial v}(v) \mid v' - v \right\rangle \quad \text{for} \quad \xi = \rho v\ , \quad b = \rho' v' - \rho v\ , \tag{140}$$

so that $\langle \frac{\partial f}{\partial \xi}(\xi) \mid - \rangle \in \mathcal{L}(\mathcal{V}_{\mathcal{W}}\,, Lin)$.

Thus

(C) *the continuous linear map* $\langle \frac{\partial f}{\partial \xi}(\xi) \mid - \rangle$ *in* (140) *is the first derivative of* $f : \mathcal{W} \to Lin$ *at* ξ . *Hence the asymptotic expansion*

$$f(\xi + b) = f(\xi) + \left\langle \frac{\partial f}{\partial \xi}(\xi) \mid b \right\rangle + o(|b|) \tag{141}$$

holds for each $b \in \mathcal{V}_W$.

Conclusion

Any result of the present paper regarding the solutions to equations (5) and (6), which has been proved in the previous sections for functionals defined on open sets, also holds for functionals of the form (132), which are defined on the nowhere dense nonconvex set (133), provided that $\mathcal{U}$ is convex. This extension of the results in the present paper is possible by the existence, set up in the paper [1], of the derivatives for functionals of the form (136) when $\mathcal{U}$ is convex. As a consequence, the constitutive functionals (132) can be differentiated at any point $(x, \xi) \in \omega \times \mathcal{W}$ even if f is not extended to an open set of $\omega \times Lin_\infty$ containing $\omega \times \mathcal{W}$.

Next we give the proofs of the above assertions (A) to (C). For $b = \rho' v' - \rho v$ we have

$$b = (\rho' - \rho)v + \rho'(v' - v) \quad \text{and} \quad \rho v + b = \rho' v + \rho'(v' - v); \tag{142}$$

thus

$$f(\rho v + b) - f(\rho v) = f(\rho v + b) - f(\rho' v) + f(\rho' v) - f(\rho v) =$$
$$= f(\rho' v + \rho'(v' - v)) - f(\rho' v) + f(\rho' v) - f(\rho v).$$

Hence by (135) and $(142)_2$ we have

$$f(\rho v + b) - f(\rho v) = f(v + (v' - v)) - f(v). \tag{143}$$

Now assume that the functional (136) is differentiable; by differentiation of the right-hand side of equation (143) we obtain

$$f(\rho v + b) - f(\rho v) = \left\langle \frac{\partial f}{\partial v}(v) \mid v' - v \right\rangle + o(|v' - v|);$$

hence assertion (B) is true. Now, by (140) and

$$\left\langle \frac{\partial f}{\partial v}(v) \mid - \right\rangle \in \mathcal{L}(\mathcal{V}_U \, , Lin) \, ,$$

we have

$$\left\langle \frac{\partial f}{\partial \xi}(\xi) \mid - \right\rangle \in \mathcal{L}(\mathcal{V}_W \, , Lin).$$

Hence (138) is equivalent to (141) and (C) holds.

Assertion (A) is a consequence of (B) and (C).

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Relazione

letta ed approvata nella seduta del 14 aprile 2000, sulla *Memoria* di Adriano Montanaro e Diego Pigozzi, presentata nella seduta del 24 aprile 1998 dal Socio A. Bressan, intitolata: *General and physically privileged solutions to certain symmetric systems of linear P.D.E.s with tensor functionals as unknowns*.

Gli Autori caratterizzano le soluzioni generali di certi sistemi simmetrici di equazioni alle derivate parziali aventi come incognite dei funzionali con valori ed argomenti tensoriali. Inoltre essi determinano tra queste soluzioni quelle fisicamente significative (in senso opportuno). Queste forniscono le condizioni sui funzionali costitutivi di due corpi termodinamici semplici e con memoria evanescente, necessarie e sufficienti affinché essi siano globalmente equivalenti, ossia, brevemente, affinché si comportino allo stesso modo in assenza di tagli; e ciò equivale, un po' più precisamente, a questa condizione: per questi corpi i problemi di evoluzione con gli stessi dati iniziali e al contorno, hanno le stesse soluzioni, comunque i dati siano scelti.

Ai suddetti sistemi di equazioni si perviene, ad esempio, quando, nella termodinamica dei continui semplici con memoria evanescente, si considerano le condizioni di bilancio locale sui funzionali costitutivi relativi a due corpi globalmente equivalenti e si assumono come incognite le differenze tra i funzionali costitutivi corrispondenti.

Fissato uno dei suddetti corpi, le suaccennate condizioni permettono di determinare tutti i corpi globalmente equivalenti a quello, che siano fisicamente realizzabili o no; e riguardo a ciò quelle condizioni hanno una certa analogia con le restrizioni (o relazioni) che la diseguaglianza dissipativa (o secondo principio della termodinamica) implica per le equazioni costitutive di un corpo di un prefissato tipo.

Il problema della suddetta determinazione non è mai stato considerato da altri autori, nemmeno per corpi termodinamici meno complessi. La sua importanza dal punto di vista fisico, o addirittura tecnico, risulta dal fatto che nel lavoro in corso di stampa su *Archive for Rational Mechanics and Analysis* Montanaro osserva, tra l'altro, che nel caso termoelastico (privo di memoria), due corpi termodinamici possono essere globalmente equivalenti ma fisicamente differenti in quanto, brevemente, due loro sottocorpi corrispondenti non siano globalmente equivalenti. I risultati della presente *Memoria* costituiscono, tra l'altro, il primo passo essenziale verso l'estensione ai considerati corpi con memoria della importante suddetta osservazione di Montanaro.

I domini dei suaccennati funzionali, coincidenti con quelli delle considerate soluzioni qui caratterizzate, sono sottoinsiemi ovunque non densi di un certo spazio Hilbertiano di dimensione infinita. Nella presente *Memoria* si dà, da un punto di vista generale, un significato rigoroso alle derivate dei detti funzionali, senza supporli estendibili a qualche insieme aperto; e tale estensione non sembra avere supporto fisico, almeno in generale. Al suddetto scopo gli Autori impongono ai funzionali di soddisfare il principio di indifferenza materiale e usano risultati di un loro precedente lavoro di Analisi matematica.

Essendo la presente *Memoria* tutta di Analisi matematica, la Commissione ha ritenuto opportuno chiedere un giudizio tecnico al noto analista Tullio Valent, esperto in applicazioni di analisi funzionale ai sistemi continui. La Commissione è lieta di poter riportare da tale giudizio quanto segue: «Il lavoro appare rigoroso dal punto di vista matematico e formalmente corretto. Si può notare come, in esso, gli Autori hanno saputo superare ostacoli sia di natura teorica sia di carattere tecnico. Infatti i problemi da loro affrontati, oltre a presentare delle difficoltà già a livello di una formulazione matematicamente rigorosa, sono piuttosto complessi e ardui da trattare, e quindi hanno richiesto una notevole abilità tecnica».

Pertanto la Commisssione ritiene il lavoro degno di essere accolto tra le *Memorie* dell'Accademia.

Giuseppe Grioli
Carlo Cercignani
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