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Regularity of solutions to stochastic Volterra equations

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Analisi matematica. — *Regularity of solutions to stochastic Volterra equations.* Nota di Anna Karczewska e Jerzy Zabczyk, presentata (*) dal Socio G. Da Prato.

Abstract. — We study regularity of stochastic convolutions solving Volterra equations on $\mathbb{R}^d$ driven by a spatially homogeneous Wiener process. General results are applied to stochastic parabolic equations with fractional powers of Laplacian.

Key words: Stochastic Volterra equations; Stochastic convolution; Function-valued solutions; Generalized and classical random fields.

Riassunto. — *Regolarità delle soluzioni di equazioni di Volterra stocastiche.* Viene studiata la regolarità di convoluzioni stocastiche risolvendo un'equazione di Volterra in $\mathbb{R}^d$ perturbata da un processo di Wiener spazialmente omogeneo. I risultati generali ottenuti sono applicati a equazioni paraboliche stocastiche con una potenza frazionaria del Laplaciano.

1. Introduction

The paper is concerned with the following stochastic Volterra equation

$$X(t,\theta) = \int_0^t v(t-\tau)AX(\tau,\theta)d\tau + X_0(\theta) + W(t,\theta)\,, \tag{1}$$

where $t \in \mathbb{R}_+$, $\theta \in \mathbb{R}^d$, $v \in L^1_{loc}(\mathbb{R}_+)$, $X_0 \in S'(\mathbb{R}^d)$ and W is a spatially homogeneous Wiener process with values in the space of real, tempered distributions $S'(\mathbb{R}^d)$. The class of operators A covered in the present paper contains, in particular, the Laplace operator Δ and its fractional powers $-(-\Delta)^{\alpha/2}$, $\alpha \in]0\,,2]$. We consider existence of solutions to (1) in $S'(\mathbb{R}^d)$ and derive conditions under which the solutions to (1) are function-valued or continuous.

The equation (1) is a generalization of stochastic heat and wave equations studied by many authors, see *e.g.*, [10, 17, 18, 21-23, 25, 30 and references therein]. In the context of infinite particle systems stochastic heat equation of a similar type has been investigated in recent papers by Bojdecki and Jakubowski [4-6] and by Dawson and Gorostiza in [12]. Stochastic heat equation with $\mathbb{R}^d$ replaced by a compact Lie group is an object of the recent paper by Tindel and Viens [29].

Taking in (1), $A = \Delta$ and $v(t) \equiv 1$, and differentiating (1) with respect to time, we obtain stochastic heat equation

$$\frac{\partial X}{\partial t}(t,\theta) = \Delta X(t,\theta) + \frac{\partial W}{\partial t}(t,\theta).$$

Similarly taking $v(t) = t$, $t \in [0\,,+\infty[$, and differentiating (1) twice results in stochastic

(*) Nella seduta del 10 marzo 2000.

wave equation

$$\frac{\partial^2 X}{\partial t^2}(t,\theta) = \Delta X(t,\theta) + \frac{\partial^2 W}{\partial t^2}(t,\theta)\,. \tag{2}$$

Let us notice however that the noise term in equation (2) differs from the one considered in papers [10, 17, 18, 21-23, 25, 30].

The stochastic Volterra equations have been treated by many authors, see [7-9] or [27, 28]. In the first three papers stochastic Volterra equations are studied in connection with problems arising in mathematical physics, particularly in viscoelasticity. It has been shown that the stochastic convolution leads to regular solutions, and next, its samples are Hölder-continuous. In the papers [27, 28] stochastic Volterra equations in the plane are studied. Authors have obtained sufficient conditions for the existence of smooth densities for the probability law of the solutions and studied also their small perturbations.

Our aim is to find conditions under which solutions to the stochastic Volterra equation (1) are function-valued or continuous with respect to the space variable. In the paper we treat the case of general dimension and the correlated, spatially homogeneous noise W_Γ of the general form.

The paper is organized as follows. In Section 2 we recall some concepts and facts needed in the paper. We formulate Proposition 1 giving conditions under which Gaussian random fields have function-valued solutions and recall from [1] a continuity criterium which will be used in the proofs of the main results. Section 3 summarizes properties of the stochastic integral with values in the space of tempered distributions $S'(\mathbb{R}^d)$. We generalize slightly a result from [6] concerned with the existence of stochastic integral. In Section 4 our main results on stochastic Volterra equations are formulated as Theorems 1 and 2. In Section 5 we provide some applications. In particular we formulate regularity and continuity results in Theorems 3 and 4.

The paper is a rewritten version of the report by Karczewska and Zabczyk [19].

2. Generalized and classical homogeneous Gaussian random fields

We start from recalling several analytical concepts needed in the paper.

Let $S(\mathbb{R}^d)$, $S_c(\mathbb{R}^d)$, denote respectively the spaces of all infinitely differentiable rapidly decreasing real and complex functions on $\mathbb{R}^d$ and $S'(\mathbb{R}^d), S'_c(\mathbb{R}^d)$ denote the spaces of real and complex, tempered distributions. The value of a distribution $\xi \in S'_c(\mathbb{R}^d)$ on a test function ψ will be written as $\langle \xi, \psi \rangle$. For $\psi \in S(\mathbb{R}^d)$ we set $\psi_{(s)}(\theta) = \overline{\psi(-\theta)}$, $\theta \in \mathbb{R}^d$. Denote by $S_{(s)}(\mathbb{R}^d)$ the space of all $\psi \in S(\mathbb{R}^d)$ such that $\psi = \psi_{(s)}$, and by $S'_{(s)}(\mathbb{R}^d)$ the space of all $\xi \in S'(\mathbb{R}^d)$ such that $\langle \xi, \psi \rangle = \langle \xi, \psi_{(s)} \rangle$ for every $\psi \in S(\mathbb{R}^d)$.

In the paper we denote by $\mathcal{F}$ the Fourier transform both on $S_c(\mathbb{R}^d)$, and on $S'_c(\mathbb{R}^d)$. In particular,

$$\mathcal{F}\psi(\theta) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \theta, \eta \rangle} \psi(\eta)\, d\eta\,, \quad \psi \in S_c(\mathbb{R}^d)\,,$$

and for the inverse Fourier transform $\mathcal{F}^{-1}$,

$$\mathcal{F}^{-1}\psi(\theta) = \int_{\mathbb{R}^d} e^{2\pi i\langle\theta\,,\eta\rangle}\psi(\eta)d\eta\,,\quad \psi \in S_c(\mathbb{R}^d).$$

Moreover, if $\xi \in S_c'(\mathbb{R}^d)$, $\langle\mathcal{F}\xi\,,\psi\rangle = \langle\xi\,,\mathcal{F}\psi\rangle$ for all $\psi \in S_c(\mathbb{R}^d)$. Let us note that $\mathcal{F}$ transforms the space of tempered distributions $S'(\mathbb{R}^d)$ into $S_{(s)}'(\mathbb{R}^d)$.

For any $h \in \mathbb{R}^d$, $\psi \in S(\mathbb{R}^d)$, $\xi \in S'(\mathbb{R}^d)$, the *translations* $\tau_h\psi$, $\tau_h'\xi$ are defined by the formulas: $\tau_h\psi(x) = \psi(x-h)$, $\langle\tau_h'\xi\,,\psi\rangle = \langle\xi\,,\tau_h\psi\rangle$, $x \in \mathbb{R}^d$.

By $\mathcal{B}(S'(\mathbb{R}^d))$ and $\mathcal{B}(S_c'(\mathbb{R}^d))$, we denote the smallest σ-algebras of subsets of $S'(\mathbb{R}^d)$ and $S_c'(\mathbb{R}^d)$, respectively, such that for any test function φ the mapping $\xi \to \langle\xi\,,\varphi\rangle$ is measurable.

Let $(\Omega\,,\mathcal{F}\,,P)$ be a complete probability space. Any measurable mapping $Y:\ \Omega \to$ $\to S'(\mathbb{R}^d)$ is called a *generalized random field*. A generalized random field Y is called *Gaussian* if $\langle Y,\varphi\rangle$ is a Gaussian random variable for any $\varphi \in S(\mathbb{R}^d)$. The definition implies that for any functions $\varphi_1,\ldots,\varphi_n \in S(\mathbb{R}^d)$ the random vector $(\langle Y,\varphi_1\rangle,\ldots,\langle Y,\varphi_n\rangle)$ is also Gaussian. One says that a generalized random field Y is *homogeneous* or *stationary* if for all $h \in \mathbb{R}^d$, the translation $\tau_h'(Y)$ of Y has the same distribution as Y.

If Y is a generalized homogeneous, Gaussian random field then for each $\psi \in S(\mathbb{R}^d)$, $\langle Y,\psi\rangle$ is a Gaussian random variable and the bilinear functional $q:\ S(\mathbb{R}^d)\times S(\mathbb{R}^d) \to \mathbb{R}$ defined by the formula,

$$q(\varphi\,,\psi) = \mathbb{E}(\langle Y,\varphi\rangle\langle Y,\psi\rangle)\quad,\quad \text{for}\quad \varphi\,,\ \psi \in S(\mathbb{R}^d)\ ,$$

is continuous and positive definite. Since $q(\varphi\,,\psi) = q(\tau_h\varphi\,,\tau_h\psi)$ for all φ, $\psi \in S(\mathbb{R}^d)$, $h \in \mathbb{R}^d$, there exists, see *e.g.* [13], a unique positive-definite distribution $\Gamma \in S'(\mathbb{R}^d)$ such that for all φ, $\psi \in S(\mathbb{R}^d)$, one has:

$$q(\varphi\,,\psi) = \langle\Gamma\,,\varphi * \psi_{(s)}\rangle\,.$$

The distribution Γ is called the *space correlation* of the field Y. By Bochner-Schwartz theorem the positive-definite distribution Γ is the inverse Fourier transform of a unique positive, symmetric, tempered measure μ on $\mathbb{R}^d : \Gamma = \mathcal{F}^{-1}(\mu)$. The measure μ is called the *spectral measure* of Γ and of the field Y.

Let $Y:\ \Omega \to S'(\mathbb{R}^d)$ be a generalized random field. When the values of Y are functions, with probability 1, then Y is called a *classical random field* or shortly *random field.*

In particular a homogeneous (stationary), Gaussian random field is a family of Gaussian random variables $Y(\theta)$, $\theta \in \mathbb{R}^d$, with Gaussian laws invariant with respect to all translations. That is, for any $\theta_1,\ldots,\theta_n \in \mathbb{R}^d$ and $h \in \mathbb{R}^d$, the law of $(Y(\theta+h),\ldots,Y(\theta_n+h))$ does not depend on $h \in \mathbb{R}^d$.

The following apparently well-known result is essential for the sequel. Its proof can be found in [19].

Proposition 1. *A generalized, homogeneous, Gaussian random field Y is classical if and only if the space correlation Γ of Y is a bounded function and if and only if the spectral measure μ of Y is finite.*

We finish the section recalling a continuity criterium which will be used in the proofs of the main continuity results, see [1, Theorem 3.4.3].

Proposition 2. *Let* $Y(\theta)$, $\theta \in \mathbb{R}^d$, *be a homogeneous, Gaussian random field with the spectral measure* μ. *If, for some* $\varepsilon > 0$,

$$\int_{\mathbb{R}^d} (\ln(1+|\lambda|))^{1+\varepsilon} \mu(d\lambda) < +\infty\,,$$

then Y *has a version with almost surely continuous sample functions.*

3. Stochastic integration

In the paper we integrate operator-valued functions $\mathcal{R}(t)$, $t \geq 0$, with respect to a Wiener process W. The operators $\mathcal{R}(t)$, $t \geq 0$, will be non-random and will act from some linear subspaces of $S'(\mathbb{R}^d)$ into $S'(\mathbb{R}^d)$. We shall assume that $W(t)$, $t \geq 0$, is taking values in $S'(\mathbb{R}^d)$. The process W is space homogeneous in the sense that, for each $t \geq 0$, random variables $W(t)$ are stationary, Gaussian, generalized random fields. We denote by Γ the covariance of $W(1)$ and the associated spectral measure by μ. To underline the fact that the distributions of W are determined by Γ we will write W_Γ. From now on we denote by q, a scalar product on $S(\mathbb{R}^d)$ given by the formula:

$$q(\phi\,,\psi) = \langle \Gamma\,, \phi * \psi_{(s)} \rangle\ ,\ \ \phi\,,\psi \in S(\mathbb{R}^d).$$

The crucial role in the theory of stochastic integration with respect to W_Γ is played by the Hilbert space $S'_q \subset S'(\mathbb{R}^d)$ called the *kernel* of W_Γ. Namely the space S'_q consists of all distributions $\xi \in S'(\mathbb{R}^d)$ for which there exists a constant C such that

$$|\langle \xi\,, \psi \rangle| \leq C\sqrt{q(\psi\,,\psi)}\,, \quad \psi \in S(\mathbb{R}^d).$$

The norm in S'_q is given by the formula:

$$|\xi|_{S'_q} = \sup_{\psi \in S} \frac{|\langle \xi\,, \psi \rangle|}{\sqrt{q(\psi\,,\psi)}}.$$

Let us assume that the stochastic integral should take values in a Hilbert space H continuously imbedded into $S'(\mathbb{R}^d)$. Let $L_{HS}(S'_q\,, H)$ be the space of Hilbert-Schmidt operators from S'_q into H. Assume that $\mathcal{R}(t), t \geq 0$, is measurable $L_{HS}(S'_q\,, H)$-valued function such that

$$\int_0^t \|\mathcal{R}(\sigma)\|^2_{L_{HS}(S'_q\,,H)} d\sigma < +\infty\,, \quad \text{for all } t \geq 0\,.$$

Then the stochastic integral

$$\int_0^t \mathcal{R}(\sigma) dW_\Gamma(\sigma)\,, \quad t \geq 0$$

can be defined in a standard way, see [16, 11] or [24].

Of special interest are operators $\mathcal{R}(t)$, $t \geq 0$, of convolution type:

$$\mathcal{R}(t)\xi = \mathbf{r}(t) * \xi\ ,\ \ t \geq 0\ ,\ \ \xi \in S'(\mathbb{R}^d)\ ,$$

with $\mathbf{r}(t) \in S'(\mathbb{R}^d)$. The convolution operator is not, in general, defined for all $\xi \in$ $\in S'(\mathbb{R}^d)$ and for the stochastic integration it is important to know under what conditions on $\mathbf{r}(\cdot)$ and ξ the convolution is well-defined. For many important examples the Fourier transform $\mathcal{F}\mathbf{r}(t)(\lambda)$, $t \geq 0$, $\lambda \in \mathbb{R}^d$, is continuous in both variables and, for any $T \geq 0$,

$$\sup_{t\in[0,T]} \sup_{\lambda\in\mathbb{R}^d} |\mathcal{F}\mathbf{r}(t)(\lambda)| = M_T < +\infty. \tag{3}$$

If this is the case then the operators $\mathcal{R}(t)$ can be defined in a non-ambiguous way using Fourier transforms:

$$\mathcal{R}(t)\xi = \mathcal{F}^{-1}(\mathcal{F}\mathbf{r}(t)\,\mathcal{F}\xi)\ ,$$

for all ξ such that $\mathcal{F}\xi$ has a representation as a function.

Proposition 3. *Assume that the function $\mathcal{F}\mathbf{r}$ is continuous in both variables and satisfies condition* (3)*. Then the stochastic convolution* :

$$\mathcal{R} * W_\Gamma(t) = \int_0^t \mathcal{R}(t-\sigma)dW_\Gamma(\sigma)\ ,\ t \geq 0\ ,$$

is a well-defined $S'(\mathbb{R}^d)$-valued stochastic process. For each $t \geq 0$, $\mathcal{R} * W_\Gamma(t)$ is a Gaussian, stationary, generalized random field with the spectral measure :

$$\mu_t(d\lambda) = \left(\int_0^t |\mathcal{F}\mathbf{r}(\sigma)(\lambda)|^2 d\sigma\right)\mu(d\lambda)\ , \tag{4}$$

and with the covariance Γ_t :

$$\Gamma_t = \int_0^t \mathbf{r}(\sigma) * \Gamma * \mathbf{r}_{(s)}(\sigma)d\sigma\ . \tag{5}$$

Proof. We show first that formula (4) defines a tempered measure μ_t and formula (5) defines a positive-definite distribution Γ_t. Since, by our assumptions, the function $|\mathcal{F}(\mathbf{r}(\sigma))(\lambda)|^2$, $\sigma \geq 0$, $\lambda \in \mathbb{R}^d$, is bounded and continuous, so the integral

$$\int_0^t |\mathcal{F}(\mathbf{r}(\sigma))(\lambda)|^2 d\sigma\ ,\ \ \lambda \in \mathbb{R}^d\ ,$$

is bounded and continuous, as well.

Since μ was a tempered measure, so it is clear that also μ_t given by (4) is a tempered measure. To show that (5) defines a positive-definite distribution it is enough to show that $\Gamma_t = \mathcal{F}^{-1}(\mu_t)$. But

$$\mathcal{F}^{-1}(\mu_t) = \int_0^t \mathcal{F}^{-1}\left[|\mathcal{F}(\mathbf{r}(\sigma))(\lambda)|^2\right] d\sigma\ .$$

However

$$|\mathcal{F}(\mathbf{r}(\sigma))(\lambda)|^2 = \mathcal{F}(\mathbf{r}(\sigma))(\lambda)\overline{\mathcal{F}(\mathbf{r}(\sigma))(\lambda)} = \mathcal{F}(\mathbf{r}(\sigma))(\lambda)\mathcal{F}(\mathbf{r}_{(s)}(\sigma))(\lambda)$$

and therefore

$$|\mathcal{F}(\mathbf{r}(\sigma))(\lambda)|^2\mu = \mathcal{F}(\mathbf{r}(\sigma))(\lambda)\,\mathcal{F}(\mathbf{r}_{(s)}(\sigma))(\lambda)\,\mathcal{F}^{-1}(\mathcal{F}(\mu)).$$

Since $\mathcal{F}^{-1}$ transforms product of distributions $\xi_1\xi_2\xi_3$ (if well-defined) onto convolutions of the inverse Fourier transforms $\mathcal{F}^{-1}(\xi_1) * \mathcal{F}^{-1}(\xi_2) * \mathcal{F}^{-1}(\xi_3)$, we have

$$\mathcal{F}^{-1}\left[|\mathcal{F}(\mathbf{r}(\sigma))(\lambda)|^2\mu\right] = \mathbf{r}(\sigma) * \mathbf{r}_{(s)}(\sigma) * \mathcal{F}^{-1}(\mu)\,.$$

Then

$$\mathcal{F}^{-1}(\mu_t) = \int_0^t \left(\mathcal{F}^{-1}[|\mathcal{F}(\mathbf{r}(\sigma))(\lambda)|^2\mu]\right) d\sigma = \int_0^t \mathbf{r}(\sigma) * \mathbf{r}_{(s)}(\sigma) * \mathcal{F}^{-1}(\mu) d\sigma\,.$$

Since $\Gamma = \mathcal{F}^{-1}(\mu)$ and $\Gamma_t = \mathcal{F}^{-1}(\mu_t)$, so

$$\Gamma_t = \int_0^t \mathbf{r}(\sigma) * \Gamma * \mathbf{r}_{(s)}(\sigma) d\sigma\,.$$

Let us notice that integrals are Riemann integrals.

Let p be an arbitrary continuous scalar product on $S(\mathbb{R}^d)$, such that the embedding: $S'_q(\mathbb{R}^d) \subset S'_p(\mathbb{R}^d)$, is Hilbert-Schmidt. Note that for $\xi \in S'_q(\mathbb{R}^d)$,

$$\xi = \mathcal{F}^{-1}(u\mu) \text{ with } u \in L^2_{(s)}(\mathbb{R}^d\ , \mu)\,.$$

By (3) , $\mathcal{F}(\mathbf{r}(t))\mathcal{F}(\xi)$ is a measure,

$$\mathcal{F}(\mathbf{r}(t))(\lambda)u(\lambda)\mu(d\lambda)\ ,$$

belonging again to $S'_q(\mathbb{R}^d)$. Moreover, for $t \in [0\,, T]$,

$$||\mathcal{R}(t)||_{L(S'_q, S'_q)} \leq \sup_{t\in[0\,,T]} \sup_{\lambda\in\mathbb{R}^d} |\mathcal{F}\mathbf{r}(t)(\lambda)| = M_T < +\infty\,.$$

Since the embedding $S'_q \subset S'_p$ is Hilbert-Schmidt, the stochastic integral, by the very definition, is an S'_p-valued random variable. Denote:

$$Z_t = \mathcal{R} * W_\Gamma(t)\,.$$

Then we may write

$$\mathbb{E}\left(\langle Z_t\,, \varphi\rangle\langle Z_t\,, \psi\rangle\right) = \mathbb{E}\left(\left\langle \int_0^t \mathcal{R}(t-\sigma)dW_\Gamma(\sigma)\,, \varphi\right\rangle \left\langle \int_0^t \mathcal{R}(t-u)dW_\Gamma(u)\,, \psi\right\rangle\right) =$$

$$= \mathbb{E}\left(\int_0^t \langle \mathbf{r}(t-\sigma) * \varphi\,, dW_\Gamma(\sigma)\rangle \int_0^t \langle \mathbf{r}(t-u) * \psi\,, dW_\Gamma(u)\rangle\right) =$$

$$= \int_0^t \langle \Gamma\,, (\mathbf{r}(\sigma) * \varphi) * (\mathbf{r}(\sigma) * \psi)_{(s)}\rangle d\sigma$$

where $\varphi,\ \psi \in S(\mathbb{R}^d)$. This implies the formula (5) of the theorem, from which (4) easily follows. $\square$

As an application consider the so called Levy-Khinchin exponent

$$a(\lambda) = \frac{1}{2}\langle Q\lambda\,, \lambda\rangle - \int_{R^d} (e^{i(\lambda\,,y)} - 1)\nu(dy) \tag{6}$$

of an infinitely divisible symmetric law. In the formula (6), Q is a symmetric, non-negative definite matrix and ν is a symmetric measure concentrated on $R^d \setminus \{0\}$ such that

$$\int_{|y|\leq 1} |y|^2 \nu(dy) <+\infty\,, \qquad \int_{|y|>1} 1\nu(dy) <+\infty\,. \tag{7}$$

From Proposition 3 we have the following proposition.

Proposition 4. *Assume that*

$$\mathcal{F}\mathbf{r}(t)(\lambda) = e^{-ta(\lambda)}\;,\;\; t \geq 0$$

where a *is the Levy-Khinchin exponent defined by* (6) *and* (7). *Then the conditions of Proposition* 3 *are satisfied.*

The proposition together with Proposition 3 strengthen slightly an earlier result by [12] which was concerned with function $a(\lambda) = |\lambda|^\alpha$, $\alpha \in]0\,,2]$. The proof in [12] was different and more functional analytic.

For more information on stochastic integral with values in the Schwartz space of tempered distributions $S'(\mathbb{R}^d)$ we refere to Itô [15, 16], Bojdecki and Jakubowski [3-6], Bojdecki and Gorostiza [2] and Peszat and Zabczyk [24, 25].

4. Stochastic Volterra equation

We study the linear, stochastic, Volterra equation in $S'(\mathbb{R}^d)$:

$$X(t) = \int_0^t v(t-\tau)AX(\tau)d\tau + X_0 + W_\Gamma(t)\,, \tag{8}$$

where $X_0 \in S'(\mathbb{R}^d)$, A is an operator given in the Fourier transform form:

$$\mathcal{F}(A\xi)(\lambda) = -a(\lambda)\,\mathcal{F}(\xi)(\lambda)\;, \tag{8a}$$

v is a locally integrable function and W_Γ is an $S'(\mathbb{R}^d)$-valued space homogeneous Wiener process.

The deterministic version of the equation has been investigated by many authors, see Prüss [26]. Stochastic Volterra equation (8) but in bounded domains has been analysed by Clément and Da Prato [7, 8].

We shall assume the following *Hypothesis* (H1):

1. For any $\gamma \geq 0$, the equation

$$\mathbf{s}(t) + \gamma \int_0^t v(t-\tau)\mathbf{s}(\tau)d\tau = 1\,, \quad t \geq 0$$

   has exactly one solution $\mathbf{s}(\cdot\,,\gamma)$ locally integrable and measurable with respect to both variables $\gamma \geq 0$ and $t \geq 0$.
2. Moreover, for any $T \geq 0, \sup_{t\in[0\,,T]} \sup_{\gamma\geq 0} |\mathbf{s}(t\,,\gamma)| <+\infty$.

For some special cases the function $\mathbf{s}(t;\gamma)$ may be found explicitely. Namely, we have (see, *e.g.*, [26]):

$$\text{(9)} \qquad \text{for} \quad v(t) \equiv 1\,, \qquad \mathbf{s}(t;\gamma) = e^{-\gamma t}\,, \qquad t \geq 0\,,\ \ \gamma \geq 0;$$

$$\text{(10)} \qquad \text{for} \quad v(t) = t\,, \qquad \mathbf{s}(t;\gamma) = \cos(\sqrt{\gamma}t)\,, \qquad t \geq 0\,,\ \ \gamma \geq 0;$$

$$\text{(11)} \qquad \text{for} \quad v(t) = e^{-t}\,, \qquad \mathbf{s}(t;\gamma) = (1+\gamma)^{-1}\left[1+\gamma e^{-(1+\gamma)t}\right]\,, \qquad t \geq 0\,,\ \ \gamma \geq 0.$$

Hypothesis (H1) is satisfied if for instance function v is nonnegative and nonincreasing, see [26, p. 39].

We introduce now the so called *resolvent* family $\mathcal{R}(\cdot)$ determined by the operator A and the function v. Namely,

$$\mathcal{R}(t)\xi = \mathbf{r}(t) * \xi\,, \quad \xi \in S'(\mathbb{R}^d)\,,$$

where,

$$\mathbf{r}(t) = \mathcal{F}^{-1}\mathbf{s}(t, a(\cdot))\,,\ \ t \geq 0\,.$$

As in the deterministic case the solution to the stochastic Volterra equation (8) is of the form:

$$X(t) = \mathcal{R}(t)X_0 + \int_0^t \mathcal{R}(t-\tau)dW_\Gamma(\tau)\,,\ \ t \geq 0\,.$$

We have the following corollaries of the previous results on stochastic integration.

THEOREM 1. *Let* W_Γ *be a spatially homogeneous Wiener process and* $\mathcal{R}(t)$, $t \geq 0$, *the resolvent for the equation* (8). *If Hypothesis* (H1) *holds then the stochastic convolution*

$$\mathcal{R} * W_\Gamma(t) = \int_0^t \mathcal{R}(t-\sigma)dW_\Gamma(\sigma)\,,\ \ t \geq 0\,,$$

is a well-defined $S'(\mathbb{R}^d)$*-valued process. For each* $t \geq 0$ *the random variable* $\mathcal{R} * W_\Gamma(t)$ *is a generalized, stationary random field on* $\mathbb{R}^d$ *with the spectral measure* μ_t:

$$\text{(12)} \qquad \mu_t(d\lambda) = \left[\int_0^t (\mathbf{s}(\sigma\,, a(\lambda)))^2\, d\sigma\right]\mu(d\lambda).$$

Proof of Theorem 1 is a direct consequence of Proposition 3.

THEOREM 2. *Assume that the Hypothesis* (H1) *holds. Then the process* $\mathcal{R} * W_\Gamma(t)$ *is function-valued for all* $t \geq 0$ *if and only if*

$$\int_{\mathbb{R}^d}\left(\int_0^t (\mathbf{s}(\sigma\,, a(\lambda)))^2\, d\sigma\right)\mu(d\lambda) < +\infty\,,\ \ t \geq 0\,.$$

If for some $\varepsilon > 0$ *and all* $t \geq 0$,

$$\int_0^t \int_{\mathbb{R}^d} (\ln(1+|\lambda|))^{1+\varepsilon}\,(\mathbf{s}(\sigma\,, a(\lambda)))^2 d\sigma\mu(d\lambda) < +\infty\,,$$

then, for each $t \geq 0$, $\mathcal{R} * W_\Gamma(t)$ *is a sample continuous random field.*

Proof of Theorem 2 comes directly from Theorem 1 and Propositions 1 and 2.

5. Applications

5.1. *Regularity and continuity in terms of* Γ.

In this subsection we provide conditions for regularity and continuity of the solutions in terms of *the covariance kernel* Γ of the Wiener process W_Γ rather than in terms of the spectral measure as we have done up to now.

We shall need an additional assumption.

Hypothesis (H2). For arbitrary $T > 0$, there exist constants c_1, $c_2 \geq 0$ such that for some $\delta \geq 0$, all $\lambda \in \mathbb{R}^d$ and $t \in [0, T]$

$$\frac{c_1}{1 + a^\delta(\lambda)} \leq \int_0^t s^2(\sigma, a(\lambda)) d\sigma \leq \frac{c_2}{1 + a^\delta(\lambda)}.$$

Let us notice that functions $s(t, a(\lambda))$ from the cases (9), (10) and (11) satisfy Hypothesis (H2) with $\delta = 1$, $\delta = 0$ and $\delta = 1$, respectively. The folllowing result is a consequence of Theorem 2.

Theorem 3. *Assume that* (H1) *and* (H2) *hold and that the operator A is given by formula* (8*a*). *Then there exists a function-valued solution to the stochastic Volterra equation* (8) *if and only if*

$$\int_{\mathbb{R}^d} \frac{1}{1 + a^\delta(\lambda)} \mu(d\lambda) <+ \infty. \tag{13}$$

In the case of stochastic wave equation (2), $v(t) = t$, $\delta = 0$ and $a(\lambda) = |\lambda|^\alpha$, $\alpha \in]0, 2]$, $A = -(-\Delta)^{\alpha/2}$, condition (13) is equivalent to the requirement that the measure μ is finite. If $\delta > 0$ and $a(\lambda) = |\lambda|^\alpha$, then condition (13) is equivalent to

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^{\delta\alpha}} \mu(d\lambda) <+ \infty. \tag{13a}$$

Moreover, if Γ is a nonnegative measure and $d \geq 3$, (13*a*) is equivalent to

$$\int_{|\lambda|<1} \frac{1}{|\lambda|^{d-\alpha\delta}} \Gamma(d\lambda) <+ \infty.$$

Theorem 4. *Let Hypothesis* (H2) *hold. Assume that $d \geq 2$, that Γ is a non-negative measure and $A = -(-\Delta)^{\alpha/2}$, $\alpha \in]0, 2]$. If*

$$\int_{|\lambda|\leq 1} \frac{1}{|\lambda|^{d-\alpha+\delta}} \Gamma(d\lambda) <+ \infty, \quad \int_{|\lambda|>1} \frac{1}{|\lambda|^{d+\alpha-\delta}} \Gamma(d\lambda) <+ \infty,$$

then the solution to the stochastic Volterra equation (8) *has continuous version.*

The proof will be based on several lemmas. For any $\gamma \in]0, 2]$ denote by p_t^γ the density of the γ-stable, rotationally invariant, density on the d-dimensional space. Thus,

$$e^{-t|\lambda|^\gamma} = \mathcal{F} p_t^\gamma(\lambda).$$

Lemma 1. *For arbitrary $t > 0$ and arbitrary $x \in \mathbb{R}^d$,*

$$p_t^\gamma(x) = t^{-d/\gamma} p_1^\gamma(x t^{-1/\gamma}).$$

LEMMA 2. *There exists a constant* $c > 0$ *such that for all* $\gamma \leq 2$,

$$G_d^\gamma(x) \stackrel{\mathrm{df}}{=} \int_0^{+\infty} e^{-t} p_t^\gamma(x) dt \leq \frac{c}{|x|^{d+\gamma}} \quad , x \in \mathbb{R}^d .$$

PROOF. It is well-known, see *e.g.*, Gorostiza and Wakolbinger [14, p. 286], that for some constant $c_1 > 0$:

$$p_1^\gamma(x) \leq \frac{c_1}{1+|x|^{d+\gamma}} \ , \quad x \in \mathbb{R}^d . \tag{14}$$

From Lemma 1 and the estimate (14) we obtain:

$$G_d^\gamma(x) = \int_0^{+\infty} e^{-t} t^{-\frac{d}{\gamma}} p_1^\gamma\left(x\, t^{-\frac{1}{\gamma}}\right) dt \leq \int_0^{+\infty} e^{-t} t^{-\frac{d}{\gamma}} \frac{c_1}{1+\left|xt^{-\frac{1}{\gamma}}\right|^{d+\gamma}} dt \leq$$

$$\leq \int_0^{+\infty} e^{-t}\, t^{-\frac{d}{\gamma}} \frac{c_1\, t^{\frac{d+\gamma}{\gamma}}}{t^{\frac{d+\gamma}{\gamma}} + |x|^{d+\gamma}} dt \leq \int_0^{+\infty} e^{-t}\, t \frac{c_1}{t^{\frac{d+\gamma}{\gamma}} + |x|^{d+\gamma}} dt \leq$$

$$\leq \frac{c_1}{|x|^{d+\gamma}} \int_0^{+\infty} e^{-t}\, t\, dt . \qquad \square$$

LEMMA 3. *If* $\gamma < d$, $\gamma \in]0\,,2]$, *then there exists a constant* $c > 0$ *such that,*

$$G_d^\gamma(x) \leq \frac{c}{|x|^{d-\gamma}} \text{ for } |x| < 1 .$$

PROOF. Since

$$G_d^\gamma(x) \leq \int_0^{+\infty} p_t^\gamma(x)\, dt \ ,$$

the result follows from the well-known formula for Riesz γ-potential, see *e.g.*, Landkof [20]. $\square$

CONCLUSION. There exists a constant $c > 0$ such that, if $\gamma < d$, $\gamma \leq 2$, then:

$$G_d^\gamma(x) \leq \frac{c}{|x|^{d-\gamma}} \text{ if } |x| \leq 1 \ ,$$

and

$$G_d^\gamma(x) \leq \frac{c}{|x|^{d+\gamma}} \text{ if } |x| \geq 1 .$$

PROOF OF THEOREM 4.

Now, we pass to the proof of the theorem. If for some $\varepsilon > 0$ and all $t > 0$,

$$\int_{\mathbb{R}^d} (\ln(1+|\lambda|))^{1+\varepsilon} \left[\int_0^t \mathbf{s}^2(\sigma\,, a(\lambda))\, d\sigma\right] \mu(d\lambda) < +\infty \ , \tag{15}$$

then by Theorem 3 for all $t > 0$, the solution X of the stochastic equation (8) has a continuous version. Taking into account that by (H2)

$$\int_0^t \mathbf{s}^2(\sigma\,, a(\lambda))d\sigma \leq \frac{c_2}{1 + a^\delta(\lambda)}\ ,$$

one can replace (15) by

$$\int_{\mathbb{R}^d} (\ln(1 + |\lambda|))^{c_2+\varepsilon} \frac{c_2}{1 + a^\delta(\lambda)}\,\mu(d\lambda) <+\infty\,. \tag{16}$$

Since we have assumed that $A = -(-\Delta)^{\alpha/2}$, then $a(\lambda) = |\lambda|^\alpha$, with $\alpha \in]0\,, 2]$ and therefore (16) becomes

$$\int_{\mathbb{R}^d} (\ln(1 + |\lambda|))^{1+\varepsilon} \frac{1}{1 + |\lambda|^{\alpha\delta}}\,\mu(d\lambda) <+\infty\,. \tag{17}$$

However, the condition (17) holds for some $\varepsilon > 0$ if for some $\delta' > 0$

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^{\alpha\delta-\delta'}}\,\mu(d\lambda) <+\infty\,.$$

In the same way as in the paper [18] by Karczewska and Zabczyk, for some constant $c > 0$:

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^\gamma}\,\mu(d\lambda) = c\int_{\mathbb{R}^d} G_d^\gamma(x)\Gamma(dx)\ ,$$

where $\gamma := \alpha\delta - \delta'$.

Taking into account Lemma 2 and Lemma 3, the result follows. □

5.2. *Some special cases.*

In this subsection we illustrate the main results obtained in the paper by several examples.

Let us recall that the linear stochastic Volterra equation (8) considered in the paper, with the operator A given in the form (8*a*), is determined by three objects: the spatial correlation Γ of the process W_Γ, the operator A and the function v or, equivalently, by the spectral measure μ, the function a and the function $\mathbf{s}$, respectively.

We apply our Theorems 2 and 4 to several special cases corresponding to particular choices of functions v, a and of the measure μ. We will assume, for instance, that $v(t) \equiv 1$ or $v(t) = t$ or $v(t) = e^{-t}$, $t \geq 0$, that $a(\lambda) = |\lambda|^\alpha$, $\alpha \in]0\,, 2]$, $\lambda \in \mathbb{R}^d$ and that the measure μ is either finite or $\mu(d\lambda) = \frac{1}{|\lambda|^\gamma}d\lambda$, $\gamma \in]0\,, d[$. Note that if $a(\lambda) = |\lambda|^2$, then $A = \Delta$ and if $a(\lambda) = |\lambda|^\alpha$, $\alpha \in]0\,, 2[$, then $A = -(-\Delta)^{\alpha/2}$ is the fractional Laplacian.

Case 1. If (H1) holds, the function a is given by (6) and (7) and the measure μ is finite then $\mathcal{R} * W_\Gamma$ is a function-valued process. To see this note that by (H1) and Theorem 1, the measure μ_t given by (12) is finite. So, the result follows from Theorem 2.

CASE 2. If (H1) and (H2) hold, the function a is given by (6) and (7) and μ is a measure such that for some $\varepsilon > 0$,

$$\int_{\mathbb{R}^d} (\ln(1 + |\lambda|))^{1+\varepsilon} \frac{1}{1 + a^\delta(\lambda)} \mu(d\lambda) <+ \infty ,$$

then for arbitrary $t > 0$, $\mathcal{R} * W_\Gamma(t)$ is a continuous random field. This follows immediately from Theorem 2.

CASE 3. Assume that $v(t) \equiv 1$ or $v(t) = t$ or $v(t) = e^{-t}$, $t \geq 0$, $A = \Delta$ and $\Gamma(x) = \Gamma_\beta(x) = \frac{1}{|x|^\beta}$, $\beta \in [0 , d[$. Then function $\mathbf{s}$ is given by formulas (9), (10) and (11), respectively. Moreover $a(\lambda) = |\lambda|^2$, and the spectral measure μ_β corresponding to Γ_β is of the form $\mu_\beta(d\lambda) = c_\beta/|\lambda|^{d-\beta}$, with c_β a positive constant. To simplify notation we assume that $d \geq 2$. Then $\mathcal{R} * W_\Gamma$ is function-valued process if and only if $\beta \in]0 , 2[$, see [18]. Moreover, if $\beta \in]0 , 2[$ then for each $t > 0$, $\mathcal{R} * W_\Gamma(t)$, is a continuous random field. To prove the latter statement we use Theorem 4 and show that for some $\delta > 0$,

$$\int_{|\lambda|<1} \frac{1}{|\lambda|^{d-2+\delta}} \Gamma_\beta(\lambda) d\lambda <+ \infty \tag{18}$$

and

$$\int_{|\lambda|\geq 1} \frac{1}{|\lambda|^{d+2-\delta}} \Gamma_\beta(\lambda) d\lambda <+ \infty . \tag{19}$$

Condition (19) is always satisfied because it is equivalent to: $\beta > \delta - 2$. Condition (18) may be replaced by the following one:

$$\int_{|\lambda|<1} \frac{1}{|\lambda|^{d-2+\delta+\beta}} d\lambda = c \int_0^1 \frac{1}{r^{d-2+\delta+\beta}} r^{d-1} dr = c \int_0^1 \frac{1}{r^{\beta-1+\delta}} dr <+ \infty ,$$

equivalent to $\beta < 2 - \delta$.

CASE 4. Assume that $v(t) \equiv 1$ and the operator A is given by the formula ($8a$), where

$$a(\lambda) = \langle Q\lambda , \lambda \rangle + \int_{\mathbb{R}^d} (1 - \cos\langle \lambda , x \rangle) \nu(dx)$$

and ν is a symmetric measure such that

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \, \nu(dx) <+ \infty .$$

Then the equation (8) has a function-valued solution if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1 + a(\lambda)} \mu(d\lambda) <+ \infty .$$

Additionally, if $X_0 = 0$ and

$$\int_{\mathbb{R}^d} (\ln(1 + |\lambda|)^{1+\varepsilon}) \frac{1}{1 + a(\lambda)} \mu(d\lambda) <+ \infty ,$$

then equation (8) has continuous version for each $t \geq 0$.

In this situation, $\mathbf{s}(\sigma\,, a(\lambda)) = e^{-\sigma a(\lambda)}$. By Theorem 2 the condition for the existence of function-valued solution of the equation (8) becomes:

$$\int_{\mathbb{R}^d} \left(\int_0^t (\mathbf{s}(\sigma\,, a(\lambda)))^2 d\sigma \right) \mu(d\lambda) = \int_{\mathbb{R}^d} \int_0^t e^{-2\sigma a(\lambda)} d\sigma\, \mu(d\lambda) <+\infty\,,$$

and it is equivalent to

$$\int_{\mathbb{R}^d} \frac{1}{1 + a(\lambda)}\, \mu(d\lambda) <+\infty\,.$$

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