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Stanislav I. Pohozaev, Alberto Tesei

Blow-up of nonnegative solutions to quasilinear parabolic inequalities

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Equazioni a derivate parziali. — *Blow-up of nonnegative solutions to quasilinear parabolic inequalities.* Nota di Stanislav I. Pohozaev e Alberto Tesei, presentata (*) dal Socio A. Tesei.

Abstract. — We investigate critical exponents for blow-up of nonnegative solutions to a class of parabolic inequalities. The proofs make use of a priori estimates of solutions combined with a simple scaling argument.

Key words: Critical exponent; Parabolic inequalities; Blow-up; Global existence.

Riassunto. — *Esplosione in tempo finito di soluzioni nonnegative di disequazioni paraboliche quasilineari.* Si studia l'esponente critico per l'esplosione in tempo finito di soluzioni nonnegative di una classe di disequazioni paraboliche. Le dimostrazioni fanno uso di stime a priori delle soluzioni, combinate con un semplice argomento di riscalamento.

1. Introduction

In this paper we investigate blow-up of nonnegative solutions to parabolic inequalities of the following type:

$$\rho(x,t)\partial_t(u^k) \geq \sum_{i,j=1}^{n} \partial_{x_i}\left[a_{ij}(x,t,u) f(|\nabla u|)\, \partial_{x_j} u\right] + c(x,t,u)\, u^q \tag{1.1}$$

in $\mathbb{R}^n \times (0,\infty)$; here $k>0$, $q>1$ and ρ, a_{ij}, f, c are given functions (ρ, f, c positive, $a_{ij}=a_{ji}$; precise assumptions are made in Section 2). In particular, our results apply to solutions of the Cauchy problem for parabolic equations of the form (1.1) with the equality sign.

Our purpose is to investigate *critical exponents* for blow-up. To describe the type of results we seek, consider the semilinear Cauchy parabolic problem:

$$\begin{cases} \partial_t u = \Delta u + u^q & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = u_0 & \text{in } \mathbb{R}^n \times \{0\}\,, \end{cases} \tag{1.2}$$

where u_0 is nonnegative, continuous and bounded in $\mathbb{R}^n$. As is well known, if

$$1 < q < 1 + 2/n \tag{1.3}$$

the only global solution to problem (1.2) is trivial; on the other hand, when $q > 1 + 2/n$ global solutions exist if the initial data u_0 is suitably small (see [1]). The number $q_c = q_c(n) := 1 + 2/n$ is called the critical exponent of problem (1.2).

The appearance of a critical exponent larger than one is a typical feature of blow-up for space dependent evolution problems; in fact, it can be regarded as the effect of

(*) Nella seduta del 10 marzo 2000.

competition between diffusion (and/or convection) effects on one side, and reaction on the other. The existence of critical exponents has been widely investigated by different methods, *e.g.* using comparison results, similarity solutions or particular functionals (see [2, 5] and references therein). In general, these methods appear to rely on the specific form of the problem under investigation.

In this paper we approach the problem from a general viewpoint, regarding blow-up as *nonexistence of global solutions* to the evolution problem. We derive sufficient conditions, which ensure nonexistence of global nonnegative solutions to parabolic inequality (1.1) (see Theorem 3.1). In particular, this nonexistence result applies to solutions of the Cauchy problem for the corresponding parabolic equation, thus giving conditions which determine the critical exponent of the problem. When applied to concrete cases dealt with in the literature, our procedure obtains the already known critical exponents (see Theorems 3.2-3.4 below and [2]). On the other hand, it can be applied to a wide class of problems, as apparent from (1.1).

Let us mention that similar methods have been used to prove nonexistence theorems of Liouville type for elliptic inequalities (see [3]). The underlying ideas of the method suggest a general approach to nonexistence problems, which leads to the concept of *nonlinear capacity* (see [4]).

2. Mathematical background

Let S_T denote the strip $\mathbb{R}^n \times (0\,, T]$ $(T \in (0\,,\infty])$; set $S \equiv S_\infty$. The following assumptions will be made throughout the paper:

(a) $\rho \in C(S_T)$, $a_{ij} \in C(S_T \times [0\,,\infty))$, $c \in C(S_T \times [0\,,\infty))$, $f \in C([0\,,\infty))$;

(b) $\rho > 0$, $c > 0$, $\rho(x\,,\cdot)$ nondecreasing for any $x \in \mathbb{R}^n$;

(c) there exist $A_0 = A_0(x\,, t\,, u)$, $A_1 = A_1(x\,, t\,, u) \in C(S_T \times [0\,,\infty))$ such that $0 \le A_0 \le$ $\le A_1$ and there holds:

$$A_0 |\,\xi\,|^2 \le \sum_{i,j=1}^{n} a_{ij}\,\xi_i\,\xi_j \ , \qquad |\sum_{i,j=1}^{n} a_{ij}\,\xi_i\,\eta_j\,| \le A_1\,|\,\xi\,||\,\eta\,|$$

for any $\xi\,,\eta \in \mathbb{R}^n$;

(d) $f \ge 0$ in $[0\,,\infty)$ and for any $t \ge 0$ there holds either

$$(i) \qquad 0 \le f(t) \le c_0 \ , \qquad \text{or} \qquad (ii) \quad c_1\, t^{p-2} \le f(t) \le c_2\, t^{p-2}$$

($c_0 > 0$; $0 < c_1 \le c_2$, $p > 2$).

The above assumptions will be collectively referred to as *Assumption* (H).

Concerning solutions to inequality (1.1) let us make the following definitions.

Definition 2.1. *By a* strong solution *to inequality* (1.1) *in* S_T *we mean any nonnegative function* $u \in C(S_T)$ *such that* (*its distributional derivatives of the first order in time and up to the second order in the space variables are defined almost everywhere in* S_T *and*) *inequality* (1.1) *is satisfied almost everywhere in* S_T.

Definition 2.2. *Let* $\alpha \in (-k, 0)$. *By a* solution of class P_α *to* (1.1) *in* S_T *we mean any nonnegative function* $u \in C(S_T)$ *such that for any test function* $\psi \geq 0$ *with support in* $\overline{S}_T$ *there holds* :

(*i*)

$$\iint_{\text{supp } u} A_1 f(|\nabla u|)\, |\nabla u|\, u^\alpha\, |\nabla \psi| < \infty \,; \tag{2.1}$$

(*ii*)

$$\begin{aligned} &|\alpha| \iint_{\text{supp } u} \Big[\sum_{i,j=1}^n a_{ij}\partial_{x_i} u\, \partial_{x_j} u\Big] f(|\nabla u|)\, u^{\alpha-1}\psi + \iint_{\text{supp } u} c\, u^{q+\alpha}\psi \leq \\ &\leq \iint_{\text{supp } u} \Big[\sum_{i,j=1}^n a_{ij}\partial_{x_i} u\, \partial_{x_j} \psi\Big] f(|\nabla u|)\, u^{\alpha} - \frac{k}{k+\alpha}\iint_{\text{supp } u} \rho\, u^{k+\alpha}\partial_t \psi \,. \end{aligned} \tag{2.2}$$

A solution of class P_α *to* (1.1) *is said to be* global, *if it is such a solution in* S_T *for any* $T > 0$.

It is easy to prove that, due to Assumption (*H*), to condition (2.1) and to the assumption $\alpha > -k$, every integral in inequality (2.2) is finite. Hence Definition 2.2 is well posed.

Concerning the relationship between the above definitions, let us note the following result.

Proposition 2.3. *Let u be a strong solution to inequality* (1.1) *in* S_T, *such that* $\partial_t u \in$ $\in L^1_{\text{loc}}(S_T)$ *and the pointwise limit* $u(\cdot\,, 0) := \lim_{t\to 0} u(\cdot\,, t)$ *is defined and continuous in* $\mathbb{R}^n$. *Let condition* (2.1) *be satisfied. Then u is a solution of class* P_α ($\alpha \in (-k, 0)$).

Proof. Multiplying (1.1) by $u^\alpha \psi$ (where $\alpha < 0$ and $\psi \geq 0$ is any test function with support in S_T) and integrating by parts we obtain:

$$\begin{aligned} &\frac{k}{k+\alpha}\int_{\mathbb{R}^n} u^{k+\alpha}(x, 0)\,\psi(x, 0)dx + \frac{k}{k+\alpha}\iint_{\text{supp } u} \rho\, u^{k+\alpha}\partial_t\psi + \\ &+ \frac{k}{k+\alpha}\iint_{\text{supp } u} (\partial_t\rho)\, u^{k+\alpha}\psi \leq \alpha \iint_{\text{supp } u} \Big[\sum_{i,j=1}^n a_{ij}\partial_{x_i} u\, \partial_{x_j} u\Big] f(|\nabla u|)\, u^{\alpha-1}\psi + \\ &\qquad + \iint_{\text{supp } u} \Big[\sum_{i,j=1}^n a_{ij}\partial_{x_i} u\, \partial_{x_j} \psi\Big] f(|\nabla u|)\, u^{\alpha} - \iint_{\text{supp } u} c\, u^{q+\alpha}\psi \,. \end{aligned}$$

The first two integrals in the left-hand side of the above inequality are finite (since $u(\cdot\,, 0)$ is locally bounded and $\alpha \in (-k, 0)$), while the third is is nonnegative by Assumption (*H*)-(*b*). As for the right-hand side, the second integral is finite by Assumption (*H*)-(*c*) and condition (2.1), while the other two are nonpositive. Hence every integral is finite, thus the claim follows. □

3. Results

In order to prove nonexistence of global solutions of class P_α to inequality (1.1) we shall argue as follows: firstly we prove suitable a priori estimates for such a solution (see Lemma 4.1); secondly, we combine the above estimates with a scaling argument to prove a general nonexistence result (see Theorem 3.1). Let us introduce the following quantities:

$$(3.1) \qquad D = D(x, t, u) := \left(\frac{A_1^p}{A_0^{p-1} c^{\frac{\mu-1}{\mu}}} \right)^{\mu} , \qquad E = E(x, t, u) := \left(\frac{\rho}{c^{\frac{\nu-1}{\nu}}} \right)^{\nu} ,$$

where

$$(3.2) \qquad \mu := \frac{q+\alpha}{q-p+1} , \qquad \nu := \frac{q+\alpha}{q-k} \qquad (\alpha < 0) ;$$

here $p = 2$ if condition (i), respectively $p > 2$ if condition (ii) of Assumption (H)-(d) holds.

Our main nonexistence result can be stated as follows.

Theorem 3.1. *Let $k > 0$, $p \geq 2$, $q > \max\{p-1, k\}$ and Assumption (H) be satisfied. Assume that for some $\alpha \in (-\min\{p-1, k\}, 0)$ there exists $\lambda > 0$ such that*:

$$(3.3) \qquad R^{n+\frac{2}{\lambda}-p\mu} \iint_{\{1 \leq \xi_\lambda \leq 2\}} \left[\sup_{u \geq 0} D(R\xi, R^{\frac{2}{\lambda}}\tau, u) \right] d\xi\, d\tau \longrightarrow 0 \quad \text{as } R \to \infty ,$$

$$(3.4) \qquad R^{n+\frac{2}{\lambda}-\frac{2\nu}{\lambda}} \iint_{\{1 \leq \xi_\lambda \leq 2\}} \left[\sup_{u \geq 0} E(R\xi, R^{\frac{2}{\lambda}}\tau, u) \right] d\xi\, d\tau \longrightarrow 0 \quad \text{as } R \to \infty ,$$

where

$$(3.5) \qquad \xi_\lambda := |\xi|^{2\eta} + |\tau|^{\lambda\eta} , \quad \eta := \max\{1/\lambda, 1\} \qquad (\xi \in \mathbb{R}^n, \tau > 0; \lambda > 0) .$$

Then the only global solution of class P_α to inequality (1.1) *is trivial.*

Let us mention some applications of the above result.

Theorem 3.2. *Let condition* (1.3) *be satisfied. Then there exists $\overline{\alpha} \in (-1, 0)$ such that for any $\alpha \in (\overline{\alpha}, 0)$ the only global solution of class P_α to the inequality*:

$$(3.6) \qquad \partial_t u \geq \sum_{i=1}^{n} \partial_{x_i} \left[\frac{1}{(1+|\nabla u|^2)^\theta} \partial_{x_i} u \right] + u^q$$

in S ($\theta \geq 0$) is trivial.

Theorem 3.3. *Let $m \geq 1$ and there hold*:

$$(3.7) \qquad m < q < m + \frac{2}{n} .$$

Then there exists $\overline{\alpha} \in (-1/m, 0)$ such that for any $\alpha \in (\overline{\alpha}, 0)$ the only global solution of class P_α to the inequality:

$$(3.8) \qquad \partial_t u \geq \Delta(u^m) + u^q$$

in S is trivial.

THEOREM 3.4. *Let $p \geq 2$ and there hold :*

$$p - 1 < q < p - 1 + \frac{p}{n} \,. \tag{3.9}$$

Then there exists $\overline{\alpha} \in (-1\,,0)$ such that for any $\alpha \in (\overline{\alpha}\,,0)$ the only global solution of class P_α to the inequality:

$$\partial_t u \geq \sum_{i=1}^{n} \partial_{x_i} \Big[\mid \nabla u \mid^{p-2} \partial_{x_i} u \Big] + u^q \tag{3.10}$$

in S is trivial.

It can be observed that both conditions (3.7) and (3.9) reduce to the Fujita condition (1.3) when $m = 1$, respectively when $p = 2$. The same condition (1.3) determines the critical exponent for inequality (3.6) for any $\theta \geq 0$.

Due to Proposition 2.3, Theorems 3.2-3.4 imply corresponding results for strong solutions to the Cauchy problem for parabolic equations - in particular, for *classical* positive solutions. We leave their formulation to the reader.

4. PROOFS

Let us first prove Theorem 3.1. For this purpose we need the following lemma (where the quantities D, E, μ, ν are defined by (3.1)-(3.2)).

LEMMA 4.1. *Let $k > 0$, $p \geq 2$, $q > \max\{p-1\,,k\}$ and Assumption (H) be satisfied. Let u be a solution of class P_α to (1.1) in S_T ($\alpha \in (-\min\{p-1\,,k\}\,,0)$). Then there exist $k_1 > 0$, $k_2 > 0$ (depending on k, p, q, α, f) such that*

$$\begin{aligned} &\frac{\mid \alpha \mid}{p} \iint_{\text{supp }u} A_0 \mid \nabla u \mid^p u^{\alpha-1}\psi + \frac{1}{\mu\nu} \iint_{\text{supp }u} c\, u^{q+\alpha}\psi \leq \\ &\qquad \leq k_1 \iint_{\text{supp }u} D\, \frac{\mid \nabla\psi \mid^{p\mu}}{\psi^{p\mu-1}} + k_2 \iint_{\text{supp }u} E\, \frac{\mid \partial_t\psi \mid^{\nu}}{\psi^{\nu-1}} \end{aligned} \tag{4.1}$$

for any test function $\psi \geq 0$ with support in $\overline{S}_T$.

PROOF. By inequality (2.2) in Definition 2.2 and Assumption (H)-(c) there holds:

$$\begin{aligned} &\mid \alpha \mid \iint_{\text{supp }u} A_0 \mid \nabla u \mid^2 f(\mid \nabla u \mid) u^{\alpha-1}\psi + \iint_{\text{supp }u} c\, u^{q+\alpha}\psi \leq \\ &\leq \iint_{\text{supp }u} A_1 \mid \nabla u \mid f(\mid \nabla u \mid) u^{\alpha} \mid \nabla\psi \mid + \frac{k}{k+\alpha} \iint_{\text{supp }u} \rho\, u^{k+\alpha} \mid \partial_t\psi \mid \,. \end{aligned} \tag{4.2}$$

Concerning the first term in the right-hand side of the above inequality, by Young inequality we have on supp ψ:

$$A_1|\nabla u| f(|\nabla u|) u^{\alpha} |\nabla\psi| \leq f(|\nabla u|) \Big\{ \frac{|\alpha|}{2} A_0 |\nabla u|^2 u^{\alpha-1}\psi + \frac{1}{2|\alpha|} \frac{A_1^2}{A_0} u^{1+\alpha} \frac{|\nabla\psi|^2}{\psi} \Big\}.$$

Integrating on supp u both members of the above inequality and inserting the resulting inequality in (4.2) we obtain:

$$
\begin{aligned}
(4.3)\qquad & \frac{|\alpha|}{2}\iint_{\text{supp } u} A_0 |\nabla u|^2 f(|\nabla u|)u^{\alpha-1}\psi + \iint_{\text{supp } u} c\,u^{q+\alpha}\psi \leq \\
& \leq \frac{1}{2|\alpha|}\iint_{\text{supp } u}\frac{A_1^2}{A_0} f(|\nabla u|)u^{1+\alpha}\frac{|\nabla\psi|^2}{\psi} + \frac{k}{k+\alpha}\iint_{\text{supp } u}\rho\, u^{k+\alpha}|\partial_t\psi|\,.
\end{aligned}
$$

(*A*) Suppose first that condition (*i*) of Assumption (*H*)-(*d*) is satisfied. In this case there holds:

$$
\begin{aligned}
\frac{1}{2|\alpha|}\frac{A_1^2}{A_0} f(|\nabla u|)u^{1+\alpha}\frac{|\nabla\psi|^2}{\psi} &\leq \frac{c_0}{2|\alpha|}\frac{A_1^2}{A_0}\,u^{1+\alpha}\frac{|\nabla\psi|^2}{\psi} \leq \\
&\leq \frac{\mu-1}{\mu}c\,u^{q+\alpha}\psi + \frac{1}{\mu}\left(\frac{c_0}{2|\alpha|}\right)^{\mu} D\frac{|\nabla\psi|^{2\mu}}{\psi^{2\mu-1}}
\end{aligned}
$$

(where c_0 is the positive constant in condition (*i*)). Integrating again and inserting the resulting inequality in (4.3) we obtain:

$$
\begin{aligned}
(4.4)\qquad & \frac{|\alpha|}{2}\iint_{\text{supp } u} A_0 |\nabla u|^2 f(|\nabla u|)u^{\alpha-1}\psi + \frac{1}{\mu}\iint_{\text{supp } u} c\,u^{q+\alpha}\psi \leq \\
& \leq \frac{1}{\mu}\left(\frac{c_0}{2|\alpha|}\right)^{\mu}\iint_{\text{supp } u} D\frac{|\nabla\psi|^{2\mu}}{\psi^{2\mu-1}} + \frac{k}{k+\alpha}\iint_{\text{supp } u}\rho\, u^{k+\alpha}|\partial_t\psi|\,.
\end{aligned}
$$

The second integral in the right-hand side of (4.4) can be similarly estimated. In fact, using again Young inequality we obtain:

$$
(4.5)\qquad \frac{k}{k+\alpha}\rho\, u^{k+\alpha}|\partial_t\psi| \leq \frac{\nu-1}{\mu\nu}c\,u^{q+\alpha}\psi + \frac{\mu^{\nu-1}}{\nu}\left(\frac{k}{k+\alpha}\right)^{\nu} E\frac{|\partial_t\psi|^{\nu}}{\psi^{\nu-1}}\,.
$$

From (4.4)-(4.5) we obtain easily:

$$
\begin{aligned}
& \frac{|\alpha|}{2}\iint_{\text{supp } u} A_0 |\nabla u|^2 f(|\nabla u|)u^{\alpha-1}\psi + \frac{1}{\mu\nu}\iint_{\text{supp } u} c\,u^{q+\alpha}\psi \leq \\
& \leq \frac{1}{\mu}\left(\frac{c_0}{2|\alpha|}\right)^{\mu}\iint_{\text{supp } u} D\frac{|\nabla\psi|^{2\mu}}{\psi^{2\mu-1}} + \frac{\mu^{\nu-1}}{\nu}\left(\frac{k}{k+\alpha}\right)^{\nu}\iint_{\text{supp } u} E\frac{|\partial_t\psi|^{\nu}}{|\psi|^{\nu-1}}\,.
\end{aligned}
$$

Then the conclusion follows in this case.

(*B*) Let condition (*ii*) of Assumption (*H*)-(*d*) be satisfied. In this case from inequality (4.2) we obtain:

$$
\begin{aligned}
(4.6)\qquad & |\alpha|\,c_1\iint_{\text{supp } u} A_0|\nabla u|^p\; u^{\alpha-1}\psi + \iint_{\text{supp } u} c\,u^{q+\alpha}\psi \leq \\
& \leq c_2\iint_{\text{supp } u} A_1|\nabla u|^{p-1}\; u^{\alpha}\,|\nabla\psi| + \frac{k}{k+\alpha}\iint_{\text{supp } u}\rho\, u^{k+\alpha}|\partial_t\psi|
\end{aligned}
$$

(where c_1 and c_2 appear in condition (*ii*)). Using repeatedly the Young inequality we have on supp ψ:

$$
\begin{aligned}
&c_2 A_1 \mid \nabla u \mid^{p-1} \; u^{\alpha} \mid \nabla \psi \mid \leq \\
&\leq \frac{\mid \alpha \mid c_1}{2} A_0 \mid \nabla u \mid^{p} \; u^{\alpha-1}\psi + \frac{c_2}{2 \mid \alpha \mid c_1} \frac{A_1^2}{A_0} \mid \nabla u \mid^{p-2} \; u^{1+\alpha} \frac{\mid \nabla \psi \mid^2}{\psi} \leq \\
&\leq \mid \alpha \mid c_1 \frac{p-1}{p} A_0 \mid \nabla u \mid^{p} \; u^{\alpha-1}\psi + \frac{c_2^{\frac{p}{2}}}{p(\mid \alpha \mid c_1)^{p-1}} \frac{A_1^p}{A_0^{p-1}} u^{p-1+\alpha} \frac{\mid \nabla \psi \mid^p}{\psi^{p-1}} \leq \\
&\leq \mid \alpha \mid c_1 \frac{p-1}{p} A_0 \mid \nabla u \mid^{p} \; u^{\alpha-1}\psi + \frac{\mu-1}{\mu} c\, u^{q+\alpha}\psi + \frac{1}{\mu}\Big[\frac{c_2^{\frac{p}{2}}}{p(\mid \alpha \mid c_1)^{p-1}}\Big]^{\mu} D \frac{\mid \nabla \psi \mid^{p\mu}}{\psi^{p\mu-1}} \,.
\end{aligned}
$$

By integrating and inserting the resulting inequality in (4.6) we obtain:

$$
\begin{aligned}
&\frac{\mid \alpha \mid c_1}{p} \iint_{\text{supp } u} A_0 \mid \nabla u \mid^{p} \; u^{\alpha-1}\psi + \frac{1}{\mu} \iint_{\text{supp } u} c\, u^{q+\alpha}\psi \leq \\
&\qquad \leq \frac{1}{\mu}\Big[\frac{c_2^{\frac{p}{2}}}{p(\mid \alpha \mid c_1)^{p-1}}\Big]^{\mu} \iint_{\text{supp } u} D \frac{\mid \nabla \psi \mid^{p\mu}}{\psi^{p\mu-1}} + \frac{k}{k+\alpha} \iint_{\text{supp } u} \rho\, u^{k+\alpha} \mid \partial_t \psi \mid .
\end{aligned}
$$

The second integral in the right-hand side of the above inequality can be dealt with as in (A) above. Then the conclusion follows. □

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. Let u be a global solution of class P_α to (1.1). We shall prove the following claim:

For any $\lambda > 0$ there exist $K_1 > 0$, $K_2 > 0$ (depending on λ, q, p, k, α, f) such that for any $R > 0$

$$
\begin{aligned}
&\frac{\mid \alpha \mid}{p} \iint_{B_{\lambda,R}} A_0 \mid \nabla u \mid^{p} \; u^{\alpha-1} dx\, dt + \frac{1}{\mu\nu} \iint_{B_{\lambda,R}} c\, u^{q+\alpha} dx\, dt \leq \\
&\qquad \leq K_1 R^{n+\frac{2}{\lambda}-p\mu} \iint_{\{1\leq \xi_\lambda \leq 2\}} \Big[\sup_{u\geq 0} D(R\xi\,,\, R^{\frac{2}{\lambda}}\tau\,,\, u)\Big] d\xi\; d\tau \; + \\
&\qquad\quad + K_2 R^{n+\frac{2}{\lambda}-\frac{2\nu}{\lambda}} \iint_{\{1\leq \xi_\lambda \leq 2\}} \Big[\sup_{u\geq 0} E(R\xi\,,\, R^{\frac{2}{\lambda}}\tau\,,\, u)\Big] d\xi\; d\tau \;,
\end{aligned} \tag{4.7}
$$

where ξ_λ is defined by (3.5) and $B_{\lambda,R} := \text{supp}\, u \bigcap \{(x, t) \in S \mid \mid x \mid^{2\eta} + t^{\lambda\eta} \leq R^{2\eta}\}$ ($\lambda > 0\,, R > 0\,, \eta := \max\{1/\lambda\,, 1\}$).

From inequality (4.7) the conclusion easily follows. In fact, let $Q \subseteq S$ be any bounded subset; choose $\lambda > 0$ such that conditions (3.3)-(3.4) are satisfied. Since the family $\{B_{\lambda,R}\}$ is nondecreasing in R and supp $u = \bigcup_{R>0} B_{\lambda,R}$, there exists $R_1 > 0$ such that supp $u \bigcap Q \subseteq B_{\lambda,R_1}$. Moreover, due to assumptions (3.3)-(3.4), for any $\epsilon > 0$ there exists $R_2 > 0$ such that for any $R > R_2$:

$$
\begin{aligned}
&R^{n+\frac{2}{\lambda}-p\mu} \iint_{\{1\leq \xi_\lambda \leq 2\}} \Big[\sup_{u\geq 0} D(R\xi\,,\, R^{\frac{2}{\lambda}}\tau\,,\, u)\Big] d\xi\; d\tau \; < \epsilon \,, \\
&R^{n+\frac{2}{\lambda}-\frac{2\nu}{\lambda}} \iint_{\{1\leq \xi_\lambda \leq 2\}} \Big[\sup_{u\geq 0} E(R\xi\,,\, R^{\frac{2}{\lambda}}\tau\,,\, u)\Big] d\xi\; d\tau < \epsilon \,.
\end{aligned}
$$

Set $\overline{R} := \max\{R_1\,,\,2R_2\}$; from (4.7) we obtain:

$$\frac{|\,\alpha\,|}{p}\iint_{\mathrm{supp}\ u\cap Q} A_0\,|\,\nabla u\,|^p\ u^{\alpha-1} + \frac{1}{\mu\nu}\iint_{\mathrm{supp}\ u\cap Q} c\,u^{q+\alpha} \leq$$
$$\leq \frac{|\,\alpha\,|}{p}\iint_{B_{\lambda,\overline{R}}} A_0\,|\,\nabla u\,|^p\ u^{\alpha-1} + \frac{1}{\mu\nu}\iint_{B_{\lambda,\overline{R}}} c\,u^{q+\alpha} < (K_1+K_2)\epsilon\ .$$

Due to the arbitrariness of Q and ϵ the conclusion follows.

It remains to prove inequality (4.7). For this purpose, consider any smooth function $\psi_0 : [0\,,\infty) \to [0\,,1]$ with the following properties:

(i) $\psi_0 \equiv 1$ in $[0\,,1]$, $\psi_0 \equiv 0$ in $[2\,,\infty)$, ψ_0 nonincreasing;

(ii) there holds:

$$\sup_{\xi\in[1\,,2]} \frac{|\,\psi_0'(\xi)\,|^{p\mu}}{\psi_0^{p\mu-1}(\xi)} < \infty\ , \qquad \sup_{\xi\in[1\,,2]} \frac{|\,\psi_0'(\xi)\,|^{\nu}}{\psi_0^{\nu-1}(\xi)} < \infty\ .$$

Define

$$\xi_{\lambda\,,R} := \frac{|\,x\,|^{2\eta} + t^{\lambda\eta}}{R^{2\eta}}\ , \qquad \psi_{\lambda\,,R}(x,\,t) := \psi_0\,(\xi_{\lambda\,,R}) \qquad (\lambda > 0\,,\,R > 0)\ .$$

Since

$$\mathrm{supp}\,\psi_{\lambda\,,R} = \{\xi_{\lambda\,,R} \leq 2\}\ ,$$

setting $\psi = \psi_{\lambda\,,R}$ in inequality (4.1) we obtain:

$$\frac{|\,\alpha\,|}{p}\iint_{B_{\lambda\,,R}} A_0\,|\,\nabla u\,|^p\ u^{\alpha-1} + \frac{1}{\mu\nu}\iint_{B_{\lambda\,,R}} c\,u^{q+\alpha} \leq$$

$$\leq \frac{|\,\alpha\,|}{p}\iint_{\mathrm{supp}\ u\cap\{\xi_{\lambda\,,R}\leq 2\}} A_0\,|\,\nabla u\,|^p\ u^{\alpha-1}\psi_{\lambda\,,R} + \frac{1}{\mu\nu}\iint_{\mathrm{supp}\ u\cap\{\xi_{\lambda\,,R}\leq 2\}} c\,u^{q+\alpha}\psi_{\lambda\,,R} \leq$$

(4.8)
$$\leq \int_{\mathrm{supp}\ u\cap\{1\leq\xi_{\lambda\,,R}\leq 2\}} D\,\frac{|\,\nabla\psi_{\lambda\,,R}\,|^{p\mu}}{\psi_{\lambda\,,R}^{p\mu-1}} + k_2\iint_{\mathrm{supp}\ u\cap\{1\leq\xi_{\lambda\,,R}\leq 2\}} E\,\frac{|\,\partial_t\psi_{\lambda\,,R}\,|^{\nu}}{\psi_{\lambda\,,R}^{\nu-1}} \leq$$

$$\leq k_1\iint_{\{1\leq\xi_{\lambda\,,R}\leq 2\}} [\sup_{u\geq 0} D]\,\frac{|\,\nabla\psi_{\lambda\,,R}\,|^{p\mu}}{\psi_{\lambda\,,R}^{p\mu-1}} + k_2\iint_{\{1\leq\xi_{\lambda\,,R}\leq 2\}} [\sup_{u\geq 0} E]\,\frac{|\,\partial_t\psi_{\lambda\,,R}\,|^{\nu}}{\psi_{\lambda\,,R}^{\nu-1}}\ .$$

It is easily seen that

$$|\,\nabla\psi_{\lambda\,,R}\,| = 2\eta|\psi_0'(\xi)|\,\frac{|x|^{2\eta-1}}{R^{2\eta}}\ ,$$
$$|\,\partial_t\psi_{\lambda\,,R}\,| = \lambda\eta|\psi_0'(\xi)|\,\frac{t^{\lambda\eta-1}}{R^{2\eta}}\ .$$

Then from (4.8) we obtain:

$$\frac{|\alpha|}{p}\iint_{B_{\lambda,R}} A_0 \,|\nabla u|^p\; u^{\alpha-1} + \frac{1}{\mu\nu}\iint_{B_{\lambda,R}} c\,u^{q+\alpha} \leq$$

$$(4.9)\qquad \leq k_1\left(\frac{2\eta}{R^{2\eta}}\right)^{p\mu}\iint_{\{1\leq\xi_{\lambda,R}\leq 2\}}\left[\sup_{u\geq 0} D\right]\frac{|\psi_0'(\xi_{\lambda,R})|^{p\mu}}{\psi_0(\xi_{\lambda,R})^{p\mu-1}}\,|x|^{(2\eta-1)p\mu} +$$

$$+ k_2\left(\frac{\lambda\eta}{R^{2\eta}}\right)^{\nu}\iint_{\{1\leq\xi_{\lambda,R}\leq 2\}}\left[\sup_{u\geq 0} E\right]\frac{|\psi_0'(\xi_{\lambda,R})|^{\nu}}{\psi_0(\xi_{\lambda,R})^{\nu-1}}\,t^{(\lambda\eta-1)\nu}\,.$$

Introducing the scaled variables

$$\xi := x/R\;, \qquad \tau := t/R^{2/\lambda}$$

there holds:

$$\xi_\lambda := |\,\xi\,|^{2\eta} + \tau^{\lambda\eta} = \xi_{\lambda,R} \qquad (\lambda > 0\,,\, R > 0)\,.$$

Then inequality (4.9) reads:

$$\frac{|\alpha|}{p}\iint_{B_{\lambda,R}} A_0\,|\nabla u|^p\; u^{\alpha-1} + \frac{1}{\mu\nu}\iint_{B_{\lambda,R}} c\,u^{q+\alpha} \leq$$

$$\leq (4\eta)^{p\mu} k_1 \sup_{\xi\in[1,2]}\frac{|\psi_0'(\xi)|^{p\mu}}{\psi_0^{p\mu-1}(\xi)}\,R^{n+\frac{2}{\lambda}-p\mu}\iint_{\{1\leq\xi_\lambda\leq 2\}}\left[\sup_{u\geq 0} D(R\xi\,,\,R^{\frac{2}{\lambda}}\tau\,,\,u)\right]d\xi\,d\tau\,+$$

$$+\,2(\lambda\eta)^{\nu} k_2 \sup_{\xi\in[1,2]}\frac{|\psi_0'(\xi)|^{\nu}}{\psi_0^{\nu-1}(\xi)}\,R^{n+\frac{2}{\lambda}-\frac{2\nu}{\lambda}}\iint_{\{1\leq\xi_\lambda\leq 2\}}\left[\sup_{u\geq 0} E(R\xi\,,\,R^{\frac{2}{\lambda}}\tau\,,\,u)\right]d\xi\,d\tau\,.$$

Then by a proper definition of K_1,K_2 the conclusion follows. □

Proving Theorems 3.2-3.4 amounts to show that the conditions of Theorem 3.1 are satisfied.

Proof of Theorem 3.2. In the present case $k=1$, $\rho = c \equiv 1$, $a_{ij}\equiv\delta_{ij}$, $f(t) = (1 + t^2)^{-\theta}$. Then Assumption (H) is satisfied; in particular, condition (i) in (H)-(d) holds. It follows that $D = E \equiv 1$, $p = 2$ and

$$\mu = \nu = \frac{q+\alpha}{q-1}\,.$$

Both conditions (3.3)-(3.4) are satisfied with $\lambda = 1$ if

$$n + 2 - \frac{2(q+\alpha)}{q-1} < 0\,.$$

Due to condition (1.3), the above inequality holds for $\alpha = 0$; hence the conclusion follows. □

Proof of Theorem 3.3. Setting $k := 1/m$ and $v := u^{1/k}$, inequality (3.8) reads

$$\partial_t(v^k) \geq k\Delta v + v^{kq}\;,$$

which is of the form (1.1) with $\rho = c \equiv 1$, $a_{ij} \equiv k\delta_{ij}$, $c \equiv 1$ and q replaced by kq. Since condition (i) in (H)-(d) holds, we have $p = 2$. It is easily seen that $D \equiv k^{\mu}$, $E \equiv 1$ and

$$\mu = \frac{kq+\alpha}{kq-1} \quad , \qquad \nu = \frac{kq+\alpha}{k(q-1)} \; .$$

Then conditions (3.3)- (3.4) are satisfied if

$$\begin{cases} n + \dfrac{2}{\lambda} - \dfrac{2(kq+\alpha)}{kq-1} < 0 \\[2ex] n + \dfrac{2}{\lambda} - \dfrac{2(kq+\alpha)}{\lambda k(q-1)} < 0 \; . \end{cases}$$

As in the proof of Theorem 3.4, we choose λ such that the above inequalities reduce to the same for $\alpha = 0$. This gives

$$\lambda = \overline{\lambda} := \frac{kq-1}{k(q-1)} > 0 \; ,$$

since $q > m = 1/k$ by condition (3.7). Hence both inequalities above are satisfied at $\alpha = 0$ for $\lambda = \overline{\lambda}$ if

$$n + \frac{2k(q-1)}{kq-1} - \frac{2kq}{kq-1} = n - \frac{2}{q-m} < 0 \; .$$

Then by condition (3.7) the conclusion follows. □

PROOF OF THEOREM 3.4. In the present case $k = 1$, $\rho = c \equiv 1$, $a_{ij} \equiv \delta_{ij}$, $f(t) = t^{p-2}$. Hence Assumption (H) is satisfied; in particular, condition (ii) in (H)-(d) holds. It is easily seen that $D = E \equiv 1$ and

$$\mu = \frac{q+\alpha}{q-p+1} \quad , \qquad \nu = \frac{q+\alpha}{q-1} \; .$$

Then conditions (3.3)-(3.4) are satisfied if

$$\begin{cases} n + \dfrac{2}{\lambda} - \dfrac{p(q+\alpha)}{q-p+1} < 0 \\[2ex] n + \dfrac{2}{\lambda} - \dfrac{2(q+\alpha)}{\lambda(q-1)} < 0 \; . \end{cases}$$

As above, we determine λ by requiring the left-hand sides of the above inequalities to be equal for $\alpha = 0$. This gives

$$\lambda = \overline{\lambda} := \frac{2(q-p+1)}{p(q-1)} > 0 \; .$$

Hence both inequalities above are satisfied at $\alpha = 0$ for $\lambda = \overline{\lambda}$ if

$$n + \frac{2}{\overline{\lambda}} - \frac{pq}{q-p+1} = n - \frac{p}{q-p+1} < 0 \; .$$

Then by condition (3.9) the conclusion follows. □

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S. I. Pohozaev:
Steklov Mathematical Institute
Russian Academy of Sciences
Gubkina, 8 - 117966 Moscow (Russia)
pohozaev@mi.ras.ru

A. Tesei:
Dipartimento di Matematica «G. Castelnuovo»
Università degli Studi di Roma «La Sapienza»
Piazzale A. Moro, 5 - 00185 Roma
tesei@mat.uniroma1.it