Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

# Rendiconti Lincei
# Matematica e Applicazioni

Luisa Mazzi, Marco Sabatini

## Commutators and linearizations of isochronous centers

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9*, Vol. **11** (2000), n.2, p. 81–98.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2000_9_11_2_81_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2000.

*Rend. Mat. Acc. Lincei*
*s. 9, v. 11:81-98 (2000)*

**Equazioni differenziali ordinarie.** — *Commutators and linearizations of isochronous centers.* Nota (*) di Luisa Mazzi e Marco Sabatini, presentata dal Socio R. Conti.

Abstract. — We study isochronous centers of some classes of plane differential systems. We consider systems with constant angular speed, both with homogeneous and nonhomogenous nonlinearities. We show how to construct linearizations and first integrals of such systems, if a commutator is known. Commutators are found for some classes of systems. The results obtained are used to prove the isochronicity of some classes of centers, and to find first integrals for a class of Liénard equations with isochronous centers.

Key words: Polynomial systems; Isochronous centers; Commuting vector fields; Linearizations; First integrals; Liénard systems.

Riassunto. — *Commutatori e linearizzazioni di centri isocroni.* Si studiano centri isocroni di alcune classi di sistemi differenziali piani. Si considerano sistemi con velocità angolare costante, sia con nonlinearità omogenee, sia con nonlinearità non omogenee. Si mostra come, a partire da un commutatore, sia possibile costruire una linearizzazione ed un integrale primo. Si trovano commutatori per alcune classi di sistemi. I risultati ottenuti vengono applicati per dimostrare l'isocronia di alcune classi di centri, e per trovare integrali primi per una classe di equazioni di Liénard con centri isocroni.

## Introduction

Let us consider an autonomous differential system in the plane:

$$(S) \qquad \begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y), \end{cases}$$

with $(x, y) \in U$, open connected subset of $\mathbb{R}^2$ containing the origin $O$, and $f, g \in C^2(U, \mathbb{R})$. We assume that $O$ is a critical point of $(S)$. $O$ is said to be a center if every orbit in a punctured neighbourhood of $O$ is a nontrivial cycle. It is said to be an isochronous center if every cycle in a punctured neighbourhood of $O$ has the same period. Several papers were devoted to study conditions under which the origin is a center, or an isochronous center (see [1, 6, 12, and references therein]).

Aside from its interest in physical applications, isochronicity is strictly related to the existence and uniqueness of solutions of some boundary value, bifurcation or perturbation problems. Moreover, isochronicity has a strong relationship to stability: a periodic solution of the central region is Liapunov stable if and only if the neighbouring periodic solutions have the same period.

Several methods have been used in attacking the isochronicity problem. One of the most effective consists in looking for a linearization $\Phi$, that is a local diffeomorphism $\Phi$ transforming $(S)$ into a linear center. By a classical result, every analytic system with an isochronous center admits a linearization [8, Theorem 5.2.8]. Recently, this result

(*) Pervenuta in forma definitiva all'Accademia il 19 ottobre 1999.

has been extended to systems of class $C^k$ having an isochronous center [18]. Looking for a linearization requires to cope with nontrivial computational problems, but has significant advantages, since this approach usually leads to find a first integral, that allows to give a description of the phase portrait of $(S)$. Recently, Mardešić *et al.* [12] found linearizations for several classes of polynomial systems.

Another method that proved to be useful in dealing with isochronous centers makes use of commutators. Let

$$(S^*) \qquad \begin{cases} \dot{x} = r(x, y) \\ \dot{y} = s(x, y), \end{cases}$$

be a second differential system defined on $U$. We say that $(S)$ and $(S^*)$ are *transversal at* $(x, y)$ if $f(x, y)s(x, y) - g(x, y)r(x, y) \neq 0$. Assume that $(S)$ and $(S^*)$ are transversal at noncritical points of $(S)$. Let us call $\alpha(t, x, y)$ (resp. $\beta(s, x, y)$) the solution of $(S)$ (resp. $S^*$) such that $\alpha(0, x, y) = (x, y)$ (resp. $\beta(0, x, y) = (x, y)$). Consider a set $\overline{U} \subset U$, positively invariant both for $\alpha$ and for $\beta$. Then $(S)$ and $(S^*)$ are said to *commute* on $\overline{U}$ if and only if, for all $s \geq 0$, $t \geq 0$:

$$\alpha(t, \beta(s, x, y)) = \beta(s, \alpha(t, x, y)).$$

Since a system has an isochronous center if and only if it has a nontrivial commutator [14], an approach to the isochronicity problem, alternative to linearization, consists in looking for nontrivial commutators. Finding commutators is sometimes easier than finding linearizations. For instance, every system of the type:

$$(S_i) \qquad \dot{z} = iP(z)$$

commutes with its orthogonal system [17, §4], while its linearization is not elementary [12, §6].

There exists a natural relationship between linearizations and commutators. If a linearization $\Phi$ is known, then a nontrivial commutator can be easily found. In fact, $\Phi$ takes the system $(S)$ into a linear system $(L_c)$, that has a linear commutator $(L_n)$. The system obtained from $(L_n)$ by the inverse transformation $\Phi^{-1}$ is a commutator of $(S)$. A general converse procedure does not yet exist. In fact, it is not known how to obtain a linearization, if a nontrivial commutator is known. In this paper, we give such a procedure for a special class of systems. We consider systems with *radial nonlinearities*:

$$(S_H) \qquad \begin{cases} \dot{x} = -y + xH(x, y) \\ \dot{y} = \phantom{-}x + yH(x, y), \end{cases} \qquad H(0, 0) = 0,$$

that is, systems with constant angular speed. In fact, using polar coordinates $\rho$, $\theta$, $(S)$ becomes:

$$\begin{cases} \dot{\rho} = \rho H(\rho\cos\theta, \rho\sin\theta) \\ \dot{\theta} = 1. \end{cases}$$

Due to such a property, for such systems it is equivalent to prove that $O$ is a center, and to prove that $O$ is an isochronous center. In other words, the isochronicity problem is equivalent to the integrability problem.

Systems of the type $(S_H)$ have already been considered in previous papers [3, 5, 10, 12]. Moreover, they appear in the study of the isochronicity of Liénard equation [16]. In fact, if $H(x, y)$ does not depend on $y$, hence $H(x, y) \equiv h(x)$, then $(S_H)$ is equivalent to the second order differential equation:

$$x'' + f(x)x' + g(x) = 0\,,$$

where $f(x) = -2h(x) - xh'(x)$, $g(x) = x + xh(x)^2$.

This paper is organized as follows.

In Section 1, we first collect some general results about couples of commuting systems. Some of them can be easily obtained from the results in [14, 15]. Theorem 1.4, dealing with linearizations of couples of commuting systems, is new. It has been independently proved in [18]. Then we show that a system of type $(S_H)$ having an isochronous center always admits a commutator of type:

$$(S_{\widetilde{H}}) \qquad \begin{cases} \dot{x} = x\widetilde{H}(x, y) \\ \dot{y} = y\widetilde{H}(x, y). \end{cases}$$

In Section 2 we consider systems of type $(S_H)$, where $H$ is a nonhomogeneous function. We show that if $(S_H)$ commutes with a system of type:

$$(S_K) \qquad \begin{cases} \dot{x} = x(1 + K(x, y)) \\ \dot{y} = y(1 + K(x, y))\,, \end{cases} \qquad K(0\,, 0) = 0\,,$$

then the origin is an isochronous center for $(S_H)$. Applying the results of Section 1, we find a *radial linearization*, that is a linearization of the form:

$$(\Phi_E) \qquad \begin{cases} u = xE(x, y) \\ v = yE(x, y)\,, \end{cases}$$

where $E$ is a suitable scalar function. Using $(\Phi_E)$, we can give a first integral of $(S_H)$.

In Section 2 we also find commutators, linearizations and first integrals for two classes of systems:

- systems of the form $(S_H)$, for which $H$ is harmonic and subject to a supplementary condition (see Section 2);

- systems of the form $(S_H)$, for which $H(x, y) = x\sigma(y)$, where $\sigma(y)$ is an arbitrary function of class $C^2$. As special cases, we have:

for $\sigma(y) \equiv$ *const.*, a class of quadratic systems considered in [10, Theorem 4];
for $\sigma(y) = ay$, a class of cubic systems studied in [13, 4, 12];
for $\sigma(y) = ay + b$, a class of cubic systems studied in [12, 3].

In Section 3 we consider systems of type $(S_H)$, where $H$ is a homogeneous function. We find conditions under which a commutator of type $(S_K)$, with $K$ homogeneous of the same degree of $H$, exists. In particular, this occurs if $H$ is a homogeneous harmonic function. We also give a commutator-based proof of some results obtained in [5]. For systems with $H$ homogeneous and polynomial, we are able to give a rational first integral of $(S_H)$. This allows to give a rational first integral for a system equivalent to the Liénard equation

$$x'' - (n+1)x'x^{n-1} + x + x^{2n-1} = 0.$$

## 1. Commuting vector fields

In Section 1 we recall some of the results obtained in [14, 15] concerning couples of commuting vector fields, and we prove some new results about them. We denote by $V(x, y)$, $W(x, y)$ two vector fields defined on an open subset $U$ of the real plane, by $(S_V)$, $(S_W)$ the associated systems of ordinary differential equations, and by $\alpha(t; x, y)$, $\beta(s; x, y)$ their local flows. Throughout this paper we deal with vector fields and systems of class $\mathcal{C}^2$, except when otherwise specified.

First of all, we recall that if $\alpha(t; x, y)$ and $\beta(s; x, y)$ commute, then the Lie brackets $[V, W]$ of $V$ and $W$ vanish identically:

$$(1.1) \qquad \begin{cases} [V, W]_1 = \left(v_1\,\partial_x w_1 - w_1\,\partial_x v_1\right) + \left(v_2\,\partial_y w_1 - w_2\,\partial_y v_1\right) \equiv 0 \\ [V, W]_2 = \left(v_1\,\partial_x w_2 - w_1\,\partial_x v_2\right) + \left(v_2\,\partial_y w_2 - w_2\,\partial_y v_2\right) \equiv 0\,. \end{cases}$$

Conversely, if $[V, W] \equiv 0$ on an open set $U$, then the local flows $\alpha$, $\beta$ commute on every subset $\overline{U}$ of $U$ that is invariant for both local flows. We say that $W$ is a commutator for $V$ if $[V, W] \equiv 0$. A commutator for $V$ on $\overline{U}$ is *nontrivial* if $V$, $W$ are transversal at their noncritical points. For sake of brevity, in general we shall not specify the set where $V$ and $W$ commute.

If $O$ is a center for $V$, we denote by $N_O$ the largest open connected region covered by cycles of $V$ surrounding $O$. When all the cycles have the same minimal period, the center is said to be *isochronous*.

Theorem 1.1. *Let $V$ have a center $O$. Then, $V$ has a non trivial commutator in a neighbourhood of $O$ if and only if $O$ is an isochronous center.*

For a proof, see [14].

For a definition of families of rotated vector fields, quoted in next theorem, see [7].

Theorem 1.2. *Let $V$ have a critical point $O$ and $W$ be a nontrivial commutator of $V$. Let us consider the family of rotated vector fields:*

$$\{V \cos\theta + W \sin\theta\,,\, \theta \in [0\,, 2\pi)\}.$$

*Then there exists a unique $\theta^* \in [0\,, \pi)$ such that $O$ is a center for $(S_V)$.*

For a proof, see [15]. As a consequence, such a family has exactly two centers, one for $\theta^* \in [0\,, \pi)$, and the other one for $\theta^* + \pi$.

As a consequence of this theorem, we have the following theorem (see also [15, Theorem 1.4 and Remark 1.4]).

Theorem 1.3. *If the system*

$$(S_v) \qquad \begin{cases} \dot{x} = -y + p(x, y) \\ \dot{y} = \phantom{-y}x + q(x, y) \end{cases}$$

*commutes with the system*

$$(S_w) \qquad \begin{cases} \dot{x} = x + r(x, y) \\ \dot{y} = y + s(x, y), \end{cases}$$

*where $p$, $q$, $r$, $s$ and their first partial derivatives vanish at the origin, then $V$ has an isochronous center at the origin.*

Proof. Let us consider the complete family of rotated vector fields of Theorem 1.2. The corresponding differential systems are:

$$(S_\theta) \qquad \begin{cases} \dot{x} = (\sin\theta)x - (\cos\theta)y + l(x, y) \\ \dot{y} = (\cos\theta)x + (\sin\theta)y + m(x, y) \end{cases}$$

where $l(x, y) = p(x, y)(\cos\theta) + r(x, y)(\sin\theta)$, $m(x, y) = q(x, y)(\cos\theta) + s(x, y)(\sin\theta)$ and their first partial derivatives vanish at the origin. When $\theta \neq 0, \pi$, the eigenvalues of the linear part of $(S_\theta)$ at $O$ have real parts different from zero. Therefore the family has its two centers at $\theta = 0, \pi$. $\square$

Definition 1.1. We say that a plane differential system $(S)$ is $C^k$-linearizable $(k = 1, \dots, \omega)$ at a point $z$ if there exists a neighbourhood $U_z$ of $z$ and a $C^k$-diffeomorphism $\phi_z : U_z \to \mathbb{R}^2$ that transforms $(S)$ into a linear system. We say that a critical point $O$ of $(S)$ is $C^k$-linearizable if $(S)$ is $C^k$-linearizable at $O$. When the order of differentiability of $\phi_z$ is not specified, we consider $C^1$-diffeomorphisms.

Theorem 1.4. *Let the system:*

$$(S_v) \qquad \begin{cases} \dot{x} = -y + p(x, y) \\ \dot{y} = \phantom{-}x + q(x, y) \end{cases}$$

*commute with the system:*

$$(S_w) \qquad \begin{cases} \dot{x} = x + r(x, y) \\ \dot{y} = y + s(x, y), \end{cases}$$

*on the open connected set $U$, where $p$, $q$, $r$, $s$ and their first partial derivatives vanish at the origin $O \in U$. If the map $\phi \in C^1(U, \mathbb{R}^2) \cap C^2(U \setminus \{O\}, \mathbb{R}^2)$ linearizes $(S_w)$ at $O$, then $\phi$ linearizes $(S_v)$ at $O$ as well.*

Proof. Without loss of generality, we can assume that $\phi(O) = O$, and that $\phi$ transforms $(S_w)$ into the system:

$$(S_w^*) \qquad \begin{cases} \dot{X} = X \\ \dot{Y} = Y. \end{cases}$$

Let us denote by

$$(S_v^*) \qquad \begin{cases} \dot{X} = aX + bY + P^*(X, Y) \\ \dot{Y} = cX + dY + Q^*(X, Y) \end{cases}$$

the system obtained from $(S_v)$ by applying $\phi$. $P^*$, $Q^*$ and their first partial derivatives vanish at the origin. Moreover, $(S_V^*)$ and $(S_W^*)$ commute. Since $(S_v^*)$ and $(S_w^*)$ are of class $\mathcal{C}^2$ but possibly at $O$, their Lie brackets have to vanish, but possibly at $O$:

$$\begin{cases} X\,\partial_X P^* + Y\,\partial_Y P^* = P^* \\ X\,\partial_X Q^* + Y\,\partial_Y Q^* = Q^*. \end{cases}$$

By continuity, such equalities hold also at the origin. By Euler's theorem, $P^*$ and $Q^*$ are homogeneous of degree 1. Their derivatives are homogeneous of degree zero, hence they are constant along the lines through the origin, where they vanish. Hence $P^*$ and $Q^*$ are identically zero. This shows that the transformation considered linearizes $(S_v)$ as well. □

In Sections 2 and 3 we shall be concerned with systems of constant angular speed. Such systems have some special properties that make easier to find commutators and linearizations. Next theorem shows that there exist commutators of a simple form.

Theorem 1.5. *The system* $(S_H)$, *of class* $C^k$, $k = 2, \dots, \infty$, *has an isochronous center at* $O$ *if and only if it has a commutator of the form*:

$$(S_{\widetilde{H}}) \qquad \begin{cases} \dot{x} = x\widetilde{H}(x, y) \\ \dot{y} = y\widetilde{H}(x, y)\,, \end{cases}$$

*of class* $C^k$, $k = 2, \dots, \infty$.

Proof. This proof is modelled on the proof of Theorem 1 in [14]. That proof consists of two parts. A geometric one, developped in Lemma 1 of [14], where a commutator of class $C^k(U^* \setminus \{O\}, R^2) \cap C^0(U^*, R^2)$ is constructed in a neighbourhood $U^*$ of $O$. An analytic one, where a commutator of class $C^k(U^*, R^2)$ is constructed. The latter has the same orbits as the former. In the proof of Lemma 1 of [14], a line segment $\Sigma_1$ is used as a section of the vector field, in order to construct the orbits of the first commutator. The system $(S_H)$ in polar coordinates $(\rho, \theta)$ has the form:

$$(S_p) \qquad \begin{cases} \dot{\rho} = \rho H(\rho\cos\theta, \rho\sin\theta) \equiv \gamma(\rho, \theta) \\ \dot{\theta} = 1. \end{cases}$$

Hence we can take a ray contained in the $x$-axis, $\{(x, y) : x > 0, y = 0\}$, as a section of the corresponding vector field. The local flow of $(S)$ transforms rays into rays, so that the orbits of the first commutator are rays. Hence the first commutator has the form:

$$\begin{cases} \dot{\rho} = \delta(\rho\cos\theta, \rho\sin\theta) \\ \dot{\theta} = 0\,, \end{cases}$$

for some function $\delta$. The procedure of Theorem 1 in [14] does not change the orbits of the commutator, hence the second commutator, of class $C^k$ in a neighbourhood of $O$, has the form:

$$(S_p^c) \qquad \begin{cases} \dot{\rho} = \widetilde{\delta}(\rho\cos\theta, \rho\sin\theta) \\ \dot{\theta} = 0\,, \end{cases}$$

for some function $\widetilde{\delta}$. This is the form of the system $(S_{\widetilde{H}})$ in polar coordinates. □

## 2. Systems with nonhomogeneous nonlinearities

We consider systems of the following type:

$$(S_H) \qquad \begin{cases} \dot{x} = -y + xH(x, y) = v_1 \\ \dot{y} = \phantom{-}x + yH(x, y) = v_2 , \end{cases}$$

with $H \in \mathcal{C}^2(U, \mathbb{R})$, $O \in U$ open connected subset of $\mathbb{R}^2$, containing the origin. We assume that $H(O) = 0$. $(S_H)$ has a unique critical point at $O$. We shall look for sufficient conditions for $O$ to be a center of $(S_H)$. Since $(S_H)$ has angular speed identically equal to one, if $O$ is a center, it is isochronous.

In order to apply the results of Section 1, we look for a commuting system of the type

$$(S_K) \qquad \begin{cases} \dot{x} = x + xK(x, y) = w_1 \\ \dot{y} = y + yK(x, y) = w_2 , \end{cases}$$

with $K \in \mathcal{C}^2(U, \mathbb{R})$, $K(O) = 0$. By Theorem 1.3, if such a system exists, then $O$ is a center for $(S_H)$. In next lemma we show that checking the commutativity condition for such systems requires to examine only one equation.

Lemma 2.1. *$(S_H)$ and $(S_K)$ commute if and only if*:

$$x\left(\partial_y K - \partial_x H + H\,\partial_x K - K\,\partial_x H\right) + y\left(-\partial_x K - \partial_y H + H\,\partial_y K - K\,\partial_y H\right) = 0. \tag{2.1}$$

*In this case, $O$ is an isochronous center of $(S_H)$.*

Proof. The Lie brackets of $(S_H)$ and $(S_K)$ vanish identically if and only if:

$$x^2\left(\partial_y K - \partial_x H + H\,\partial_x K - K\,\partial_x H\right) + xy\left(-\partial_x K - \partial_y H + H\,\partial_y K - K\,\partial_y H\right) = 0$$

$$xy\left(\partial_y K - \partial_x H + H\,\partial_x K - K\,\partial_x H\right) + y^2\left(-\partial_x K - \partial_y H + H\,\partial_y K - K\,\partial_y H\right) = 0.$$

Since $H$ and $K$ are of class $\mathcal{C}^2$, the Lie brackets vanish identically on $U$ if and only if (2.1) holds identically on $U$. □

A first consequence is an existence theorem for general systems, not necessarily polynomial. We recall that two real harmonic functions $H$ and $K$ are said to be *conjugate* if $H + iK$ is holomorphic and $H(0, 0) = K(0, 0) = 0$.

Theorem 2.1. *Let $H$ and $K$ be conjugate harmonic functions. Then $(S_H)$ and $(S_K)$ commute if and only if there exists a scalar function $\chi$ such that*:

$$H^2(x, y) + K^2(x, y) = \chi(x^2 + y^2). \tag{2.2}$$

Proof. Let $(S_H)$ and$(S_K)$ commute. By Cauchy-Riemann equations, formula (2.1) can be written as follows,

$$0 = x\left(\partial_y K - \partial_x H + H\,\partial_x K - K\,\partial_x H\right) + y\left(-\partial_x K - \partial_y H + H\,\partial_y K - K\,\partial_y H\right) =$$

$$= x\left(H\,\partial_x K - K\,\partial_x H\right) + y\left(H\,\partial_y K - K\,\partial_y H\right) =$$

$$= x\left(-H\,\partial_y H - K\,\partial_y K\right) + y\left(H\,\partial_x H + K\,\partial_x K\right) =$$

$$= \frac{1}{2}\nabla(H^2 + K^2)\cdot(y, -x).$$

Here, $\cdot$ denotes the scalar product. Since $\nabla(H^2 + K^2)$ is orthogonal to the circumferences centred at the origin, the level sets of $H^2 + K^2$ are circumferences, and there exists $\chi$ such that $H^2(x, y) + K^2(x, y) = \chi(x^2 + y^2)$.

Viceversa, if $H$ and $K$ are conjugate harmonic functions, and there exists a scalar function $\chi$ such that $H^2(x, y) + K^2(x, y) = \chi(x^2 + y^2)$, we have,

$$\nabla(H^2 + K^2)\cdot(y, -x) = 0$$

and we can read backwards the proof of the previous point, up to the commutativity formula. □

In order to find a linearization, and then a first integral of $(S_H)$, we look for a map that linearizes both systems. By Theorem 1.4, every map linearizing $(S_K)$ linearizes $(S_H)$ as well. Let us consider the linear systems:

$$(L_c) \qquad \begin{cases} \dot u = -v \\ \dot v = \phantom{-}u, \end{cases} \qquad\qquad (L_n) \qquad \begin{cases} \dot u = u \\ \dot v = v, \end{cases}$$

and the following couple of functions:

$$A(x, y) = \begin{cases} 0 & (x, y) = (0, 0) \\ \dfrac{yH - xK}{(x^2 + y^2)(1 + K)} & (x, y) \neq (0, 0), \end{cases}$$

$$B(x, y) = \begin{cases} 0 & (x, y) = (0, 0) \\ \dfrac{-xH - yK}{(x^2 + y^2)(1 + K)} & (x, y) \neq (0, 0). \end{cases}$$

Since $H$ and $K$ and their first partial derivatives vanish at $O$, $A$ and $B$ are continuous in a neighbourhood $V_0$ of $O$ and $\mathcal{C}^2$ on $V_0 \setminus \{O\}$. In next theorem we show that if $(S_H)$ and $(S_K)$ commute, then both systems admit a radial linearization. We shall use the following elementary lemma. We denote by $D(O, \epsilon)$ an open disc of radius $\epsilon$ centerd at $O$.

Lemma 2.2. *Let* $A, B \in \mathcal{C}^0(D(O, \epsilon), \mathbb{R}) \cap \mathcal{C}^1(D(O, \epsilon) \setminus \{O\}, \mathbb{R})$. *If the differential form* $Adx + Bdy$ *is closed in* $D(O, \epsilon) \setminus \{O\}$, *then it is exact.*

THEOREM 2.2. *Let $H$ and $K$ be as above. If $(S_H)$ and $(S_K)$ commute, then there exists an open neighbourhood $V_0$ of $O$, $V_0 \subset U$, and a map $\phi_E$ of class $C^1$ on $V_0$ of the form*:

$$\begin{cases} u = xE(x, y) \\ v = yE(x, y), \end{cases}$$

*with $E(x, y) > 0$, such that*:
(1) *$\phi_E$ is a local diffeomorphism at the origin,*
(2) *$\phi_E$ transforms $(S_H)$ into $(L_c)$ and $(S_K)$ into $(L_n)$.*

PROOF. First we show that formula (2.1) is equivalent to the existence of a map $\phi_E$ satisfying (2), and we give a costructive formula to get $E$. Then we show that $\phi_E$ is a local diffeomorphism at the origin.

A map $\phi_E$ transforms $(S_H)$ into the following system:

$$\begin{cases} \dot u = \dot x E + x\partial_x E\ \dot x + x\partial_y E\ \dot y = \\ \quad = (-y + xH)E + x\partial_x E(-y + xH) + x\partial_y E(x + yH) = \\ \quad = -yE + xHE + x(-y\partial_x E + x\partial_y E) + xH(x\partial_x E + y\partial_y E), \\ \dot v = \dot y E + y\partial_x E\ \dot x + y\partial_y E\ \dot y = \\ \quad = (x + yH)E + y\partial_x E(-y + xH) + y\partial_y E(x + yH) = \\ \quad = xE + yHE + y(-y\partial_x E + x\partial_y E) + yH(x\partial_x E + y\partial_y E). \end{cases}$$

Since $u = xE$, $v = yE$, the new system coincides with $(L_c)$ if and only if:

$$\begin{cases} \dot u = -yE = -yE + xHE + x(-y\partial_x E + x\partial_y E) + xH(x\partial_x E + y\partial_y E) \\ \dot v = \quad xE = \quad xE + yHE + y(-y\partial_x E + x\partial_y E) + yH(x\partial_x E + y\partial_y E). \end{cases}$$

For a continuous $E$, these equalities hold if and only if:

$$HE + H(x\partial_x E + y\partial_y E) - y\partial_x E + x\partial_y E = 0. \tag{2.3}$$

On the other hand, $\phi_E$ transforms $(S_K)$ into the following system:

$$\begin{cases} \dot u = \dot x E + x\partial_x E\ \dot x + x\partial_y E\ \dot y = \\ \quad = (x + xK)E + x\partial_x E(x + xK) + x\partial_y E(y + yK) = \\ \quad = xE + xKE + x(x\partial_x E + y\partial_y E)(1 + K), \\ \dot v = \dot y E + y\partial_x E\ \dot x + y\partial_y E\ \dot y = \\ \quad = (y + yK)E + y\partial_x E(x + xK) + y\partial_y E(y + yK) = \\ \quad = yE + yKE + y(x\partial_x E + y\partial_y E)(1 + K). \end{cases}$$

The new system coincides with $(L_n)$ if and only if:

$$\begin{cases} \dot u = xE = xE + xKE + x(x\partial_x E + y\partial_y E)(1 + K) \\ \dot v = yE = yE + yKE + y(x\partial_x E + y\partial_y E)(1 + K). \end{cases}$$

As above, for a continuous $E$, these equalities hold if and only if:

$$KE + (x\partial_x E + y\partial_y E)(1 + K) = 0. \tag{2.4}$$

(2.3) and (2.4) yield the system:

$$\left\{ \begin{array}{l} (-y + xH)\, \partial_x E + (x + yH)\, \partial_y E = -HE \\ x(1 + K)\, \partial_x E + y(1 + K)\, \partial_y E = -KE. \end{array} \right. \tag{2.5}$$

We can solve (2.5) as a linear system in the unknowns $\partial_x E$ and $\partial_y E$:

$$\left\{ \begin{array}{l} \partial_x E = E \dfrac{yH - xK}{(x^2 + y^2)(1 + K)} = EA \\ \partial_y E = E \dfrac{-xH - yK}{(x^2 + y^2)(1 + K)} = EB. \end{array} \right. \tag{2.6}$$

Since we are looking for a positive $E$, we can set $L(x, y) := \log E(x, y)$, so that equations (2.6) become:

$$\left\{ \begin{array}{l} \partial_x L = A \\ \partial_y L = B. \end{array} \right. \tag{2.7}$$

Hence, $E$ exists if and only if the differential form $Adx + Bdy$ is exact. By Lemma 2.2, it is sufficient to show that $Adx + Bdy$ is closed in a punctured neighbourhood of the origin. It is easy to check that the closedness condition $\partial_y A = \partial_x B$ holds if and only if formula (2.1) holds. Since $(S_H)$ and $(S_K)$ commute, it holds and there exists $L$ such that $dL = Adx + Bdy$. The regularity properties of $A$ and $B$ ensure that $L$ and $E(x, y) = \exp L(x, y)$ are of class $\mathcal{C}^2(V_0 \setminus \{O\}, \mathbb{R}) \cap \mathcal{C}^1(V_0, \mathbb{R})$.

Since $E$ is an exponential function, $E(0, 0) \neq 0$. The jacobian matrix of $\phi_E$ is invertible at the origin, since its determinant is equal to $E^2(0, 0)$. Hence $\phi_E$ is a local diffeomorphism at the origin. □

We denote by $(\rho, \theta)$ the polar coordinates of the point $(x, y)$: $x = \rho \cos\theta$, $y = \rho \sin\theta$. For sake of simplicity, in next corollary the function $E(x, y)$ will be expressed in polar coordinates, rather than in rectangular ones.

Corollary 2.1. *Under the hypotheses of Theorem 2.2, the function $E$ is given by*:

$$E(x, y) = E(\rho\cos\theta, \rho\sin\theta) = \exp\left( -\int_0^\rho \frac{K(r\cos\theta, r\sin\theta)}{r(1 + K(r\cos\theta, r\sin\theta))} dr \right)$$

*and a first integral of $(S_H)$ is given by*:

$$I(x, y) = (x^2 + y^2) E(x, y)^2 .$$

Proof. By Theorem 2.2, $E$ exists, and (2.4) holds. Equation (2.4) in polar coordinates becomes:

$$\rho \partial_\rho E = -E \frac{K}{1 + K} . \tag{2.8}$$

For every $\theta$, this linear equation can be integrated with respect to $\rho$:

$$E(\rho\cos\theta, \rho\sin\theta) = E(O) \exp\left( -\int_0^\rho \frac{K(r\cos\theta, r\sin\theta)}{r(1 + K(r\cos\theta, r\sin\theta))} dr \right).$$

Choosing $E(O) = 1$, we get the function of the statement.

Since $u^2 + v^2$ is a first integral of $(L_c)$, the function $(x^2 + y^2)E(x, y)^2$ is a first integral of $(S_H)$. $\square$

In next corollary we consider a special class of systems, whose orbits are symmetric with respect to the $y$-axis.

COROLLARY 2.2. *Let $\sigma \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$. Then the systems:*

$$(\Sigma_x) \quad \begin{cases} \dot{x} &= -y + x^2\sigma(y) \\ \dot{y} &= x + xy\sigma(y) \end{cases} \qquad (\Sigma_y) \quad \begin{cases} \dot{x} &= x + xy\sigma(y) \\ \dot{y} &= y + y^2\sigma(y) \end{cases}$$

*commute. Moreover, there exists a neighbourhood of the origin where the transformation:*

$$(\phi_\sigma) \qquad \begin{cases} u = x \exp\left( - \displaystyle\int_0^y \frac{\sigma(r)}{1 + r\sigma(r)} dr \right) \\ v = y \exp\left( - \displaystyle\int_0^y \frac{\sigma(r)}{1 + r\sigma(r)} dr \right) \end{cases}$$

*linearizes both $(\Sigma_x)$ and $(\Sigma_y)$. Moreover, the function:*

$$I(x, y) = (x^2 + y^2) \exp\left( - \int_0^y \frac{2\sigma(r)}{1 + r\sigma(r)} dr \right)$$

*is a first integral of $(\Sigma_x)$.*

PROOF. $(\Sigma_x)$ and $(\Sigma_y)$ are of the form $(S_H)$ and $(S_K)$, with $H(x, y) = x\sigma(y)$, $K(x, y) = y\sigma(y)$. Elementary computations show that in this case formula (2.1) holds. The differential form $\omega = Adx + Bdy$ of Theorem 2.2 is:

$$\omega = Adx + Bdy = -\frac{\sigma(y)}{1 + y\sigma(y)} dy.$$

$\Omega$ is of class $\mathcal{C}^2$ and closed on $\mathbb{R}^2 \setminus \{(x, y) : y\sigma(y) = -1\}$. Hence $\omega$ is exact on the connected component of $\mathbb{R}^2 \setminus \{(x, y) : y\sigma(y) = -1\}$ containing the origin, and we can apply Theorem 2.2. Integrating (2.8) we can choose $E(O) = 1$, so that:

$$E(x, y) = \exp\left( - \int_0^y \frac{\sigma(r)}{1 + r\sigma(r)} dr \right).$$

In order to find a first integral of $(S_H)$, we take a first integral of $(L_c)$, $u^2 + v^2$, and we write it as a function of $x$ and $y$, obtaining the function $I$ of the statement. $\square$

Corollary 2.2 extends previous results about cubic systems contained in [3, 12] to systems of arbitrary degree.

In [3] systems for which $H(x, y) = \alpha x + \beta y + Ax^2 + Bxy + Cy^2$ are studied. The origin is proved to be an isochronous center if and only if:

$$\begin{cases} A + C = 0 \\ A\alpha^2 + B\alpha\beta + C\beta^2 = 0. \end{cases}$$

Elementary algebraic computations show that in this case we have:

$$H(x, y) = (\alpha x + \beta y) \left[ l + m(\beta x - \alpha y) \right] ,$$

where $l$ and $m$ are real numbers. Rotating the axes, the system can be taken into a system for which $H(x, y) = x\sigma(y) = x(l + my)$, so that in this case $\sigma(y)$ is a polynomial of degree 1.

In [12, §5], the same class of systems considered in [3] is studied and linearizations are given.

## 3. Systems with homogeneous nonlinearities

In this section, we consider again systems of type $(S_H)$, assuming $H$ to be homogeneous. As in Section 2, we look for a commuting system of type $(S_K)$, trying to find conditions under which $K$ can be taken homogeneous. In this case, the commutativity condition becomes even simpler than (2.1).

Lemma 3.1. *Let $H$ and $K$ be homogeneous functions of degree $d$. Then $(S_H)$ and $(S_K)$ commute if and only if*

$$x\partial_y K - y\partial_x K = dH. \tag{3.1}$$

Proof. By Euler's theorem, we have:

$$H\left(x\partial_x K + y\partial_y K\right) - K\left(x\partial_x H + y\partial_y H\right) = HdK - KdH = 0.$$

Hence equation (2.1) becomes:

$$\left(x\partial_y K - y\partial_x K\right) - \left(x\partial_x H + y\partial_y H\right) = 0.$$

Applying again Euler's theorem, we obtain formula (3.1). □

Theorem 3.1. *Let $H$ be a harmonic function, homogeneous of degree $d$. Then $O$ is an isochronous center of $(S_H)$.*

Proof. Let $K$ be the conjugate harmonic function of $H$. By Cauchy-Riemann equations, $K$ is a homogeneous function of degree $d$. From Lemma 3.1, $(S_H)$ and $(S_K)$ commute if and only if

$$x\partial_y K - y\partial_x K = dH,$$

that is

$$\left(x\partial_y K - y\partial_x K\right) - \left(x\partial_x H + y\partial_y H\right) = 0,$$

or

$$x\left(\partial_y K - \partial_x H\right) - y\left(\partial_x K + \partial_y H\right) = 0,$$

that holds, by Cauchy-Riemann equations. □

When $H$ and $K$ are homogeneous, it is not necessary for $H$ and $K$ to be conjugate harmonic functions for $(S_H)$ and $(S_K)$ to commute. An example is given by the following couple of systems, where $H(x, y) = 2xy$ is harmonic, and $K(x, y) = 2y^2$ is not:

$$(S_H)\quad \begin{cases} \dot{x} = -y + 2x^2 y \\ \dot{y} = \ \ x + 2xy^2, \end{cases} \qquad\qquad (S_K)\quad \begin{cases} \dot{x} = x + 2xy^2 \\ \dot{y} = y + 2y^3. \end{cases}$$

If $H$ and $K$ are homogeneous polynomials, the analysis of the commutativity condition can be reduced to a problem in linear algebra.

Let us consider again system $(S_H)$, where $H$ is a polynomial of degree $n-1$. At first, we do not assume $H$ to be homogeneous. We look for a polynomial commutator $(S_K)$, where $K$ has degree $n-1$ as well.

If we write $H$ and $K$ as the sum of their homogeneous polynomials $H_i$ and $K_i$, $i = 1, \dots, n$, equation (2.1) becomes:

$$\left(x\partial_y K - y\partial_y K\right) - \left(\sum_{i=1}^{n-1} iH_i\right) + H\left(\sum_{i=1}^{n-1} iK_i\right) - K\left(\sum_{i=1}^{n-1} iH_i\right) = 0. \tag{3.2}$$

The first two parentheses contain only terms up to degree $n-1$, while the second two contain terms from degree 2 to degree $2(n-1)$. If we write:

$$H\sum_{i=1}^{n-1} iK_i = \sum_{j=1}^{n-1}\sum_{i=1}^{n-1} iH_jK_i\,,$$

$$K\sum_{j=1}^{n-1} jH_j = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} jH_jK_i\,,$$

then equation (3.2) becomes:

$$\left(x\partial_y K - y\partial_y K\right) - \left(\sum_{i=1}^{n-1} iH_i\right) + \sum_{i,j=1}^{n-1}(i-j)H_jK_i = 0\,, \tag{3.3}$$

which is an equation of degree $2n-3$, since the only term of degree $2n-2$ is $(n-1-n+$ $+\,1)\ H_{n-1}K_{n-1} = 0$.

If we decompose equation (3.3) into its homogeneous parts, we get $2n-3$ equations:

$$\begin{cases} x\partial_y K_1 - y\partial_x K_1 - H_1 = 0 \\ x\partial_y K_2 - y\partial_x K_2 - 2H_2 = 0 \\ x\partial_y K_3 - y\partial_x K_3 - 3H_3 + \left(H_1K_2 - H_2K_1\right) = 0 \\ \dots\dots \\ x\partial_y K_{n-1} - y\partial_x K_{n-1} - (n-1)H_{n-1} + \sum_{\substack{i+j=n-1 \\ i,j\neq 0,\,n-1}} (j-i)H_jK_i = 0 \\ \sum_{\substack{i+j=d \\ i,j\neq 0}} (j-i)H_jK_i = 0\,, \qquad n \le d \le 2n-3. \end{cases}$$

When $H$ is homogeneous, such a system is much simpler. We have the following lemma:

LEMMA 3.2. *If $H$ is a homogeneous polynomial, then any polynomial $K$ of the same degree such that $(S_H)$ and $(S_K)$ commute is homogeneous as well.*

Proof. By assumption, we have $H_1 = H_2 = \cdots = H_{n-2} \equiv 0$, hence

$$\begin{cases} x\partial_y K_1 - y\partial_x K_1 = 0 \\ x\partial_y K_2 - y\partial_x K_2 = 0 \\ \dots\dots \\ x\partial_y K_{n-1} - y\partial_x K_{n-1} - (n-1)H_{n-1} = 0 \\ (2n-2-d)H_{n-1}K_{d-n+1} = 0\,, \qquad n \le d \le 2n-3. \end{cases}$$

The equations in the last row imply that $K_1 = K_2 = \cdots = K_{n-2} \equiv 0$. Hence $K$ has to be homogeneous of the same degree as $H$. □

Lemma 3.2 allows us to apply Lemma 3.1, so that in order to find a commutator of degree $n$ for $(S_H)$, it is sufficient to find a polynomial $K$ solving the equation

$$x\partial_y K_{n-1} - y\partial_x K_{n-1} - (n-1)H_{n-1} = 0. \tag{3.4}$$

If we set

$$H(x, y) = H_{n-1}(x, y) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2}y + \cdots + a_1xy^{n-2} + a_0y^{n-1}\,,$$
$$K(x, y) = K_{n-1}(x, y) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2}y + \cdots + b_1xy^{n-2} + b_0y^{n-1}\,,$$

and we write (3.4) explicitely, we get a linear system of $n$ equations in the $n$ unknowns $b_{n-1}, \ldots, b_0$, which can be written in matrix form as:

$$\begin{pmatrix} 0 & -1 & 0 & & & & \dots & 0 \\ n-1 & 0 & -2 & 0 & & & \vdots & 0 \\ 0 & n-2 & 0 & -3 & 0 & & \vdots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \dots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \dots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & & & & 2 & 0 & -n+1 \\ 0 & \dots & & & & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_{n-1} \\ b_{n-2} \\ b_{n-3} \\ \vdots \\ \vdots \\ \vdots \\ b_1 \\ b_0 \end{pmatrix} = (1-n) \begin{pmatrix} a_{n-1} \\ a_{n-2} \\ a_{n-3} \\ \vdots \\ \vdots \\ \vdots \\ a_1 \\ a_0 \end{pmatrix}.$$

For sake of brevity, we write this matrix equation as:

$$Ab = (1-n)a. \tag{3.5}$$

It is easy to prove:

Lemma 3.3.
(*i*) *If $n$ is even, $A$ has rank $n$.*
(*ii*) *If $n$ is odd, $A$ has rank $n-1$.*

Theorem 3.2. *Let $H$ be a homogeneous polynomial of degree $n-1$. We have:*
(*i*) *if $n$ is even, then there exists a unique homogeneous polynomial commutator of degree $n$.*
(*ii*) *if $n$ is odd, and $\operatorname{rank} A = \operatorname{rank}(A, a)$, then $(S_H)$ has $\infty^1$ homogeneous polynomial commutators of degree $n$. Otherwise, $(S_H)$ has no polynomial commutators of degree $n$.*

Proof. If $n$ is odd, $A$ is invertible and the linear system has a unique solution.

If $n$ is even, by Rouché-Capelli theorem the linear system has solutions if and only if rank $A$ = rank $(A, a)$ = $n-1$. When this is the case, the system has $\infty^1$ solutions. □

Remark 3.1. In [2, Corollary 4.2; 5, Theorem 2.1], the authors prove that, if $H$ is polynomial and homogeneous, then the isochronicity of the center is equivalent to:

$$\int_0^{2\pi} H(\cos t, \sin t)dt = 0. \tag{3.6}$$

When $n$ is even, (3.6) always holds. When $n$ is odd, formula (3.6) gives a linear condition $\mathcal{L}_1$ on the coefficients of $H$. Also, when $n$ is odd, Theorem 3.2 (*ii*) gives a linear condition $\mathcal{L}_2$ on the coefficients of $H$ for the existence of a homogeneous polynomial commutator. Since the existence of a commutator is equivalent to the isochronicity of the center, one has $\mathcal{L}_1 \equiv \mathcal{L}_2$. As a consequence, if $H$ is polynomial and homogeneous, and $O$ is a center (hence an isochronous center) of $(S_H)$, then Theorem 3.2 ensures the existence of a commutator $(S_K)$ with $K$ polynomial and homogeneous.

Remark 3.2. The coefficients of $K$ can be found by solving the linear system (3.5). When $n$ is even, it is possible to determine the coefficients $\{b_j, j = 0, \dots, n-1\}$ by means of a recursive procedure. In fact, (3.5) holds if and only if

$$\begin{cases} -b_1 & = (n-1)a_0 \\ (n-j)b_{j-1} - (j+1)b_{j+1} & = (n-1)a_j, \qquad\qquad j = 1, \dots, n-2 \\ b_{n-2} & = (n-1)a_{n-1}. \end{cases} \tag{3.7}$$

The coefficients $b_j$, $j$ odd, can be computed by recurrence, by making $j$ increase from 1 to $n-1$. We obtain the following:

$$\begin{cases} b_1 & = -(n-1)a_0 \\ b_{j+2} & = \dfrac{(n-1-j)b_j - (n-1)a_{j+1}}{j+2}, \qquad\qquad j \text{ odd}, j = 1, \dots, n-1. \end{cases} \tag{3.8}$$

Similarly for the coefficients $b_j$, $j$ even, by making $j$ decrease from $n-2$ to 0. We obtain:

$$\begin{cases} b_{n-2} & = (n-1)a_{n-1} \\ b_{j-2} & = \dfrac{jb_j + (n-1)a_{j-1}}{n-j+1}, \qquad\qquad j \text{ even}, j = n-2, \dots, 0. \end{cases} \tag{3.9}$$

Remark 3.3. J. Devlin (personal communication) communicated us a way to find the homogeneous polynomial $K$ by means of an integration. The expression of equation (3.1) in polar coordinates is:

$$\partial_\theta K = (n-1)H$$

so that one has:

$$K(\rho\cos\theta, \rho\sin\theta) = \int_0^\theta (n-1)H(\rho\cos r, \rho\sin r)dr = (n-1)\rho^{n-1}\int_0^\theta H(\cos r, \sin r)dr.$$

The same holds for a nonpolynomial homogeneous $H$, if we assume the existence of a homogeneous $K$ such that $(S_H)$ and $(S_K)$ commute.

When $H$ and $K$ are homogeneous and $(S_H)$ and $(S_K)$ commute, we can find a simple linearization of $(S_H)$.

THEOREM 3.3. *Let $H$ and $K$ be homogeneous of degree $d$. If $(S_H)$ and $(S_K)$ commute, then the transformation*

$$(\phi_K) \qquad \begin{cases} u = x[1 + K(x, y)]^{-1/d} \\ v = y[1 + K(x, y)]^{-1/d} \end{cases}$$

*linearizes $(S_H)$. A first integral is given by*:

$$I(x, y) = \frac{(x^2 + y^2)^d}{(1 + K(x, y))^2}.$$

PROOF. $\phi_K$ is a transformation of the type given in Theorem 2.2, with $E(x, y) = [1 + K(x, y)]^{-1/d}$. Therefore, $\phi_K$ linearizes $(S_H)$ and $(S_K)$ if $[1 + K(x, y)]^{-1/d}$ satisfies equations (2.5). This follows easily from the homogeneity of $H$ and $K$ and from formula (3.1). A first integral can be obtained from $u^2 + v^2$, as in the previous cases. The first integral of the statement is $(u^2 + v^2)^d$, expressed as a function of $x$ and $y$. □

REMARK 3.4. Our technique and the one given in [12] do not necessarily lead to the same linearization. As special cases of systems considered in Theorem 3.3, we have systems $(S_2)$ in [12, p. 82], where $H(x, y) = x$, and $(S_2^*)$ in [12, p. 85], where $H(x, y) = xy$ (in $(S_2^*)$ a misprint changed the correct form of the system: $\dot{y}$ should be $x(1 + y^2)$).

The commutator of $(S_2)$ we find by using the conjugate function of $H$ is $(S_K)$, with $K = y$, so that the linearization provided by Theorem 2.3 coincides with the one given in [12].

On the other hand, the commutator of system $(S_2^*)$ we find in the same way is $(S_K)$, with $K = (-x^2 + y^2)/2$, so that we find a linearization that does not coincide with the one given in [12]. Anyway, $(S_2^*)$ commutes also with any system we get choosing $K = -\mu x^2 + (1 - \mu)y^2$, $\mu \in \mathbb{R}$, so that we find infinitely many distinct linearizations. Choosing $\mu = 1$ we get the one given in [12].

Theorem 3.3 shows that, if $H$ is a homogeneous polynomial of degree $n - 1$, then $(S_H)$ has a rational first integral. As a consequence, the integral curves of $(S_H)$ are contained in algebraic curves of degree $2n - 2$.

Theorem 3.3 allows also to give a rational first integral for a special class of Liénard differential equations. Let us consider the equation

$$(E_n) \qquad x'' - (n + 1)x'x^{n-1} + x + x^{2n-1} = 0.$$

As in [16], we can study the behaviour of its solutions by means of an equivalent differential system of a special form:

$$(S_n) \qquad \begin{cases} \dot{x} &= -y + x^n \\ \dot{y} &= x + yx^{n-1}. \end{cases}$$

This is a special case of system $(S_H)$, with $H(x, y) = x^{n-1}$. If $n$ is an even positive integer, by Lemma 3.3, Theorem 3.2 and Remark 3.1, there exists a unique homogeneous polynomial $K_{n-1}(x, y)$ of degree $n-1$ such that the system

$$\begin{cases} \dot{x} &= x + xK_{n-1}(x, y) \\ \dot{y} &= y + yK_{n-1}(x, y) \end{cases}$$

commutes with $(S_n)$. In order to determine the coefficients of $K_{n-1}$, we can apply the procedure described in Remark 3.2. We get a homogeneous polynomial where only even powers of $x$ appear:

$$K_{n-1}(x, y) = (n-1) \sum_{l=0}^{\frac{n-2}{2}} \frac{(n-2)!!}{(n-2-2l)!!(2l+1)!!} x^{n-2-2l} y^{2l+2}.$$

Applying Theorem 3.3 one obtains the first integral of $(S_n)$:

$$I_n(x, y) = \frac{(x^2 + y^2)^{(n-1)}}{\left(1 + (n-1) \sum_{l=0}^{\frac{n-2}{2}} \frac{(n-2)!!}{(n-2-2l)!!(2l+1)!!} x^{n-2-2l} y^{2l+2}\right)^2}.$$

Systems with homogeneous nonlinearities have been considered also in [9]. In [11], cubic systems with rational first integrals have been studied.

## Acknowledgements

This research has been performed under the auspices of the group «Equazioni differenziali ordinarie ed applicazioni». Most of the results of this paper have been presented at the yearly meeting of the group «Equazioni differenziali ed applicazioni», Napoli, 1996, and at the «Symposium on Planar Vector Fields», Lleida, 1996.

The authors would like to thank proff. R. Conti and M. Villarini for several useful conversations on the subject of this paper, and dr. J. Devlin for a careful reading of a previous version of this paper.

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Pervenuta il 23 aprile 1999,
in forma definitiva il 19 ottobre 1999.

L. Mazzi:
Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi, 24 - 10129 Torino
mazzi@polito.it

M. Sabatini:
Dipartimento di Matematica
Università degli Studi di Trento
38050 Povo TN
sabatini@science.unitn.it