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Stochastic invariance and consistency of financial models

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Analisi matematica. — *Stochastic invariance and consistency of financial models.* Nota (*) di Jerzy Zabczyk, presentata dal Socio G. Da Prato.

Abstract. — The paper is devoted to a connection between stochastic invariance in infinite dimensions and a consistency question of mathematical finance. We derive necessary and sufficient conditions for stochastic invariance of Nagumo's type for stochastic equations with additive noise. They are applied to Ornstein-Uhlenbeck processes and to specific financial models. The case of evolution equations with general noise is discussed also and a comparison with recent results obtained by geometric methods is presented as well.

Key words: Stochastic equations; Invariant sets; Forward curve; Consistency.

Riassunto. — *Invarianza stocastica e consistenza dei modelli finanziari.* Questo lavoro riguarda la connessione fra l'invarianza stocastica in dimensione infinita e un problema di consistenza in finanza matematica. Vengono date condizioni necessarie e sufficienti di tipo Nagumo per l'invarianza di equazioni stocastiche con rumore additivo. Esse sono applicate a processi di Ornstein-Uhlenbeck e specifici modelli finanziari. Vengono anche discusse equazioni di evoluzione con rumore generale e viene fatto un paragone con recenti risultati ottenuti con metodi geometrici.

1. Introduction

Consider a nonlinear Ito evolution equation on a separable Hilbert space H,

$$dX(t) = \big(AX(t) + F(X(t))\big)\,dt + B(X(t))\,dW(t)\ , \quad X(0) = x \in H\,. \tag{1}$$

The Wiener process W takes values in a separable Hilbert space U, is defined on a probability space $(\Omega\,, \mathcal{F}\,, \mathbb{P})$ and has covariance Q. The operator A generates a C_0-semigroup of linear operators $S(t)\ , t \geq 0\ ,\ F$ is a Lipschitz mapping from H into H, and B is a Lipschitz mapping from H into the space $L(U, H)\ ,$ of all linear transformations from U into H. The solution to (1) will be denoted by X^x. A set $K \subset H$ is said to be *invariant* for (1) if and only if,

$$\mathbb{P}(X^x(t) \in K) = 1 \ \text{ for all }\ x \in K\ ,\ t \geq 0\,.$$

The literature on stochastic invariance is rather extensive, both for finite dimensional systems, see [2], and for infinite dimensional ones, see [28, 18, 19], for a discussion of recent results. Deterministic theory is presented in [1]. In the present paper we concentrate on the approach which uses a connection between stochastic equations and deterministic control theory, see a discussion in §8.

Assume that the equation (1) is used as a model of a real phenomena, say of physical or economical nature. In concrete cases the space H consists of functions. Assume that for a curve fitting one applies a statistical procedure which uses functions from a given

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set K. One says that the model (1) is *consistent* with the set K if and only if the set K is invariant for (1). The importance of stochastic invariance for mathematical finance was stressed in the pioneering work [4]. In [4] and in the following papers [5, 7] the stochastic invariance was discussed from various points of view.

We first discuss the case of the stochastic evolution equations

$$dX(t) = \big(AX(t) + F(X(t))\big)\,dt + BdW(t)\ , \quad X(0) = x \in H\ , \tag{2}$$

where $B \in L(U, H)$. We reduce the invariance question for (2) to the invariance problem for a controlled deterministic equation

$$\frac{d}{dt}y(t) = Ay(t) + F(y(t)) + BQ^{1/2}u(t)\ ,\ \ y(0) = x \in H\ , \tag{3}$$

and formulate necessary and sufficient conditions for stochastic invariance in the Nagumo form. Of special interest are invariant, finite dimensional manifolds K. They are precisely those, which should be used for curve fitting. We establish if and only if conditions for existence of an invariant finite dimensional linear subspace for Ornstein-Uhlenbeck processes. Although this situation has been studied in [5, 7] our Theorem 4 gives the first complete answer to the invariance question even in the specific case of HJM models. The results are then applied to the HJM model in the Musiela parametrization both on infinite and finite intervals and to a recent model elaborated by El Karoui and collaborators [10, 11]. The final application concerns the model for LIBOR rates introduced by Brace, Gątarek and Musiela [8] see also [15]. The paper ends with some sufficient conditions for stochastic invariance for general equations (1) using approximations of Wong-Zakai type. Comparison with the approaches proposed in [5, 7, 14] is discussed as well.

The main results of the paper are Theorem 3 and Theorem 4 and specific applications worked out in §4 - §7.

The paper is a rewritten version of the preprint [34].

2. Invariance for equations with additive noise

We are concerned here with the invariance of a closed set K with respect to the equation (2). Without any loss of generality we can assume that $Q = I$. To assure existence of a continuous solution to (2) we will impose, see [12] or [13], the following condition:

$$\text{For some } \alpha \text{ and all } T > 0 \qquad \int_0^T t^{-\alpha} ||S(t)B||^2_{HS}\,dt < +\ \infty\ , \tag{4}$$

where $||\cdot||^2_{HS}$ stands for the Hilbert-Schmidt norm.

Theorem 1. *Assume that A generates a C_0-semigroup on a Hilbert space H, F is a Lipschitz transformation from H into H, $B \in L(U, H)$ and the condition* (4) *holds. Then a closed set $K \subset H$ is invariant for* (2) *if and only if it is invariant for* (3).

Let us recall that a closed set $K \subset H$ is said to be invariant for (3) if for arbitrary

$x \in K$ and arbitrary, locally square integrable, U-valued function $u(\cdot)$ the solution $y^{x,u(\cdot)}(t)$, $t \geq 0$, of the equation (3) stays in K for all $t \geq 0$.

Proof. Denote by Z the stochastic convolution:

$$Z(t) = \int_0^t S(t-r)BdW(r) , \quad t \geq 0 . \tag{5}$$

Since (4) holds we can assume that Z has continuous paths. Fix $T > 0$. It follows from [13, p. 141], that the support of the law $\mathcal{L}(Z)$, on the space $C[0, T; H]$ is identical with the closure in $C[0, T; H]$, of the set S_T of all functions f,

$$f(t) = \int_0^t S(t-r)Bu(r)dr , \quad t \in [0, T] , \quad u \in L^2[0, T; U] .$$

Let c be the Lipschitz constant of F, $M = \sup_{t\in[0,T]} ||S(t)||$. For arbitrary $u \in L^2[0, T; U]$ we have,

$$X^x(t) - y^{x,u(\cdot)}(t) = \int_0^t S(t-r)\big(F(X^x(r)) - F(y^{x,u(\cdot)}(r))\big)\,dr + Z(t) - \int_0^t S(t-r)Bu(r)dr ,$$

and therefore,

$$|X^x(t) - y^{x,u(\cdot)}(t)| \leq cM \int_0^t |X^x(r) - y^{x,u(\cdot)}(r)|dr + \sup_{t\in[0,T]} |Z(t) - \int_0^t S(t-r)Bu(r)dr| .$$

By Gronwall's lemma, for all $t \in [0, T]$,

$$|X^x(t) - y^{x,u(\cdot)}(t)| \leq e^{TcM} \sup_{t\in[0,T]} |Z(t) - \int_0^t S(t-r)Bu(r)dr| . \tag{6}$$

Assume that K is invariant for (2) and let $u(\cdot) \in L^2[0, T; U]$. It follows from the description of the support of the process Z that for arbitrary $\delta > 0$

$$\mathbb{P}\left(\sup_{t\in[0,T]} |Z(t) - \int_0^t S(t-r)Bu(r)dr| < \delta \right) > 0 .$$

But

$$\mathbb{P}(X^x(t) \in K) = 1 \text{ for all } t \in [0, T] ,$$

and by (6),

$$\operatorname{dist}(y^{x,u(\cdot)}(t) , K) \leq \delta \exp(cTM) .$$

Since δ was an arbitrary positive number

$$y^{x,u(\cdot)}(t) \in K \quad \text{for all } t \in [0, T] .$$

Conversely, assume that K is invariant for (3). Since the law $\mathcal{L}(Z)$ is tight in $C[0, T; H]$, for arbitrary $\delta > 0$ and arbitrary $\epsilon > 0$, there exists a finite number of functions $u_1, \ldots, u_N \in L^2[0, T; U]$ such that with a probability greater than $1-\epsilon$, for at least one $n = 1, \ldots, N$,

$$\sup_{t\in[0,T]} |Z(t) - \int_0^t S(t-r)Bu_n(r)dr| < \delta .$$

Consequently, by (6), with a probability greater than $1-\epsilon$, for at least one $n = 1, \dots, N$:

$$\sup_{t\in[0,T]} |X^x(t) - y^{x,u_n(\cdot)}(t)| \le e^{TcM}\delta.$$

This easily implies the result. □

Assume now that $B = 0$, then explicit conditions for invariance are known. In particular the following result was proved by Pavel [26, 27], in the case of compact operators $S(t)\, t \ge 0$, and for general C_0-semigroups by Jachimiak [17, 18].

Theorem 2. *Assume that the operator A generates a C_0-semigroup on a Banach space H and F is a Lipschitz transformation from H into H. A closed set $K \subset H$ is invariant for*

$$\frac{d}{dt}y(t) = Ay(t) + F(y(t)), \quad y(0) = x \in H, \tag{7}$$

if and only if for arbitrary $x \in H$

$$\liminf_{t\downarrow 0} \frac{1}{t} \operatorname{dist}[S(t)x + tF(x), K] = 0. \tag{8}$$

If the set K is contained in the domain $D(A)$ of the generator A then the condition (8) can be replaced by the classical Nagumo's condition

$$\liminf_{t\downarrow 0} \frac{1}{t} \operatorname{dist}[x + t(Ax + F(x)), K] = 0, \quad \text{for all } x \in K. \tag{9}$$

As a corollary from Theorem 1 and the above theorem we obtain the following basic result.

Theorem 3. *Assume that the assumptions of Theorem 1 are satisfied. A closed set $K \subset H$ is invariant for (2) if and only if for arbitrary $x \in K$, $v \in U$,*

$$\liminf_{t\downarrow 0} \frac{1}{t} \operatorname{dist}[S(t)x + t(F(x) + Bv), K] = 0. \tag{10}$$

Proof. It is not difficult to see that the set K is invariant for (3) if and only if $y^{x,u(\cdot)}(t) \in K$, $t \ge 0$ for all $x \in K$ and all piecewise constant, U-valued functions $u(\cdot)$. Therefore the set K is invariant for (3) if and only if $y^{x,v} \in K$, $t \ge 0$ for all $x \in K$ and all constant control functions $u(\cdot) \equiv v \in U$. So by the theorem due to Pavel and Jachimiak the result follows. □

The condition (9) can be rewritten, using the concept of the *tangent cone* $T_K(x)$ to the closed set K at point $x \in K$:

$$T_K(x) = \left\{ v \in U : \liminf_{t\downarrow 0} \frac{1}{t} \operatorname{dist}[(x + tv), K] = 0 \right\},$$

as

$$Ax + F(x) + Bv \in T_K(x), \qquad \text{for all } x \in K, \quad v \in U. \tag{11}$$

In particular if K is a differentiable manifold then the tangent cone coincides with the tangent space.

3. Invariance for Ornstein-Uhlenbeck processes

We apply Theorem 1 to the equation:

$$dX(t) = (AX(t) + a)dt + BdW(t) \ , \quad X(0) = x \ , \tag{12}$$

with $a \in H$. Together with (12) it is convenient to introduce a simpler model:

$$dX(t) = (AX(t) + \sigma_0)dt + \sum_{j=1}^{d} \sigma_j dW_j(t) \ , \quad X(0) = x, \tag{13}$$

where σ_j, $j=0, 1, 2, \ldots, d<\infty$ are given elements from H and W_j, $j=1, 2, \ldots, d$, are one dimensional Wiener processes. The following theorem is concerned with the important question of existence of a *finite* dimensional linear subspace invariant for (12).

Theorem 4. *i) If there exists a finite dimensional linear space $K \subset H$ invariant for* (12) *then the system* (12) *can be rewritten in the form* (13).

ii) There exists a finite dimensional linear space $K \subset H$ invariant for the equation (13) *if and only if $\sigma_j \in D(A)$, $j = 0, 1, \ldots, d$ and there exist elements $\sigma_j \in D(A)$, $j = d + 1, \ldots, D <+ \infty$ such that,*

$$A\sigma_k = \sum_{j=0}^{D} \alpha_{kj}\sigma_j \ , \quad k = 0, \ldots, D. \tag{14}$$

Proof. Consider the controlled system,

$$\frac{d}{dt}y(t) = Ay(t) + a + Bu(t) \ , \quad y(0) = x \in H \ , \tag{15}$$

corresponding to (12). Then, for $x \in K$, and for a constant control function $u(t) = v \in U$, $t \geq 0$,

$$y^{x,v}(t) = S(t)x + \int_0^t S(r)adr + \int_0^t S(r)Bv\, dr \in K \, .$$

In particular, taking $x = 0$, $v = 0$ one gets that:

$$\int_0^t S(r)a\, dr \in K \ , \quad t \geq 0 \, .$$

Consequently also:

$$t^{-1} \int_0^t S(r)Bv\, dr \in K \ , \quad t > 0 \ ,$$

and this implies that $Bv \in K$. Since the space K was assumed to be finite dimensional, the operator B should be of finite rank, so the part *i*) follows.

To show *ii*) denote by K_0 the closure of the linear space of all elements of the form:

$$u_0 \int_0^t S(r)\sigma_0\, dr + \sum_{j=1}^{d} \int_0^s S(r)\sigma_j u_j(r)\, dr \ ,$$

where $u_0 \in R^1$, u_j are square integrable functions and $t, s \geq 0$. It is not difficult to see that K_0 is the smallest closed linear subspace of H containing all σ_j, $j = 0, \dots, d$ and invariant with respect to all operators $S(t)$, $t \geq 0$. Since $K_0 \subset K$, therefore K_0 is finite dimensional space. However a finite dimensional linear space K_0 is invariant for the semigroup S if and only if it is contained in the domain $D(A)$ of the generator A of S and is invariant for A. To see this note that if K_0 is invariant for all $S(t)$ then the restriction of $S(t)$ to K_0 is a C_0 semigroup on a finite dimensional space. Since the domain of its generator is a linear and dense subset of a finite dimensional linear space it should be identical with the whole K_0. It is clear that the generator of the restricted semigroup is identical with the restriction of the generator. Consequently

$$K_0 \subset D(A) \text{ and } A(K_0) \subset K_0 . \tag{16}$$

If the vectors σ_j, $j = d + 1, \dots, D$ are chosen in such a way that the space K_0 is spanned by all σ_j, $j = 0, \dots, D$ then they will have the required properties.

To prove the converse implication assume that the vectors σ_j, $j = d+1, \dots, D$ have the desired properties. Then the space K_0 spanned by all σ_j, $j = 0, \dots, D$ is contained in the domain $D(A)$ and is invariant for A. Since it is finite dimensional it is also invariant for the semigroup S and consequently it is invariant for the system (15). □

Remark 1. If one is looking for necessary and sufficient conditions that guarantee existence of a finite dimensional manifold (containing the vector 0) invariant for the system (13), which is not necessarily a linear space, one arrives at identical conditions but not involving the vector σ_0, see [34]. The required manifold is composed of finite dimensional linear spaces attached to any point of the curve:

$$c(t) = \int_0^t S(r)\sigma_0 dr , \quad t \geq 0.$$

Remark 2. A finite realization question, similar to the one considered here, has been posed and solved in [5]. However it was assumed implicitely in [5] that the vectors σ_j , were in the intersection of all the domains $D(A^n)$, $n = 1, \dots$. There is no such assumption in our theorem which applies to all linear systems.

4. HJM model in Musiela parametrization. Unbounded maturities

We apply Theorem 4 to the Gaussian version of the HJM model in Musiela parametrization. Thus in the present situation the equation (13) is of the form:

$$dX(t, \xi) = \left(\frac{\partial X}{\partial \xi}(t, \xi) + \sum_{j=1}^{d} \sigma_j(\xi) \int_0^{\xi} \sigma_j(\eta) d\eta \right) dt + \sum_{j=1}^{d} \sigma_j(\xi) dW_j(t) , \tag{17}$$

$$X(0, \xi) = x(\xi) , \quad x \in L^2_\gamma(0, +\infty) , \quad t, \xi \geq 0. \tag{18}$$

We assume that H is equal to the space $L^2_\gamma(0, +\infty)$, $\gamma \geq 0$, of measurable functions

x defined on $[0, +\infty)$ such that,

$$||x||_{\gamma}^{2} = \int_{0}^{\infty} e^{-\gamma\xi}|x(\xi)|^{2}d\xi < +\infty . \tag{19}$$

The semigroup S is given by the shift operators:

$$S(t)x(\xi) = x(t+\xi) , \quad t, \xi \in [0, +\infty) , \quad x \in L_{\gamma}^{2}(0, +\infty) .$$

The domain $D(A)$ of the infinitesimal generator A consists of all absolutely continuous functions x with the first derivative in $L_{\gamma}^{2}(0, +\infty)$. We denote the domain by H_{γ}^{1}. The generator A on H_{γ}^{1} is identical with the first derivative operator. For $j = 1, \dots, d$, $\xi \in [0, +\infty)$, define

$$\sigma_{0}(\xi) = \sum_{j=1}^{d} a_{j}(\xi) , \text{ where } a_{j}(\xi) = \sigma_{j}(\xi) \int_{0}^{\xi} \sigma_{j}(\eta)d\eta . \tag{20}$$

THEOREM 5. *Assume that functions* σ_j *and* a_j, $j = 1, \dots, d$ *are in* $L_{\gamma}^{2}(0, +\infty)$. *There exists a finite dimensional, linear subspace invariant for* (17) *if and only if the functions* σ_j, $j = 1, \dots, d$, *are linear combinations of*

$$\xi^{k}e^{\alpha\xi}\cos(\beta\xi) , \quad \xi^{k}e^{\alpha\xi}\sin(\beta\xi) , \tag{21}$$

for some

$$k = 0, 1, \dots , \quad \beta \in R^{1} , \quad \alpha < \gamma/2 .$$

PROOF. To proof necessity remark that by Theorem 4 the functions σ_j, $j = 0, \dots, d$, are in H_{γ}^{1}. In particular they are absolutely continuous. Moreover, again by Theorem 4 there exists absolutely continuous functions σ_j, $j = d+1, \dots, D$, such that for $k = 0, 1, \dots, D$, and some numbers α_{kj},

$$\frac{d}{d\xi}\sigma_{k}(\xi) = \sum_{j=0}^{D} \alpha_{kj}\sigma_{j}(\xi) , \quad \xi \geq 0. \tag{22}$$

The linear system (22), by the Jordan decomposition theorem, has solutions which are necessarily of the required form. The condition on the parameter α is a consequence of the fact that the functions σ, are in the space H.

To prove the converse implication assume that the functions σ_j, $j = 1, \dots, d$, are of the prescribed form. By an elementary induction argument one deduces that then functions a_j, $j = 1, \dots, d$, are also of the form (21) but with $\alpha < \gamma$, only. This is however enough to deduce that these functions belong to H. Let $\alpha_1, \dots, \alpha_m$ be all the exponents which are present in the representations of the functions σ_j and all a_j, $j = 1, \dots, d$, and $\beta_1, \dots, \beta_n$, all the parameters β. Let in addition M be the higest degree of the polynomials present in the representations. Then the finite dimensional subspace $K \subset H$ spanned by all functions of the form:

$$p(\xi)e^{\alpha_{l}\xi}\cos(\beta_{k}\xi) , \quad p(\xi)e^{\alpha_{l}\xi}\sin(\beta_{k}\xi) , \tag{23}$$

where p are arbitraray polynomials of degree not greater than M, and $l = 1, \ldots, m$, $k = 1, \ldots, n$ is invariant for the operator A of the first order differentiation. □

5. HJM model in Musiela parametrization. Bounded maturities

We pass now to the model in which the maturities are assumed to be bounded by, say 1. It seems that the semigroup treatment of the model presented below is new in the financial literature. The system (17)-(18) now becomes

$$(24) \quad dX(t,\xi) = \left(\frac{\partial X}{\partial \xi}(t,\xi) + \sum_{j=1}^{d} \sigma_j(\xi) \int_0^{\xi} \sigma_j(\eta) d\eta \right) dt + \sum_{j=1}^{d} \sigma_j(\xi) dW_j(t),$$

$$(25) \quad X(0,\xi) = x(\xi), \quad x \in L^2(0,1), \quad X(t,1) = l(t), \quad t \geq 0, \quad \xi \in [0,1].$$

The boundary process $l(\cdot)$ is called the *long rate* of the market and can be any (adapted) continuous stochastic process. It is convenient to write the solution to (24), which depends now not only on x but also on the process l, in the familiar integral form:

$$X^{x,l}(t) = S(t)(x) + \int_0^t S(r)\sigma_0 dr + \sum_{j=1}^{d} \int_0^t S(t-r)\sigma_j dW_j(r) + R_t(l), \quad t \geq 0,$$

where

$$S(t)x\xi = x(t+\xi), \ \text{if } t+\xi \in [0,1], \quad S(t)x\xi = 0, \ \text{if } t+\xi \notin [0,1],$$

and

$$R_t(l)(\xi) = l((t+\xi-1)^+), \quad \xi \in [0,1], \quad t \geq 0.$$

Moreover the generater A of the semigroup $S(t)$ is again the first derivative operator, $A = \partial/\partial\xi$, but with the domain $D(A)$, consisting of absolutely continuous functions on $[0,1]$ vanishing at 1, see [33]. We refer the reader to [13] where general stochastic PDEs with random boundary data are treated by the semigroup approach. (In particular $R_t(l) = -A\big(\int_0^t S(t-s)\mathcal{D}l(s)\big) ds$ where the boundary operator $\mathcal{D}$, in the present case, acts from R^1 into H, by the formula $\mathcal{D}r(\xi) = r$ for all $\xi \in [0,1]$).

Proposition 1. *Assume that $l(t) = 0$, $t \geq 0$. There exists an invariant, linear, finite dimensional space for the equation (25) only in the trivial case when $\sigma_j = 0$, $j = 0, \ldots, d$.*

Proof. If K is an invariant closed space containing 0 then for all $j = 0, 1, \ldots, d$, and all $t \geq 0$, $S(t)\sigma_j \in K$. But if $a \in H$ and $a \neq 0$ then the vectors $S(t)a$, $t \in (0,1)$ span an infinite dimensional space. To see this let $[0,\delta] \subset [0,1]$ be the smallest interval, with the left end 0, in which the function a is not zero on a subset of positive measure. Then the functions $S_{\frac{\delta}{n}}a$, $n = 1, \ldots$ are linearly independent. □

It is more natural to ask here for the existence of finite dimensional spaces $K \subset \subset L^2(0\,,1) \times L_\gamma(0\,,+\infty)$ such that if $(x, l) \in K$, where now l denotes a deterministic function, then $(X^{x,l}(t)\,, l(t+\cdot)) \in K$ for all $t \geq 0$. To construct examples for which there exist nontrivial finite dimensional spaces of this new form one can consider, following a suggestion by D. Filipovic, the same equation but on the infinite interval and for which there exists a finite dimensional invariant subspace $K_0 \subset L_\gamma(0\,,+\infty)$. It is then enough to define K as the set of pairs: $(Px\,,Lx)\,, x \in K_0$, where $Px(\xi) = x(\xi)$, $\xi \in [0\,,1]\,, Lx(t) = x(t+1)\,, t \in [0\,,+\infty]$. Whether all invariant subspaces for the equation on $[0\,,1]$ can be obtained this way is not clear.

6. Application to second order term structure models

In papers [10, 11] different, still infinite dimensional, models of term structure were proposed. To describe them assume that the maximal maturity of bonds is 1 and denote by $s(t)$ and $l(t)$ respectively the short rate $X(t, 0)$ and the long rate $X(t, 1)$ at time $t \geq 0$. Let $m(\xi)\,, \xi \in [0\,,1]$, be the average profile of the term structure, with $m(0) = 0$, $m(1) = 1$, and let $Y(t, \xi)$, $t \geq 0$, $\xi \in [0\,,1]$, be the flactuation process, with $Y(t, 0) = Y(t, 1) = 0$. Then,

$$X(t, \xi) = s(t) + (l(t) - s(t))(m(\xi) + Y(t, \xi))\,, \quad t \geq 0\,, \quad \xi \in [0\,,1]\,. \tag{26}$$

To calibrate the model it was assumed in [11], that

$$dY(t, \xi) = \left(\frac{\partial Y}{\partial \xi}(t, \xi) + \frac{\kappa}{2}\frac{\partial^2 Y}{\partial \xi^2}(t, \xi)\right) dt + dW(t, \xi)\,, \quad t \geq 0\,, \quad \xi \in [0\,,1]\,. \tag{27}$$

In (27) W stands for $H = L^2([0\,,1])$ valued Wiener process, «white» in the space variable ξ, and κ a positive number. The first and the second partial derivatives of Y were interepreted in [11] as respectively the steepnes and the curvature of the forward rate function. In the present section we assume however that

$$W(t) = \sum_{j=1}^{d} \sigma_j W_j(t)\,, \quad t \geq 0\,, \tag{28}$$

where $\sigma_j \in H$ and W_j, $j = 1\,, \ldots, d$, are independent, real valued, Wiener processes. We ask under what conditions on the functions σ_j there exists a finite dimensional subspace of H invariant for the process Y? An answer to the problem might be of some importance in the modelling practice. It is natural to assume that

$$A = \frac{\kappa}{2}\frac{\partial^2}{\partial \xi^2} + \frac{\partial}{\partial \xi}\,,$$

with the domain $D(A)$ consisting of functions defined on $[0\,,1]$ which vanish at points 0 and 1, are absolutely continuous together with the first derivative and with the second derivative belonging to H. The corresponding semigroup will be denoted by S.

We have the following answer.

Theorem 6. *There exists a finite dimensional linear space* K *invariant for the solution* Y *of the equation* (27) *with the Wiener process* (28), *if and only if*

$$\sigma_k(\xi) = e^{-\frac{\xi}{\kappa}} \sum_{j=1}^{n_k} \alpha_{kj} \sin(j\pi\xi) \ , \quad k = 1, \dots, d \ , \quad \xi \in [0, 1] \tag{29}$$

for some natural numbers n_k, $k = 1, \dots, d$, *and some real numbers* α_{kj}, $k = 1, \dots, d$, $j = 1, \dots, n_k$.

Proof. We introduce a new, equivalent scalar product in H:

$$\langle x , y \rangle_0 = \int_0^1 x(\xi)(y\xi) e^{2\frac{\xi}{\kappa}} d\xi \ ,$$

and denote by H_0 the set H equipped with the new scalar product. It is well known that the semigroup S is self adjoint in H_0 and its generator A has a complete set of eigenvectors:

$$\sqrt{2} \sin(j\pi\xi) e^{-\frac{\xi}{\kappa}} \ j = 1, 2, \dots, \quad \xi \in [0, 1] \ ,$$

corresponding to eigenvalues $\lambda_j = \frac{1}{2\kappa}(1 + j^2\pi^2\kappa^2)$, $j = 1, 2, \dots$. However a finite dimensional space is invariant for a self-adjoint semigroup if and only if this space is spanned by eigenvectors of the generator. This follows from the fact that the semigroup restricted to that finite dimensional space is self-adjoint and therefore its generator is a self-adjoint operator on a finite dimensional space. But then its eigenvectors spann that space. This easily implies the result. □

7. Invariance for a LIBOR model

We show that our Theorem 3 is in principle applicable to an important model of LIBOR rates proposed recently by Brace, Gątarek and Musiela [8], see also [15]. Let us recall that if $L(t, \xi)$, $t, \xi \geq 0$ denotes the forward LIBOR rate at time t, then for a positive number δ and a function $\sigma(\xi)$, $\xi \geq 0$,

$$dL(t, \xi) = \left(\frac{\partial L}{\partial \xi}(t, \xi) + L(t, \xi) G(L(t))(\xi) \right) dt + L(t, \xi)\sigma(\xi) dW(t) \ , \tag{30}$$

where W is a $1-$ dimensional Wiener process and,

$$G(x)(\xi) = \sigma(\xi) \sum_{k=0}^{[\delta^{-1}\xi]} \frac{\delta x(\xi - k\delta)}{1 + \delta x(\xi - k\delta)} \sigma(\xi - k\delta) \ , \quad \xi \geq 0 \, . \tag{31}$$

The noise term in the equation (30) is not additive and therefore our Theorem 1 and Theorem 3 are not directly applicable. If we define however a new process $Y(t, \xi) = \ln L(t, \xi)$, $t, \xi \geq 0$ and apply Ito's formula we arrive at the following equation:

$$dY(t, \xi) = \left(\frac{\partial Y}{\partial \xi}(t, \xi) + F(Y(t))(\xi) \right) dt + \sigma(\xi) dW(t) \ , \tag{32}$$

where,

$$F(y)(\xi) = \sigma(\xi)\, G(e^y)(\xi) - \frac{1}{2}\sigma^2(\xi)\ , \quad \xi \geq 0\,. \tag{33}$$

One can easily check the following proposition,

PROPOSITION 2. *Assume that* $\sigma\ , \sigma^2 \in L^2_\gamma(0\,, +\infty)$, *and for a constant* M,

$$\sigma^2(\xi) \sum_{k=0}^{[\delta^{-1}\xi]} \sigma^2(\xi - k\delta) \leq M\ , \quad \xi \geq 0\ , \tag{34}$$

then the transformation F *is Lipschitz on* $L^2_\gamma(0\,, +\infty)$.

Thus under the assumptions of the propositions invariant sets for the LIBOR model are characterized by Theorem 3. Whether there exist non-trivial invariant sets for the model requires an additional work.

8. EQUATIONS WITH GENERAL PERTURBATIONS

We come back now to the general equation (1). Some invariance results for (1) can be obtained by an extension of the control methods used in §2. The extensions are related to the *Stroock-Varadhan support* theorems and to the *Wong-Zakai approximation* results. The idea to use support theorems to invariance questions has appeared in Milian's PhD Thesis [20], see also [21]. It allowed to obtain necessary and sufficient conditions for stochastic invariance, see [20, 21], analogous to the ones formulated in the following Theorem 7.

Let $(\lambda_j\ , e_j\ , j = 1\,, 2\,, \ldots)$ be the sequence of eigenvalues and eigenvectors of the covariance operator Q of the Wiener process W. Denoting by $DB(x, h)$ the Gateaux derivative of B at point x and in the direction h, we define the so called *Wong-Zakai correction* term by the formula:

$$B_{WZ}(x) = \frac{1}{2} \sum_{1}^{+\infty} \lambda_j DB(x\ , B(x)e_j)\, e_j\ , \quad x \in H\,. \tag{35}$$

Denote by $X^x(t)$, $t \geq 0$, the solution to (1) and by $X^x_n(t)$, $t \geq 0$, the mild solution of the following equation:

$$dX_n = (AX_n + (F - B_{WZ})(X_n))dt + B(X_n(t))dW_n\ , \tag{36}$$

with the same initial condition $X_n(0) = x$. Here W_n is the piecewise linear approximation of W with the derivative:

$$\frac{d}{dt}\, W_n(t) = 2^n \left(W\left(\frac{k+1}{2^n}\right) - W\left(\frac{k}{2^n}\right)\right)\ , \quad t \in \left[\frac{k}{2^n}\ , -\frac{k+1}{2^n}\right)\,. \tag{37}$$

One can prove, under various sets of conditions, that:

$$\mathcal{L}X_n \Rightarrow \mathcal{L}X\ , \text{weakly}\ , \text{as } n \to +\infty\ , \text{on } C([0\,, T]\,, H)\ , \ \text{ for any } \ T > 0\,. \tag{38}$$

Here $\mathcal{L}$ stands for the probability law of a random variable. For classical finite dimensional results we refer the reader to [31] and for a probabilistic treatment see *e.g.* [22, 3, 29].

THEOREM 7. *Assume that* (38) *holds. A closed set* $K \subset H$ *is invariant for the equation* (1) *if for each* $u \in U$ *and for each* $x \in K$ *:*

$$\liminf_{t \downarrow 0} \frac{1}{t} \operatorname{dist}[S(t)x + t(F(x) - B_{WZ}(x) + B(x)Q^{1/2}u), K] = 0 . \tag{39}$$

PROOF. If the condition (39) holds then, by theorems from [26], and from [17], the set K is invariant for the equation:

$$\frac{d}{dt}y(t) = Ay(t) + F(y(t)) - B_{WZ}(y(t)) + B(y(t))Q^{1/2}u , \quad y(0) = x \in H , \tag{40}$$

for all $u \in U$. In particular the set K is invariant for all solutions X_n of the equation (36). It is an easy consequence of the weak convergence (38) that the set K is invariant also for the limiting process X. $\square$

Although for «practical purposes» sufficient conditions for invariance are more important than the necessary ones we formulate now also necessary and sufficient conditions for invariance.

Let us recall that for several classes of stochastic equations one can identify the support of their solutions, compare [31]. Let $\mathcal{S}_T(x)$ be the closure in $C([0, T]; H)$,of the set of all functions $y(t) = y^{x,u(\cdot)}(t)$, $t \in [0, T]$, which are solutions of:

$$\frac{d}{dt}y(t) = Ay(t) + (F - B_{WZ})y(t) + B(y(t))Q^{1/2}u(t) , \quad y(0) = x , \tag{41}$$

for arbitrary piecewise constant functions $u(\cdot)$ taking values in U. Under some conditions, see [22, 23, 3] one can show that on arbitrary interval $[0, T]$ the law $\mathcal{L}X^x$ of the solution X^x of the equation (1) is identical with the set $\mathcal{S}_T(x)$. It is clear that if this is the case then the condition specified in the Theorem 7 is also necessary for the invariance.

Although the Nagumo theorem is true for arbitrary semilinear equations, the Wong-Zakai theorem and the support type theorems have not been proved for general equations. In particular they have not yet been established for equations with generators corresponding to shift semigroups. Therefore the results in [14, 7], obtained by the geometric methods, can not be obtained that way, using the existing results. However the described approach seems to be natural and can lead to more general results, see also discussions in [18, 19]. On the other hand geometric methods can give additional information on Nagumo's type conditions for deterministic equations, see [34] for more information.

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