Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Gareth P. Parry, Miroslav Šilhavý

Invariant line integrals in the theory of defective crystals

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **11** (2000), n.2, p. 111–140.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2000_9_11_2_111_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2000.

Rend. Mat. Acc. Lincei
s. 9, v. 11:111-140 (2000)

Fisica matematica. — *Invariant line integrals in the theory of defective crystals.* Nota di Gareth P. Parry e Miroslav Šilhavý, presentata (*) dal Socio G. Capriz.

Abstract. — In a continuum theory of crystals with defects, invariant line integrals measure the line defects of the lattice structure. It is shown that the integrands of invariant line integrals can always be taken to have the transformation properties of covariant vector-valued functions.

Key words: Invariant; Integral; Defect.

Riassunto. — *Invarianti integrali di linea in una teoria dei cristalli difettivi.* In una teoria che modellizza i cristalli con difetti come mezzi continui, gli invarianti integrali di linea quantificano proprio i difetti di linea nella struttura cristallina. Si mostra in questo lavoro che le integrande di tali invarianti integrali possono sempre essere scelte in modo da avere la proprietà di transformazione di un vettore covariante.

1. Introduction

The Bürgers' integrals $\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3$ of the theory of defects are

$$\mathbf{B}^a = \oint_c \mathbf{d}^a \cdot d\mathbf{x}, \quad a = 1, 2, 3, \tag{1}$$

where $\mathbf{d}^1, \mathbf{d}^2, \mathbf{d}^3$ are vector fields defined in a region $\Omega \subset \mathsf{R}^3$, and $c \subset \Omega$ is a circuit. These three integrals have the following invariance property: if $\mathbf{u} : \Omega \to \overline{\Omega}$ is a C^∞ diffeomorphism, and the fields are covariant in the sense that transformed fields $\overline{\mathbf{d}}^1, \overline{\mathbf{d}}^2, \overline{\mathbf{d}}^3$ are defined in $\overline{\Omega}$ via

$$\overline{\mathbf{d}}^a(\mathbf{u}(\mathbf{x})) = [\nabla \mathbf{u}(\mathbf{x})]^{-T} \mathbf{d}^a(\mathbf{x}), \quad a = 1, 2, 3, \tag{2}$$

where $\overline{\mathbf{x}} = \mathbf{u}(\mathbf{x})$, then

$$\oint_{\overline{c}} \overline{\mathbf{d}}^a \cdot d\overline{\mathbf{x}} = \oint_c \mathbf{d}^a \cdot d\mathbf{x}, \quad a = 1, 2, 3,$$

where $\overline{c} = \mathbf{u}(c)$.

A partial converse of this invariance property is proved in Davini [1]. The result states that if $\mathbf{P}$ is a vector-valued function of $\mathbf{d}^1, \mathbf{d}^2, \mathbf{d}^3$ (written as $\mathbf{P}(\{\mathbf{d}^a\})$), such that $\oint_c \mathbf{P}(\{\mathbf{d}^a\}) \cdot d\mathbf{x}$ is an invariant line integral in the sense that

$$\oint_{\overline{c}} \mathbf{P}(\{\overline{\mathbf{d}}^a\}) \cdot d\overline{\mathbf{x}} = \oint_c \mathbf{P}(\{\mathbf{d}^a\}) \cdot d\mathbf{x}$$

whenever (2) holds, with $\overline{\mathbf{x}} = \mathbf{u}(\mathbf{x}), \overline{c} = \mathbf{u}(c)$, for all $c \subset \Omega$, then

$$\mathbf{P} = c_a \mathbf{d}^a + \mathbf{w},$$

(*) Nella seduta del 25 febbraio 2000.

where c_1, c_2, c_3 and $\mathbf{w}$ are constants. It follows that

$$\oint_c \mathbf{P}(\{\mathbf{d}^a\}) \cdot d\mathbf{x} = c_a \oint_c \mathbf{d}^a \cdot d\mathbf{x},$$

so that all invariant line integrals of the specified form may be taken as constant linear combinations of the Bürgers' integrals. In particular notice that the integrand of any invariant line integral of this type can always be taken to have the transformation property (2) (specifically $c_a\mathbf{d}^a$ satisfies (2)).

Here we characterize invariant line integrals where the integrands have a general functional form, and so provide a broad generalization of Davini's result. Let $\mathbf{l} = (\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3)$ be three linearly independent vector fields defined over Ω and let $\mathbf{d} = (\mathbf{d}^1, \mathbf{d}^2, \mathbf{d}^3)$ be dual to $\mathbf{l}$. Let $\Delta^{(r)}$ consist of the vectors $\mathbf{l}$ together with all their spatial gradients up to order r;

$$\Delta^{(r)} = \{\mathbf{l}, \nabla\mathbf{l}, \dots, \nabla^r\mathbf{l}\},$$

and let $\overline{\Delta}^{(r)}$ be defined analogously via the equivalent of relation (2) for the duals of the fields $\overline{\mathbf{d}}$. Suppose now that $\mathbf{P} = \mathbf{P}(\Delta^{(r)})$ and consider invariant line integrals which, by definition, have the property

$$\oint_{\overline{c}} \mathbf{P}(\overline{\Delta}^{(r)}) \cdot d\overline{\mathbf{x}} = \oint_c \mathbf{P}(\Delta^{(r)}) \cdot d\mathbf{x} \tag{3}$$

for all $c \subset \Omega$, whenever (2) holds. Note that the transformation properties of the quantities $\nabla^k\mathbf{l}$, $0 \le k \le r$, obtained from the analogue of (2) by successive differentiation, generally involve nonlinear functions of the gradients $\nabla^p\mathbf{u}$, $1 \le p \le k+1$. The central result of the paper is that if $\mathbf{P}$ satisfies (3), then there exists a covariant vector field $\overline{\mathbf{P}} = \overline{\mathbf{P}}(\Delta^{(r)})$, satisfying

$$\overline{\mathbf{P}}(\overline{\Delta}^{(r)}(\mathbf{x})) = [\nabla\mathbf{u}(\mathbf{x})]^{-T}\overline{\mathbf{P}}(\Delta^{(r)}(\mathbf{x})),$$

and a scalar valued function $\phi = \phi(\Delta^{(r-1)})$, such that

$$\mathbf{P} = \overline{\mathbf{P}} + \nabla\phi.$$

It follows that

$$\oint_c \mathbf{P}(\Delta^{(r)}) \cdot d\mathbf{x} = \oint_c \overline{\mathbf{P}}(\Delta^{(r)}) \cdot d\mathbf{x},$$

so that the integrand of an invariant line integral can always be taken to have the transformation property (2).

The result does not imply that the functional dependence of $\overline{\mathbf{P}}$ on $\Delta^{(r)}$ is particularly simple, for one can construct covariant vector fields which depend nonlinearly upon $\nabla^p\mathbf{l}$, for arbitrary $1 \le p \le r$. However, the result is sufficiently powerful to give an explicit representation theorem for all invariant line integrals of this type. By combining Proposition 3.3 with Theorem 6.1, it follows that any invariant line integrand may be taken as

$$\oint_c \tau_a(E^{(r)})\mathbf{d}^a \cdot d\mathbf{x},$$

where $E^{(r)}$ is a set of suitably symmetrized covariant derivatives of the quantities $(\nabla \wedge \mathbf{d}^a \cdot \mathbf{d}^b / \det\{\mathbf{d}^a\})$, and τ_a are arbitrary functions.

We believe that the result has an intrinsic interest, but the motivation for the problem comes from a model of defective crystals [1-7, 11-15], where the state of the crystal is specified by three independent lattice vectors $\mathbf{l} = (\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3)$ defined over a region Ω occupied by the crystal. Invariant integrals

$$(4) \qquad \oint_c \mathbf{P}(\Delta^{(r)}) \cdot d\mathbf{x}, \quad \int_S \mathbf{G}(\Delta^{(r)}) \cdot \mathbf{n}\, dS(\mathbf{x}), \quad \int_V h(\Delta^{(r)})\, dV(\mathbf{x}),$$

where $\mathbf{P}$, $\mathbf{G}$ and h are vector- and scalar-valued functions of the lattice vectors and their spatial gradients up to order r, c is a circuit, $S \subset \Omega$ is a closed surface with outer normal $\mathbf{n}$ and V is a region in Ω, play an important role in the theory. These integrals are said to be elastic invariants if they remain invariant under elastic changes of state, to be defined in Section 2; for the line integrals, invariance under elastic changes of state amounts precisely to condition (3) above. Cataloguing the invariant integrals is necessary, in this theory, because of the interpretation of the integrals as measures of the «defectiveness» of the crystal. The reader is referred to the above references for more detailed motivation.

2. Basic notions of the theory of defective crystals

We start with a review of the basic notions of the theory of defective crystals [1-5]. Also some results of Parry and Šilhavý [15] are adapted.

2.1. *States and transformation rules.* A *state* of a defective crystal is $\Sigma = \{\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \Omega\}$ where $\Omega \subset \mathsf{R}^3$ is an open simply connected region and $\mathbf{l}_a$, $a = 1, 2, 3$, are C^∞ functions on Ω with values in R^3 such that for each $\mathbf{x} \in \Omega$, $\mathbf{l}_a(\mathbf{x})$ are linearly independent. We denote by $\mathbf{d}^a$ the dual basis and write $\mathbf{l} = \{\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3\}$, $\mathbf{d} = \{\mathbf{d}^1, \mathbf{d}^2, \mathbf{d}^3\}$, $\Sigma = \{\mathbf{l}, \Omega\}$. With this notation, we write $\mathbf{l} = \mathbf{1}$ or $\mathbf{d} = \mathbf{1}$ where $\mathbf{1}$ is the identity matrix to express the fact that $\mathbf{l}_a = \mathbf{e}^a$ or $\mathbf{d}^a = \mathbf{e}^a$ where $\mathbf{e}^a$ is the canonical basis in R^3. Two states $\{\mathbf{l}, \Omega\}$ and $\{\bar{\mathbf{l}}, \overline{\Omega}\}$ are said to be *elastically related* if there exists a C^∞ diffeomorphism $\mathbf{u} : \Omega \to \overline{\Omega}$ such that

$$(5) \qquad \bar{\mathbf{l}}_a(\mathbf{u}(\mathbf{x})) = \nabla\mathbf{u}(\mathbf{x})\mathbf{l}_a(\mathbf{x})$$

for each $\mathbf{x} \in \Omega$ and $a \in \{1, 2, 3\}$; hence

$$(6) \qquad \overline{\mathbf{d}}^a(\mathbf{u}(\mathbf{x})) = [\nabla\mathbf{u}(\mathbf{x})]^{-T}\mathbf{d}^a(\mathbf{x}).$$

The number of cells per unit volume n and the Burgers' vectors $\mathbf{b}^a$, given by

$$(7) \qquad n = (\mathbf{d}^1 \times \mathbf{d}^2) \cdot \mathbf{d}^3, \quad \mathbf{b}^a = \operatorname{curl} \mathbf{d}^a,$$

transform according to the rules

$$(8) \qquad \overline{\mathbf{b}}^a(\mathbf{u}(\mathbf{x})) = [\det \nabla\mathbf{u}(\mathbf{x})]^{-1}\nabla\mathbf{u}(\mathbf{x})\mathbf{b}^a(\mathbf{x}), \quad \overline{n}(\mathbf{u}(\mathbf{x})) = [\det \nabla\mathbf{u}(\mathbf{x})]^{-1} n(\mathbf{x}).$$

Throughout, r denotes a nonnegative integer. For each state $\Sigma = \{\mathbf{l}, \Omega\}$ and each $\mathbf{x} \in \Omega$ we denote

$$\Delta^{(r)}(\mathbf{x}) = \Delta_\Sigma^{(r)}(\mathbf{x}) := \{\mathbf{l}(\mathbf{x}), \nabla\mathbf{l}(\mathbf{x}), \dots, \nabla^r\mathbf{l}(\mathbf{x})\} \tag{9}$$

the collection of gradients of the lattice vectors up to order r. We denote by $\mathcal{D}^{(r)}$ the set of all possible values of $\Delta^{(r)}$, which is the set of all $r+1$ tuples $\{\mathbf{l}^{(0)}, \dots, \mathbf{l}^{(r)}\}$ respecting the symmetry of partial derivatives. Thus the components of $\mathbf{l}^{(k)} = \nabla^k\mathbf{l}$ are $l^{(k)}_{ai_1, i_2 \dots i_{k+1}}$ where the indices following the comma denote the partial differentiation with respect to the corresponding components of $\mathbf{x}$. The summation convention is used for repeated indices, including the indices enumerating the lattice vectors. We use the multiindex notation to abbreviate collections of indices; thus a multiindex I of order r is an ordered r tuple

$$I = (i_1, \dots, i_r), \quad i_1, \dots, i_r = 1, 2, 3. \tag{10}$$

We denote by M^r the set of all multiindices of order r. If I is as in (10) and $i \in \{1, 2, 3\}$ we write $I \cup i = (i_1, \dots, i_r, i)$, $i \cup I = (i, i_1, \dots, i_r)$ and often abbreviate $I \cup i = Ii$, $i \cup I = iI$. The multiplicity $m(i, I)$ of i in I is the number of occurrences of i among $i_1, \dots, i_r$; if $m(i, I) > 0$ we define $I \setminus i$ to be the multiindex of order $r-1$ obtained from I by deleting the last occurrence of i among $i_1, \dots, i_r$. The minimal multiplicity $M(I)$ of I is the minimum of $m(1, I), m(2, I), m(3, I)$. The summation convention is extended for twice repeated multiindices; however, if the formula contains the symbol $m(i, I)$ then no summation is executed over i and I unless explicitly indicated; cf., *e.g.*, (35) and (36).

2.2. *Differential functions, elastic scalars and elastic vectors.* A function $f : \mathcal{D}^{(r)} \to M$ with values in any set is said to be a *differential function* of order r. (We borrow this terminology from Olver [9]). A differential function f of order r is said to be a *homogeneous differential function* of order r if it depends exclusively on $\nabla^r\mathbf{d}$,

$$f(\Delta^{(r)}) = \overline{f}(\nabla^r\mathbf{d}).$$

If f is a differential function of order r and $\Delta_\circ^{(r-1)} \in \mathcal{D}^{(r-1)}$ is a fixed element then the *freeze* $f_{[\Delta_\circ^{(r-1)}]}$ of f at $\Delta_\circ^{(r-1)}$ is a homogeneous differential function of order r given by

$$f_{[\Delta_\circ^{(r-1)}]}(\Delta^{(r)}) = f(\Delta_\circ^{(r-1)}, \nabla^r\mathbf{d})$$

for each $\Delta^{(r)}$ of the form (9). Our considerations will often involve the analysis of the highest order terms in the equations. To simplify the statements and notation in these situations, we say that two differential functions f, g are r-equivalent if both f and g are differential functions of order r with values in some vector space and there exists a differential function h of order $r-1$ such that $f - g = h$; we denote this fact by writing

$$f \overset{(r)}{=} g. \tag{11}$$

A real-valued differential function f of order r is said to be an *elastic scalar invariant* (briefly, a *scalar,* or an *invariant*) of order r if for any two elastically related states $\{\mathbf{l}\,,\Omega\}$ and $\{\bar{\mathbf{l}}\,,\overline{\Omega}\}$ and any $\mathbf{x}\in\Omega$ we have

$$f(\overline{\Delta}^{(r)}(\mathbf{u}(\mathbf{x})))=f(\Delta^{(r)}(\mathbf{x}))$$

where $\overline{\Delta}^{(r)}$ is given by

$$\overline{\Delta}^{(r)}=\overline{\Delta}^{(r)}(\mathbf{y})=\{\bar{\mathbf{l}}\,,\overline{\nabla}\,\bar{\mathbf{l}}\,,\dots,\overline{\nabla}^{k}\,\bar{\mathbf{l}}\}\,, \tag{12}$$

$\overline{\nabla}$ denotes the differentiation with respect to $\mathbf{y}=\mathbf{u}(\mathbf{x})\in\overline{\Omega}\,,$ and $\mathbf{u}$ is the deformation mapping $\{\mathbf{l}\,,\Omega\}$ to $\{\bar{\mathbf{l}}\,,\overline{\Omega}\}$. An R^3-valued differential function $\mathbf{Q}$ of order r is said to be an *elastic contravariant vector* of order r if

$$\mathbf{Q}(\overline{\Delta}^{(r)}(\mathbf{u}(\mathbf{x})))=\nabla\mathbf{u}(\mathbf{x})\mathbf{Q}(\Delta^{(r)}(\mathbf{x})) \tag{13}$$

and an R^3-valued differential function $\mathbf{P}$ of order r is said to be an *elastic covariant vector* of order r if

$$\mathbf{P}(\overline{\Delta}^{(r)}(\mathbf{u}(\mathbf{x})))=[\nabla\mathbf{u}(\mathbf{x})]^{-T}\mathbf{P}(\Delta^{(r)}(\mathbf{x})) \tag{14}$$

where $\Delta^{(r)}$ and $\overline{\Delta}^{(r)}$ are as above and (13), (14) hold for any two elastically related states. An R^3-valued differential function $\mathbf{M}$ of order r is said to be a *weighted elastic contravariant vector* of order r if $\mathbf{M}/n$ is an elastic contravariant vector. All differential funcions are assumed to be inifinitely differentiable with respect to the components of $\Delta^{(r)}$. This assumption is consistent: all the differential functions derived below from given ones by various operations will be infinitely differentiable, too. If f is any smooth differential function with values in some finite-dimensional vector space we denote by ∇f the *full gradient* of f calculated by formally applying the chain rule, *i.e.*,

$$\nabla f(\Delta^{(r+1)}):=\sum_{p=0}^{k}\partial_{\nabla^{p}\mathbf{l}}f(\Delta^{(r)})\nabla^{p+1}\mathbf{l},\quad \Delta^{(r+1)}\in\mathcal{D}^{(r+1)}\,, \tag{15}$$

which is a differential function of order $r+1$. Similarly, if $\mathbf{M}$ is an R^3-valued differential function of order r then its full curl and full div are the differential functions of order $r+1$ defined by

$$(\operatorname{curl}\mathbf{M})_i=\epsilon_{ijk}M_{j,k}\,,\quad \operatorname{div}\mathbf{M}=M_{i,i}$$

where $M_{j,k}$ is the k-th component of the full gradient ∇M_j of M_j. (For conventional reasons we use an unusual sign in the definition of curl). If f is a scalar, $\mathbf{M}$ a weighted elastic contravariant vector and $\mathbf{P}$ an elastic covariant vector then

∇f	is an elastic covariant vector,
$\operatorname{div}\mathbf{M}/n$	is a scalar, and
$\operatorname{curl}\mathbf{P}$	is a weighted elastic contravariant vector.

2.3. *Integral invariants.* Consider the integrals (4) where $\mathbf{P}\,,\mathbf{G}$ are R^3-valued differential functions of order r, h is a real-valued function of order $r,$ c is a closed curve in

Ω, S is a boundary of a Lipschitz region in Ω and V is a Lipschitz region in Ω. The function $\mathbf{P}$ is said to be an *elastic invariant line integrand* if for each pair $\{\mathbf{l}, \Omega\}$ and $\{\bar{\mathbf{l}}, \overline{\Omega}\}$ of elastically related states with the elastic change $\mathbf{u}$ we have

$$\int_c \mathbf{P}(\Delta^{(r)}) \cdot d\mathbf{x} = \int_{\mathbf{u}(c)} \mathbf{P}(\overline{\Delta}^{(r)}) \cdot d\mathbf{y} \tag{16}$$

where $\Delta^{(r)}, \overline{\Delta}^{(r)}$ are given by (9) and (12), respectively, and (16) holds for each closed curve in Ω. The function $\mathbf{G}$ is said to be an *elastic invariant surface integrand* if

$$\int_S \mathbf{G}(\Delta^{(r)}) \cdot d\mathbf{x} = \int_{\mathbf{u}(S)} \mathbf{G}(\overline{\Delta}^{(r)}) \cdot d\mathbf{y}\,, \tag{17}$$

for each boundary S of a Lipschitz region in Ω; h is said to give an *elastic invariant volume integrand* if

$$\int_V h(\Delta^{(r)}) \cdot d\mathbf{x} = \int_{\mathbf{u}(V)} h(\overline{\Delta}^{(r)}) \cdot d\mathbf{y} \tag{18}$$

for each open subset V of Ω. In this situation, we also say that (4) are elastic invariant integrals. The functions $\mathbf{P}$ and $\mathbf{G}$ are not uniquely determined by the values of the integrals $(4)_{1,2}$. Namely, the passages $\mathbf{P} \mapsto \mathbf{P}^*$, $\mathbf{G} \mapsto \mathbf{G}^*$ where

$$\mathbf{P}^* = \mathbf{P} + \nabla F, \quad \mathbf{G}^* = \mathbf{G} + \operatorname{curl} \mathbf{W}\,, \tag{19}$$

where F is any real-valued differential function and $\mathbf{W} = \mathbf{W}(\Delta^{(r-1)})$ is any R^3-valued differential function, leaves the values of $(4)_{1,2}$ unchanged. We say that $\mathbf{P}$ and $\mathbf{P}^*$ ($\mathbf{G}$ and $\mathbf{G}^*$) are equivalent if $(19)_1$ holds for some F ($(19)_2$ holds for some $\mathbf{W}$).

Proposition 2.4. *The integrals* $(4)_{1,2,3}$ *are elastic invariants if and only if*

$\operatorname{curl} \mathbf{P}$	*is a weighted elastic contravariant vector,*
$\operatorname{div} \mathbf{G}/n$	*is a scalar, and*
h/n	*is a scalar, respectively.*

Proof. Apply the Stokes and Gauss theorems to convert the line and surface integrals in (16) and (17) into the surface and volume integrals, then apply the change of variable formulas for the surface and volume integrals to eliminate the images of the surfaces and volumes under $\mathbf{u}$ and use the arbitrariness of the surfaces and volumes. $\square$

3. Preliminaries

3.1. It is convenient to divide the gradient of lattice vectors

$$\mathbf{d}^{(r)a} := \nabla^r \mathbf{d}^a = \{d^{(r)a}_{i_1,\dots i_{r+1}} : i_1, \dots, i_{r+1} = 1, 2, 3\}\,,$$

into a *symmetric part of the gradient of lattice vectors* $\mathbf{s}^{(r)a}$ and a *skew part of the gradient of* *lattice vectors* $\mathbf{g}^{(r)a}$ where

$$\mathbf{s}^{(r)a} = \{s^{(r)a}_{i_1 \dots i_{r+1}} : i_1, \dots, i_{r+1} = 1, 2, 3\},$$

$$\mathbf{g}^{(r)a} = \{g^{(r)a}_{i_1 \dots i_r} : i_1, \dots i_r = 1, 2, 3\},$$

are defined by

$$(20) \qquad \begin{cases} s^{(r)a}_{i_1 \dots i_{r+1}} &= \dfrac{d^{(r)a}_{i_1, \dots i_{r+1}} + d^{(r)a}_{i_2, i_1 i_3 \dots i_{r+1}} + \dots + d^{(r)a}_{i_{r+1}, i_1 \dots i_r}}{r+1}, \\ g^{(r)a}_{i_1 \dots i_r} &= \epsilon_{i_1 jk} d^{(r)a}_{j, k i_2 \dots i_{r+1}}. \end{cases}$$

It is noted that $g^{(r)a}_{i_1 \dots i_r}$ is symmetric in $i_2, \dots, i_r$ and for $r > 2$, $\mathbf{g}^{(r)a}$ is traceless in the sense

$$(21) \qquad g^{(r)a}_{mmi_3 \dots i_r} = g^{(r)a}_{mi_2 mi_3 \dots i_r} = \dots = g^{(r)a}_{mi_2 \dots i_{r-1} m} = 0.$$

Generally, we say that a system $f = \{f_I, I \in \mathsf{M}^r\}$ is *traceless* (with respect to I) if

$$f_{mmi_3 \dots i_r} = f_{mi_2 mi_4 \dots i_r} = \dots = f_{mi_2 i_3 \dots i_{r-1} m} = 0$$

and that f is *symmetric in the* **last** *indices* if f is symmetric in the last $r-1$ indices. Note

$$\mathbf{g}^{(1)a} = \mathbf{b}^a = \operatorname{curl} \mathbf{d}^a \text{ and } \mathbf{g}^{(r)a} = \nabla^{r-1} \mathbf{b}^a \text{ for } r > 1.$$

Define $\mathbf{s}^{(0)} = \mathbf{d}$. We also use the notation

$$\mathbf{d}^{(r)} = \{d^{(r)a}_{i_1 \dots i_{r+1}} : a, i_1, \dots, i_{r+1} = 1, 2, 3\}, \text{ and similarly for } \mathbf{s}^{(r)}, \mathbf{g}^{(r)}.$$

We have

$$(22) \quad d^{(r)a}_{i_1, \dots i_{r+1}} = \frac{1}{r+1} \{\epsilon_{i_1 i_2 s} g^{(r)a}_{s i_3 \dots i_{r+1}} + \epsilon_{i_1 i_3 s} g^{(r)a}_{s i_2 i_4 \dots i_{r+1}} + \dots + \epsilon_{i_1 i_{r+1} s} g^{(r)a}_{s i_2 \dots i_r}\} + s^{(r)a}_{i_1 \dots i_{r+1}},$$

$$(23) \quad s^{(r)a}_{i_1 \dots i_{r+1}, i} = s^{(r+1)a}_{i_1 \dots i_{r+1} i} - \frac{1}{(r+1)(r+2)} \{\epsilon_{ii_1 s} g^{(r+1)a}_{s i_2 \dots i_{r+1}} + \epsilon_{ii_2 s} g^{(r+1)a}_{s i_1 i_3 \dots i_{r+1}} + \dots + \epsilon_{ii_{r+1} s} g^{(r+1)a}_{s i_1 \dots i_r}\}.$$

Moreover, $\nabla^r \mathbf{d}^a$ can be reconstructed from $\mathbf{g}^{(r)a}$ and $\mathbf{s}^{(r)a}$ and these two collections can be chosen independently and arbitrarily subject to the above restrictions. However, the roles of $\mathbf{g}^{(r)a}$ and $\mathbf{s}^{(r)a}$ are entirely different from the point of view of the elastic invariance: It is shown in Parry and Šilhavý [15, Remark 3.3] that at a given point, $\bar{\mathbf{s}}^{(\alpha)a}$, $1 \le \alpha \le r$, can be made vanishing by an appropriate elastic change of state from the original one, while there is no way to annihilate the Burgers' vector $\mathbf{b}^a$ and its gradients $\mathbf{g}^{(\alpha)a}$ by an elastic change of state. Moreover, there exists a fully invariant version of the lattice components of $\mathbf{g}^{(r)a}$, namely the components of the scalar invariants $W^{(r)}$ to be now introduced.

3.2. *Basic scalar invariants.* Let $\Sigma = \{\mathbf{l}, \Omega\}$ and $\mathbf{x} \in \Omega$ be given and for each $r \ge 1$ define inductively the objects $Z^{(r)}, Y^{(r)}, W^{(r)}$ at $\mathbf{x}$ as follows. For $r = 1$, $Z^{(1)}$ is a collection

$$Z^{(1)} = \{Z^{(1)ab}, a, b = 1, 2, 3\},$$

and $Y^{(1)}$, $W^{(1)}$ are similarly defined collections of $Y^{(1)ab}$, $W^{(1)ab}$, respectively, where

(24) $$Z^{(1)ab} := Y^{(1)ab} := W^{(1)ab} := \mathbf{b}^a \cdot \mathbf{d}^b / n.$$

For each $r > 1$ let further $Z^{(r)}$ be the collection

$$Z^{(r)} = \{ Z^{(r)ab}_{c_1 \dots c_{r-1}}, a, b, c_1, \dots, c_{r-1} = 1, 2, 3 \},$$

and $Y^{(r)}$, $W^{(r)}$ similarly defined collections of $Y^{(r)ab}_{c_1 \dots c_{r-1}}$, $W^{(r)ab}_{c_1 \dots c_{r-1}}$, respectively, where

$$Z^{(r)ab}_{c_1 \dots c_{r-1}} := \mathbf{l}_{c_{r-1}} \cdot \nabla Z^{(r-1)ab}_{c_1 \dots c_{r-2}},$$
$$Y^{(r)ab}_{c_1 \dots c_{r-1}} := \text{the symmetrization with respect to } c_1, \dots, c_{r-1} \text{ of } Z^{(r)ab}_{c_1 \dots c_{r-1}},$$

and

$$W^{(r)ab}_{c_1 \dots c_{r-1}} := Y^{(r)ab}_{c_1 \dots c_{r-1}} - \frac{1}{r+1} (\delta^b_{c_1} Y^{(r)am}_{mc_2 \dots c_{r-1}} + \dots + \delta^b_{c_{r-1}} Y^{(r)am}_{mc_1 \dots c_{r-2}}).$$

The last is the traceless part of $Y^{(r)ab}_{c_1 \dots c_{r-1}}$ in the sense

(25) $$W^{(r)am}_{mc_2 \dots c_{r-1}} = \dots = W^{(r)am}_{c_1 \dots c_{r-2} m} = 0.$$

We also denote

$$E^{(r)} = (W^{(1)}, \dots, W^{(r)}) \quad \text{if} \quad r \geq 1 \text{ and } E^{(0)} = \emptyset$$

and write $E^{(r)}_\Sigma(\mathbf{x})$ to emphasize that this object is associated with the state Σ and the point $\mathbf{x}$. It is also possible to interpret $Z^{(r)}, \dots, E^{(r)}$ as differential functions of order r and if necessary, we write

$$W^{(r)} = \mathsf{W}^{(r)}(\Delta^{(r)}), \quad E^{(r)} = \mathsf{E}^{(r)}(\Delta^{(r)})$$

to emphasize that $W^{(r)}$, $E^{(r)}$ are the values associated with $\Delta^{(r)}$. Let $\mathcal{W}^{(r)}$ be the space of all collections $W^{(r)} = \{ W^{(r)ab}_{c_1 \dots c_{r-1}}, a, b, c_1, \dots, c_{r-1} = 1, 2, 3 \}$, symmetric with respect to $c_1 \dots c_{r-1}$ and traceless in the sense (25), and let $\mathcal{E}^{(r)}$ be the space of all $E^{(r)}$, which is

$$\mathcal{E}^{(r)} = \mathcal{W}^{(1)} \times \mathcal{W}^{(2)} \times \dots \times \mathcal{W}^{(r)}.$$

Each component of $Z^{(r)}$, $Y^{(r)}$ and $W^{(r)}$ is a scalar and, in the notation (11),

(26) $$\begin{cases} \mathbf{g}^{(r)a} \overset{(r)}{=} n W^{(r)ab}_{c_1 \dots c_{r-1}} \mathbf{l}_b \otimes \mathbf{d}^{c_1} \otimes \dots \otimes \mathbf{d}^{c_{r-1}}, \\ W^{(r)ab}_{c_1 \dots c_{r-1}} \overset{(r)}{=} n^{-1} \mathbf{d}^b \cdot \mathbf{g}^{(r)a}[\mathbf{l}_{c_1}, \dots, \mathbf{l}_{c_{r-1}}] \end{cases}$$

where

$$\mathbf{g}^{(r)a}[\mathbf{l}_{c_1}, \dots, \mathbf{l}_{c_{r-1}}]_i = g^{(r)a}_{ii_1 \dots i_{r-1}} l_{c_1 i_1}, \dots, l_{c_{r-1} i_{r-1}},$$

see Parry and Šilhavý [15, Remark 3.5 and Lemma 3.6]. Furthermore, $W^{(\alpha)}$, $1 \leq \alpha \leq r$, and $\mathbf{s}^{(\alpha)}$, $0 \leq \alpha \leq r$, can be chosen independently and they determine uniquely all the gradients $\nabla^\alpha \mathbf{d}^a$ for all $0 \leq \alpha \leq r$. For each $E^{(r)}_\circ \in \mathcal{E}^{(r)}$ there exists a unique $\Delta^{(r)}_\circ \in \mathcal{D}^{(r)}$ such that the objects $\mathbf{d}, \mathbf{s}^{(\alpha)}$, $\alpha = 1, \dots, r$, and $E^{(r)}$ corresponding to $\Delta^{(r)}_\circ$ satisfy

(27) $$\mathbf{d} = \mathbf{1}, \quad \mathbf{s}^{(\alpha)} = 0, \quad \alpha = 1, \dots, r, \text{ and } E^{(r)}_\circ = E^{(r)}.$$

If $\mathsf{D}^{(r)}(E_\circ^{(r)})$ denotes this unique $\Delta_\circ^{(r)}$, then the components of $\mathsf{D}^{(r)}(E_\circ^{(r)})$ are polynomials in the components of $E_\circ^{(r)}$ [15, Remark 3.7].

Proposition 3.3.

(*a*) *Differential functions* $f, \mathbf{P}, \mathbf{Q}$ *are an elastic scalar, elastic covariant vector, and elastic contravariant vector, respectively, if and only if*

$$f(\Delta^{(r)}) = \overline{f}(E^{(r)}), \quad \mathbf{P}(\Delta^{(r)}) = \tau_a(E^{(r)})\mathbf{d}^a, \quad \mathbf{Q}(\Delta^{(r)}) = \omega_a(E^{(r)})\mathbf{l}_a, \tag{28}$$

respectively, where $\overline{f}$, τ_a *and* ω_a *are suitable functions*;

(*b*) *if* f *is a polynomial in the components of* $\Delta^{(r)}$ *then* $\overline{f}$ *is a polynomial in the components of* $E^{(r)}$; *if* f *depends affinely (quadratically) on* $\nabla^r\mathbf{d}$ *with the coefficients depending on* $\Delta^{(r-1)}$ *then* $\overline{f}$ *depends affinely (quadratically) on* $W^{(r)}$ *with the coefficients depending on* $E^{(r-1)}$. *The same applies to* $\mathbf{P}, \mathbf{Q}$.

Proof. Everything except the statements about the affine (quadratic) dependencies is a restatement of Parry and Šilhavý [15, Theorems 4.1 and 4.2]. Let us prove the assertion on the affine dependence. Thus let f be of the form

$$f(\Delta^{(r)}) = G(\Delta^{(r-1)})[\nabla^r\mathbf{d}] + H(\Delta^{(r-1)}). \tag{29}$$

Here and in the sequel we write $\alpha(\Delta^{(r-1)})[\beta]$ to indicate that the expression depends linearly on β with the coefficients depending on $\Delta^{(r-1)}$ where β is any quantity. By (20) and (22) there is a linear correspondence between $\nabla^r\mathbf{d}$ and the pairs $(\mathbf{s}^{(r)}, \mathbf{g}^{(r)})$. By (26), for each fixed $\Delta^{(r-1)}$ there is a linear correspondence between $W^{(r)}$ and $\mathbf{g}^{(r)}$. Thus for each fixed $\Delta^{(r-1)}$ there is a linear correspondence between $\nabla^r\mathbf{d}$ and the pairs $(\mathbf{s}^{(r)}, W^{(r)})$. Hence by (29),

$$f(\Delta^{(r)}) = \overline{G}_{\mathbf{s}}(\Delta^{(r-1)})[\mathbf{s}^{(r)}] + \overline{G}_W(\Delta^{(r-1)})[W^{(r)}] + \overline{H}(\Delta^{(r-1)}).$$

On the other hand, by the representation theorem $(28)_1$ we have

$$\overline{G}_{\mathbf{s}}(\Delta^{(r-1)})[\mathbf{s}^{(r)}] + \overline{G}_W(\Delta^{(r-1)})[W^{(r)}] + \overline{H}(\Delta^{(r-1)}) = \overline{f}(E^{(r)}). \tag{30}$$

Using that $\mathbf{s}^{(r)}$ and $W^{(r)}$ may be chosen independently we see that the first term on the left-hand side of (30) must vanish identically and

$$\overline{G}_W(\Delta^{(r-1)})[W^{(r)}] + \overline{H}(\Delta^{(r-1)}) = \overline{f}(E^{(r)}).$$

Hence $\overline{f}(E^{(r)})$ is affine in $W^{(r)}$. The assertion about the quadratic dependence is proved similarly. □

Remark 3.4. *For each* $r \geq 1$ *there exists a function* $F^{(r)}$ *such that*

$$\mathbf{l}_a \cdot \nabla W_I^{(r)pq} = W_{I\cup a}^{(r+1)pq} + F_{Ia}^{(r)pq}(E^{(r)}); \tag{31}$$

moreover, $F^{(r)}$ *is a polynomial in the components of* $E^{(r)}$ *that for* $r \geq 2$ *depends affinely on* $W^{(r)}$.

For example, for $r = 1$ we have

$$\mathbf{l}_a \cdot \nabla W^{(1)pq} = W_a^{(2)pq} - \tfrac{1}{3}\delta_a^q \epsilon_{bcd} W^{(1)pb} W^{(1)cd}; \tag{32}$$

(see [15]).

Proof. Prove first that for each $r \geq 1$ we have

$$W_I^{(r)pq} = Z_I^{(r)pq} + h_I^{(r)pq}(E^{(r-1)}) \tag{33}$$

where $h_I^{(r)pq}$ is a polynomial in $E^{(r-1)}$ that for $r \geq 3$ depends affinely on $W^{(r-1)}$. We use that by Parry and Šilhavý [15, Lemma 3.6, eq. (31)],

$$W_{c_1 \ldots c_{r-1}}^{(r)ab} = Z_{c_1 \ldots c_{r-1}}^{(r)ab} + \mu_{c_1 \ldots c_{r-1}}^{(r)ab}(\Delta^{(r-1)}). \tag{34}$$

where $\mu^{(r)}$ is a polynomial in the components of $\Delta^{(r-1)}$ that for $r \geq 3$ depends affinely on $\nabla^{r-1}\mathbf{d}$. Observe that since $W_I^{(r)pq}$ and $Z_I^{(r)pq}$ are elastic scalars, so also is $\mu_I^{(r)pq}$. By the representation theorem (28)

$$\mu_I^{(r)pq}(\Delta^{(r-1)}) = h_I^{(r)pq}(E^{(r-1)})$$

and the last is a polynomial in $E^{(r-1)}$ that for $r \geq 3$ depends affinely on $W^{(r-1)}$ since $\mu^{(r)}$ is a polynomial in $\Delta^{(r-1)}$ that for $r \geq 3$ depends affinely on $\nabla^{r-1}\mathbf{d}$ and the polynomiality and affine dependence are preserved by Proposition 3.3. Let us now prove (31). For $r = 1$ this is (32). Assume that the assertion is true for all orders $\leq r - 1$. Differentiating (33) in the direction $\mathbf{l}_a$ we obtain

$$\mathbf{l}_a \cdot \nabla W_I^{(r)pq} = Z_{I \cup a}^{(r+1)pq} + \mathbf{l}_a \cdot \nabla h_I^{(r)pq}(W^{(1)}, \ldots, W^{(r-1)})$$

and eliminating $Z_{I \cup a}^{(r+1)pq}$ by (33) for $r = r + 1$

$$\mathbf{l}_a \cdot \nabla W_I^{(r)pq} = W_{I \cup a}^{(r+1)pq} - h_{I \cup a}^{(r+1)pq}(E^{(r)}) + \mathbf{l}_a \cdot \nabla h_I^{(r)pq}(W^{(1)}, \ldots, W^{(r-1)}).$$

Now $h_{I \cup a}^{(r+1)pq}(E^{(r)})$ is a polynomial in $E^{(r)}$ that for $r \geq 2$ depends affinely on $W^{(r)}$ as part of the assertion (33). Furthermore, expanding

$$T := \mathbf{l}_a \cdot \nabla h_I^{(r)pq}(W^{(1)}, \ldots, W^{(r-1)})$$

we obtain a polynomial in the variables

$$W^{(1)}, \ldots, W^{(r-1)}, \text{ and } \mathbf{l}_a \cdot \nabla W^{(1)}, \ldots, \mathbf{l}_a \cdot \nabla W^{(r-1)}$$

which is affine in the last set of variables. By the induction hypothesis,

$$\mathbf{l}_a \cdot \nabla W^{(1)} = G^{(1)}(E^{(2)}), \quad \ldots, \quad \mathbf{l}_a \cdot \nabla W^{(r-1)} = G^{(r-1)}(E^{(r)})$$

where $G^{(1)}, \ldots, G^{(r-1)}$ denote the whole right-hand side of (31) for $r = 1, \ldots, r-1$, respectively. From that we see in particular that $G^{(r-1)}(E^{(r)})$ depends on $W^{(r)}$ affinely while $G^{(1)}, \ldots, G^{(r-2)}$ are independent of $W^{(r)}$. Hence also T is a polynomial in $E^{(r)}$ that depends on $W^{(r)}$ affinely. □

4. Auxiliary results

This section is devoted to deriving some results whose proofs would inappropriately break the considerations in the subsequent sections, or which are used more than once in different situations. Thus the reader may skip this section at a first reading, leaving it for reference as needed.

After introducing some notation, the first part of the section provides remarks on the algebra of systems with unequal number of indices. The notion of multiplicity is central here. Then, using these remarks, some simple assertions are proved for homogeneous R^3-valued differential functions with vanishing curl-type expressions. The basic moral here is that such homogeneous functions have a potential which is again a homogeneous function. Hence the spatial potential, whose existence is immediate from the elementary vector analysis, is again given by a *constitutive equation* (in the jargon of continuum mechanics), and it is exactly this fact that is crucial for our considerations. Assertions of this type might also be derived by specializing, to this situation, the homotopy formula for differential forms with coefficients which are differential functions, as treated by Olver [8, 9, Chapter 5].

We denote by $\mathbf{C}^r$ the set of all collections $C = \{C_I : I \in \mathsf{M}^r\}$, by $\mathbf{S}^r$ the subset of $\mathbf{C}^r$ consisting of all collections C symmetric in the indices of I, *i.e.*,

$$C_{\pi(I)} = C_I$$

for each permutation $\pi(I)$ of I, by $\mathbf{C}^r_{\mathsf{L}}$ the subset of $\mathbf{C}^r$ consisting of all C which are symmetric in the last $r-1$ indices and by $\mathbf{C}^r_{\mathsf{TL}}$ the subset of $\mathbf{C}^r_{\mathsf{L}}$ whose elements are traceless.

Remark 4.1.
(*a*) *Let $C \in \mathbf{S}^r$ and define the collection $\widetilde{C} \in \mathbf{C}^{r+1}$ by*

$$\widetilde{C}_{iL} = \frac{m(i, L)}{r+1} C_{L\setminus i}. \tag{35}$$

Then $\widetilde{C} \in \mathbf{C}^{r+1}_{\mathsf{L}}$ and

$$C_{i_1\ldots i_r} s_{i_1\ldots i_r i} = \widetilde{C}_{iL} s_L = \frac{1}{r+1} \sum_{L\in \mathsf{M}^{r+1}} m(i, L) C_{L\setminus i} s_L \tag{36}$$

for each $s \in \mathbf{S}^{r+1}$.
(*b*) *Let $S \in \mathbf{C}^{r+1}_{\mathsf{L}}$ and define $S^\circ \in \mathbf{C}^{r+1}$ by*

$$S^\circ_{aJ} = S_{aJ} - \frac{m(a, J)}{r+2} T_{J\setminus a}, \quad a = 1, 2, 3, \; J \in \mathsf{M}^r,$$

where $T \in \mathbf{S}^{r-1}$ is the trace defined by $T_K = S_{aaK}$, $K \in \mathsf{M}^{r-1}$. Then $S^\circ \in \mathbf{C}^{r+1}_{\mathsf{TL}}$ and $S^\circ_I F_I = S_I F_I$ for each $F \in \mathbf{C}^{r+1}_{\mathsf{TL}}$.

If $m(i, L) = 0$ then $L \setminus i$ and hence $C_{L\setminus i}$ in (36) are undefined; however, we consistently interpret the product $m(i, L) C_{L\setminus i}$ as 0 throughout the paper. Recall also the convention that no summation is executed over i and L in (35) since the formula contains $m(i, L)$.

Proof. (*a*): We write

$$C_{i_1\ldots i_r} s_{i_1\ldots i_r i} = C_{i_1\ldots i_r} \delta_{i i_{r+1}} s_{i_1\ldots i_r i_{r+1}};$$

in view of the symmetry of s we can symmetrize the coefficient in front of $s_{i_1\dots i_r i_{r+1}}$, which gives the symmetrized coefficient

$$\widetilde{C}_{ii_1\dots,i_r i_{r+1}} := \frac{1}{r+1}\Big\{C_{i_{r+1}i_2\dots i_r}\delta_{ii_1} + C_{i_{r+1}i_1 i_3\dots i_r}\delta_{ii_2} + \dots + C_{i_1\dots i_r}\delta_{ii_{r+1}}\Big\}.$$

Let $L = (i_1, \dots, i_r, i_{r+1})$. Then the first term in the curly brackets is nonzero only if $i = i_1$ and then $(i_{r+1}, i_2, \dots i_r)$ is a permutation of $L \setminus i$. In view of the symmetry of C we have

$$C_{i_{r+1}i_2\dots i_r} = C_{L\setminus i}.$$

Similarly for the terms that follow. Since there are exactly $m(i, L)$ indices in L which are equal to i, we obtain (36). (b): Clearly, S° has the desired symmetry. Let us check that S° is traceless. We have

$$S^\circ_{aaK} = S_{aaK} - \frac{1}{r+2}\sum_{a=1}^{3}(m(a, K) + 1)T_K = T_K - \frac{T_K}{r+2}(r - 1 + 3) = 0$$

where we have used

$$\sum_{a=1}^{3} m(a, K) = r - 1.$$

The equality $S^\circ_I F_I = S_I F_I$ follows from $F \in \mathsf{C}^{r+1}_{\mathsf{TL}}$. $\square$

Remark 4.2.

Let $r \geq 1$ *and* $C \in \mathsf{C}^{r+1}$.

(a) *If* $C \in \mathsf{C}^{r+1}_{\mathsf{L}}$ *and*

$$C_{iI}s_{I\cup j} = C_{jI}s_{I\cup i}, \quad i, j = 1, 2, 3, \tag{37}$$

for each $s \in \mathsf{S}^{r+1}$ *then there exists a* $D \in \mathsf{S}^{r-1}$ *such that*

$$C_{iI} = \frac{m(i, I)}{r} D_{I\setminus i}; \tag{38}$$

in other words,

$$C_{iI}t_I = D_{j_1\dots j_{r-1}} t_{j_1\dots j_{r-1} i} \tag{39}$$

for each $t \in \mathsf{S}^r$.

(b) *If* $r \geq 2$, *if* C *is symmetric in the last* $r - 1$ *indices and* (37) *holds for each* $s \in \mathsf{C}^{r+1}_{\mathsf{L}}$ *then there exists a* $D \in \mathsf{C}^{r-1}_{\mathsf{L}}$ *such that*

$$C_{ikI} = \frac{m(i, I)}{r-1} D_{k(I\setminus i)} \tag{40}$$

and (39) *holds for each* $t \in \mathsf{C}^r_{\mathsf{L}}$.

Proof. (a): By Remark 4.1(a), (37) may be rewritten as

$$\sum_{\widetilde{I}\in\mathsf{M}^{r+1}} m(j, \widetilde{I})C_{i\cup(\widetilde{I}\setminus j)}s_{\widetilde{I}} = \sum_{\widetilde{I}\in\mathsf{M}^{r+1}} m(i, \widetilde{I})C_{j\cup(\widetilde{I}\setminus i)}s_{\widetilde{I}}.$$

The arbitrariness of s implies

(41) $$m(j,\widetilde{I})\,C_{i\cup(\widetilde{I}\setminus j)} = m(i,\widetilde{I})\,C_{j\cup(\widetilde{I}\setminus i)}.$$

Setting $\widetilde{I} = I\cup j$, where $I\in \mathsf{M}^r$, gives

(42) $$(m(j,I)+1)\,C_{i\cup I} = (m(i,I)+\delta_{ij})\,C_{j\cup((I\cup j)\setminus i)}.$$

Define

$$D_J = \frac{r}{r+2}C_{jJ\cup j}, \quad J\in \mathsf{M}^{r-1},$$

and sum (42) over $j=1,2,3$ to obtain

(43) $$(r+3)\,C_{i\cup I} = m(i,I)\sum_{j=1}^{3} C_{j\cup((I\cup j)\setminus i)} + C_{i\cup I} = \frac{r+2}{r}m(i,I)D_{I\setminus i} + C_{i\cup I}$$

where we have used

$$\sum_{j=1}^{3} m(j,I) = r.$$

Equation (43) gives (38). Remark 4.1(a) shows that (39) is a consequence. (b): The analogue of equation (37) may be written as

(44) $$C_{ikM}s_{k\cup M\cup j} = C_{jkI}s_{k\cup M\cup i}$$

and by Remark 4.1(a) we have

$$C_{ikM}s_{k\cup M\cup j} = \sum_{\widetilde{M}\in\mathsf{M}^r}\frac{m(j,\widetilde{M})}{r-1}C_{ik(\widetilde{M}\setminus j)}s_{k\widetilde{M}};$$

hence (44) reads

$$m(j,\widetilde{M})\,C_{ik(\widetilde{M}\setminus j)}s_{k\widetilde{M}} = m(i,\widetilde{M})\,C_{jk(\widetilde{M}\setminus i)}s_{k\widetilde{M}}$$

and the arbitrariness of s gives

$$m(j,\widetilde{M})\,C_{ik(\widetilde{M}\setminus j)} = m(i,\widetilde{M})\,C_{jk(\widetilde{M}\setminus i)}.$$

Thus for each k fixed, we have an equation of the same structure as (41). This in particular gives that for each k there exists a collection D_{kS} such that (40) holds. The rest of the proof is identical. $\square$

REMARK 4.3.

(a) *Let $M\in \mathsf{C}^{r+1}_{\mathsf{TL}}$. If for some $i_\circ\in\{1,2,3\}$ we have*

(45) $$M_{ii_1\ldots i_r}g_{ii_1\ldots i_r\cup i_\circ} = 0$$

for all $g\in\mathsf{C}^{r+2}_{\mathsf{TL}}$ then

(46) $$M_{ii_1\ldots i_r} = 0.$$

(*b*) *Let* $C = \{C_{I;J}, I \in \mathsf{M}^r, J \in \mathsf{M}^s\}$ *be a system that is symmetric in the indices of* I, *traceless in* J *and symmetric in the last* $s-1$ *indices of* J. *If*

$$C_{I;J} s_{I\cup i} g_{J\cup j} = C_{I;J} s_{I\cup j} g_{J\cup i}$$

for each i, j, *each* $s \in \mathbf{S}^{r+1}$ *and each* $g \in \mathbf{C}^{s+1}_{\mathsf{TL}}$ *then*

(47) $$C_{I;J} = 0$$

identically.

Proof. (*a*): Using Remark 4.1(*a*) to the last r indices of M one gets

$$M_{ii_1\ldots i_r} g_{ii_1\ldots i_r\cup i_\circ} = \frac{S_{iJ} g_{i\cup J}}{r+1}$$

where

$$S_{iJ} = m(i_\circ, J) M_{i(J\setminus i_\circ)};$$

S_{iJ} is symmetric in the indices of J. Then (45) reads

(48) $$S_{iJ} g_{i\cup J} = 0.$$

Next, since g is traceless, we take the traceless part of S_{iJ} by using Remark 4.1(*b*). The trace of S is

$$\begin{aligned} T_L = M_{iiL} &= \sum_{i=1}^{3} m(i_\circ, i\cup L) M_{i\cup((i\cup L)\setminus i_\circ)} = \\ &= \sum_{i=1}^{3} (m(i_\circ, L) + \delta_{i_\circ i}) M_{i\cup((i\cup L)\setminus i_\circ)} = \\ &= \sum_{i=1}^{3} m(i_\circ, L) M_{i\cup((i\cup L)\setminus i_\circ)} + M_{i_\circ\cup((i_\circ\cup L)\setminus i_\circ)} = M_{i_\circ\cup L}. \end{aligned}$$

Thus the traceless part S°_{iJ} of S_{iJ} is given by

(49) $$S^\circ_{iJ} = m(i_\circ, J) M_{i\cup(J\setminus i_\circ)} - \frac{m(i, J)}{r+3} M_{i_\circ\cup(J\setminus i)}.$$

By Remark 4.1(*b*), $S_{iJ} g_{i\cup J} = S^\circ_{iJ} g_{i\cup J}$ and hence by (48),

$$S^\circ_{iJ} g_{i\cup J} = 0$$

for all $g \in \mathbf{C}^{r+2}_{\mathsf{TL}}$. Since $S^\circ \in \mathbf{C}^{r+2}_{\mathsf{TL}}$, this implies $S^\circ = 0$, *i.e.*, (see (49))

$$m(i_\circ, J) M_{i\cup(J\setminus i_\circ)} - \frac{m(i, J)}{r+3} M_{i_\circ\cup(J\setminus i)} = 0.$$

Writing $K = J \setminus i_\circ$ then

$$(m(i_\circ, K) + 1) M_{i\cup K} - \frac{m(i, K) + \delta_{ii_\circ}}{r+3} M_{i_\circ\cup((K\cup i_\circ)\setminus i)} = 0.$$

For $i = i_\circ$ this provides $M_{i_\circ K} = 0$ for arbitrary K which in turn implies that the second term vanishes, giving (46) generally. (*b*): By Remark 4.1(*a*) this may be rewritten as

$$\sum_{\widetilde{I}} m(i, \widetilde{I})\, C_{\widetilde{I}\setminus i;J}\, s_{\widetilde{I}}\, g_{J\cup j} = \sum_{\widetilde{I}} m(j, \widetilde{I})\, C_{\widetilde{I}\setminus j;J}\, s_{\widetilde{I}}\, g_{J\cup i}$$

which implies

$$m(i, \widetilde{I})\, C_{\widetilde{I}\setminus i;J}\, g_{J\cup j} = m(j, \widetilde{I})\, C_{\widetilde{I}\setminus j;J}\, g_{J\cup i}$$

and hence

$$(m(i, I) + 1)\, C_{I;J}\, g_{J\cup j} = (m(j, I) + \delta_{ij})\, C_{(I\cup i)\setminus j;J}\, g_{J\cup i}. \tag{50}$$

If the minimal multiplicity of I is 0 then $m(j, I) = 0$ for some j and (50) gives

$$C_{I;J}\, g_{J\cup j} = 0 \tag{51}$$

for all g and this particular j. Item (*a*) then gives that (47) holds for such I. If the minimal multiplicity of I is 1 then $(I \cup i) \setminus j$ has the minimal multiplicity 0 for some i, j and (50) gives that (51) holds for this particular j. Item (*a*)then gives (47). Thus the assertion is proved for all I with minimal multiplicity 1 and proceeding inductively one obtains (47) generally. □

The following three lemmas deal essentially with homogeneous differential functions, *i.e.*, with differential functions depending only on the highest-order derivatives of the lattice vectors (see Subsection 2.2). In Section 5, they will be applied, without further notice, to the freezes of differential functions (Subsection 2.2), *i.e.*, to the differential functions obtained by freezing all lower-order derivatives and considering the dependence on the highest-order derivatives. Thus, *e.g.*, what appears as a constant in this section will be typically a function of the lower order derivatives of the lattice vectors in Section 5.

LEMMA 4.4.

(*a*) *Let* $\mathbf{P} = \mathbf{P}(\mathbf{s}^{(r)})$ *be an* R^3*-valued homogeneous function of order* r *that depends only on* $\mathbf{s}^{(r)}$*. If*

$$\epsilon_{ijk} \frac{\partial P_j}{\partial s_I^{(r)a}}\, s_{I\cup k}^{(r+1)a} = 0 \tag{52}$$

for each $\mathbf{s}^{(r+1)a} \in \mathbf{S}^{r+2}$, $a = 1, 2, 3$, *then* $\mathbf{P}$ *is of the form*

$$P_i = \sigma_I^a s_{I\cup i}^{(r)a} + \beta_i$$

where σ_I^a, β_i *are constants, with* σ_I^a *symmetric in the indices of* I.

(*b*) *Let* $\mathbf{P} = \mathbf{P}(\mathbf{d}^{(r)})$ *be an* R^3*-valued homogeneous function of order* r*. If*

$$\operatorname{curl} \mathbf{P} = \mathbf{0} \tag{53}$$

identically then $\mathbf{P}$ *is of the form*

$$P_i = \theta_I^a d_{I\cup i}^{(r)a} + \beta_i \tag{54}$$

where θ_I^a , β_i *are constants, with* θ_i^a *symmetric in the last indices of* I.

(*c*) *Let* $\mathbf{P} = \mathbf{P}(\mathbf{g}^{(r)})$ *be an* R^3*-valued homogeneous function of order* r *that depends only on* $\mathbf{g}^{(r)}$. *If*

$$\epsilon_{ijk} \frac{\partial P_j}{\partial g_N^{(r)a}} g_{N\cup k}^{(r+1)a} = 0 \tag{55}$$

for each $\mathbf{g}^{(r+1)a} \in \mathsf{C}_{\mathsf{TL}}^{r+1}$, $a = 1 , 2 , 3$, *then* $\mathbf{P}$ *is of the form*

$$\mathbf{P} = \text{const} \quad \text{if} \quad r = 1$$

and

$$P_i = \gamma_N^a g_{N\cup i}^{(r)a} + \beta_i \quad \text{if} \quad r \geq 2 \tag{56}$$

where γ_N^a , β_i *are constants, with* γ_N^a *symmetric in the last* $r-1$ *indices of* N *and moreover traceless if* $r \geq 3$.

Proof. (*a*): To simplify the notation, write $S_I^a = s_I^{(r)a}$; furthermore, let us fix a which therefore appears as a 'parameter.' The condition (52) reads

$$\frac{\partial P_i}{\partial S_I^a} s_{I\cup j}^{(r+1)a} = \frac{\partial P_j}{\partial S_I^a} s_{I\cup i}^{(r+1)a}$$

and thus by Remark 4.2(*a*) the partial derivative is of the form

$$\frac{\partial P_i}{\partial S_I^a} = \frac{m(i, I)}{r} \sigma_{I\setminus i}^a$$

from which

$$\frac{\partial^2 P_i}{\partial S_I^a \partial S_J^b} = \frac{m(i, I)}{r} \frac{\partial \sigma_{I\setminus i}^a}{\partial S_J^b} .$$

The symmetry of the second partial derivatives then provide

$$m(i, I) \frac{\partial \sigma_{I\setminus i}^a}{\partial S_J^b} = m(i, J) \frac{\partial \sigma_{J\setminus i}^b}{\partial S_I^a} .$$

For $I = K \cup i$ this gives

$$\frac{\partial \sigma_K^a}{\partial S_J^b} = \frac{m(i, J)}{m(i, K) + 1} \frac{\partial \sigma_{J\setminus i}^b}{\partial S_{K\cup i}^a} . \tag{57}$$

Let $j \neq i$ be arbitrary and write (57) with the choices $i = j$, $K = J \setminus i$, $J = K \cup i$, $a = b$, $b = a$ to obtain

$$\frac{\partial \sigma_{J\setminus i}^b}{\partial S_{K\cup i}^a} = \frac{m(j, K)}{m(j, J) + 1} \frac{\partial \sigma_{(K\cup i)\setminus j}^a}{\partial S_{(J\setminus i)\cup j}^b} \quad \text{(no summation on } i\text{)}. \tag{58}$$

Equations (57) and (58) give

$$\frac{\partial \sigma_K^a}{\partial S_J^b} = \frac{m(i, J) m(j, K)}{(m(i, K) + 1)(m(j, J) + 1)} \frac{\partial \sigma_{(K\cup i)\setminus j}^a}{\partial S_{(J\setminus i)\cup j}^b} . \tag{59}$$

If the minimal multiplicity $M(K)$ of K satisfies $M(K) = 0$ then $M(j, K) = 0$ for some $j \in \{1, 2, 3\}$ and the above equation gives

$$\frac{\partial \sigma^a_K}{\partial S^b_J} = 0 \tag{60}$$

for each a, b, J. If $M(K) = 1$ then $M((K \cup i) \setminus j) = 0$ for some $i \neq j$ and (60) proved for all K with $M(K) = 0$ combined with (59) gives that (60) holds for all K with $M(K) = 1$. The induction gives that (60) holds generally. Thus σ^a_K are constants and the integration gives (54). (*b*): To simplify the notation, write $D^a_{kI} = d^{(r)a}_{kI}$. The condition (53) reads

$$\frac{\partial P_i}{\partial D^a_{kI}} d^{(r+1)a}_{kI\cup j} = \frac{\partial P_j}{\partial D^a_{kI}} d^{(r+1)a}_{kI\cup i}$$

and thus by Remark 4.2(*b*) the partial derivative is of the form

$$\frac{\partial P_i}{\partial D^a_{jJ}} = \frac{m(i, J)}{r-1} \theta^a_{jJ\setminus i}$$

from which

$$\frac{\partial^2 P_i}{\partial D^a_{jJ} \partial D^b_{kK}} = \frac{m(i, J)}{r-1} \frac{\partial \theta^a_{jJ\setminus i}}{\partial D^b_{kK}}.$$

The symmetry of the second partial derivatives leads, in the same way as in the proof of (*a*), to the following identity:

$$\frac{\partial \theta^a_{jJ}}{\partial D^b_{kK}} = \frac{m(i, K) m(n, J)}{(m(i, J) + 1)(m(n, K) + 1)} \frac{\partial \theta^a_{j\cup((J\cup i)\setminus n)}}{\partial D^b_{k\cup(K\setminus i)\cup n}} \tag{61}$$

which must hold for arbitrary $j, k, J, K, i, n, i \neq n$. Fixing $j, k \in \{1, 2, 3\}$, we see that (61) has the same structure as (59) and thus (61) implies that $\partial \theta^a_{jJ} / \partial D^b_{kK} = 0$. The rest of the proof is identical. (*c*): Let $\mathbf{P}$ be as in (*c*) and define $\overline{\mathbf{P}} = \overline{\mathbf{P}}(\mathbf{d}^{(r)})$ to be a homogeneous differential function given by

$$\overline{\mathbf{P}}(\mathbf{d}^{(r)}) = \mathbf{P}(\mathbf{g}^{(r)}).$$

The application of (*b*) gives that

$$P_i(\mathbf{g}^{(r)}) = \theta^a_I d^{(r)a}_{I\cup i} + \beta_i \tag{62}$$

with θ^a_I, β_i constants. Since P_i depends on $\mathbf{d}^{(r)}$ only through $\mathbf{g}^{(r)}$, one sees that

$$\theta^a_I s^{(r)a}_{I\cup i} = 0 \tag{63}$$

for each $s^{(r)}$ completely symmetric. For $r = 1$ this gives immediately that θ^a_I vanish, leading to the assertion of (*c*) in the case $r = 1$. If $r \geq 2$, (63) shows that the complete symmetrization of θ^a_I with respect to the indices of I vanishes and this knowledge enables one to rewrite (62) in the form (56); the details are left to the reader. □

LEMMA 4.5.

(*a*) *Let* $E = \{E^{ab}_{JK} : a, b = 1, 2, 3, J, K \in \mathsf{M}^r\}$ *be a system symmetric under the permutations of the indices in* J *and under the permutations of the indices in* K. *If*

$$\epsilon_{ijk} E^{ab}_{JK} s^a_{J\cup j} s^b_{K\cup k} = 0$$

for all systems $\mathbf{s} = \{s^a_L : a = 1, 2, 3, L \in \mathsf{M}^r\}$ *symmetric in the indices of* L *then*

$$E^{ab}_{JK} = E^{ba}_{KJ}.$$

(*b*) *Let* $F = \{F^{ab}_{JK} : a, b = 1, 2, 3, J, K \in \mathsf{M}^{r+1}\}$ *be a system symmetric under the permutations of the last* r *indices in* J *and under the permutations of the last* r *indices in* K. *If*

$$\epsilon_{ijk} F^{ab}_{JK} e^a_{J\cup j} e^b_{K\cup k} = 0 \tag{64}$$

for all systems $\mathbf{e} = \{e^a_L : a = 1, 2, 3, L \in \mathsf{M}^{r+1}\}$ *symmetric in the last* r *indices of* L *then*

$$F^{ab}_{JK} = F^{ba}_{KJ}.$$

In the applications, $\mathbf{e}$ will be the gradient $\mathbf{d}^{(r+1)}$ of order $r+1$ of the lattice vectors.

PROOF. (*a*): Set

$$W^{ab}_{J;K} = E^{ab}_{JK} - E^{ba}_{KJ} \text{ and } \mathsf{B}_i[\mathbf{s}, \mathbf{t}] = \epsilon_{ijk} E^{ab}_{JK} s^a_{J\cup j} t^a_{K\cup K}$$

for all systems $\mathbf{s}, \mathbf{t} = \{t^a_L : a = 1, 2, 3, L \in \mathsf{M}^{r+1}\}$ symmetric in the indices of L. The hypothesis says $\mathsf{B}_i[\mathbf{s}, \mathbf{s}] = 0$ and the polarization identity

$$\mathsf{B}_i[\mathbf{s}+\mathbf{t}, \mathbf{s}+\mathbf{t}] = \mathsf{B}_i[\mathbf{s}, \mathbf{s}] + \mathsf{B}_i[\mathbf{t}, \mathbf{t}] + \mathsf{B}_i[\mathbf{s}, \mathbf{t}] + \mathsf{B}_i[\mathbf{t}, \mathbf{s}]$$

implies that B_i is skew: $\mathsf{B}_i[\mathbf{s}, \mathbf{t}] + \mathsf{B}_i[\mathbf{t}, \mathbf{s}] = 0$. This reads

$$\epsilon_{ijk} E^{ab}_{JK} s^a_{J\cup j} t^b_{K\cup k} + \epsilon_{ijk} E^{ab}_{JK} t^a_{J\cup j} s^b_{K\cup k} = 0$$

which can be rewritten as (changing the names of the multiindices)

$$W^{ab}_{J;K} s^a_{J\cup j} t^b_{K\cup k} = W^{ab}_{J;K} s^a_{J\cup k} t^b_{K\cup j}.$$

By Remark 4.1(*a*) this means

$$\sum_{\widetilde{J}, \widetilde{K}} m(j, \widetilde{J}) m(k, \widetilde{K}) W^{ab}_{\widetilde{J}\setminus j; \widetilde{K}\setminus k} s^a_{\widetilde{J}} t^b_{\widetilde{K}} = \sum_{\widetilde{J}, \widetilde{K}} m(k, \widetilde{J}) m(j, \widetilde{K}) W^{ab}_{\widetilde{J}\setminus k; \widetilde{K}\setminus j} s^a_{\widetilde{J}} t^b_{\widetilde{K}}$$

and from this

$$m(j, \widetilde{J}) m(k, \widetilde{K}) W^{ab}_{\widetilde{J}\setminus j; \widetilde{K}\setminus k} = m(k, \widetilde{J}) m(j, \widetilde{K}) W^{ab}_{\widetilde{J}\setminus k; \widetilde{K}\setminus j} \tag{65}$$

For $\widetilde{J} = J \cup j$, $\widetilde{K} = K \cup k$ and $k \neq j$, this gives

$$W^{ab}_{JK} = \frac{m(k, J) m(j, K)}{(m(j, J) + 1)(m(k, K) + 1)} W^{ab}_{(J\cup j)\setminus k; (K\cup k)\setminus j}. \tag{66}$$

This equation has the same structure as (59). Thus if the minimal multiplicity $M(J)$ of J is 0, then choosing k such that $m(k, J) = 0$, (66) gives

$$W^{ab}_{J;K} = 0 \tag{67}$$

for all J, a, b. Proceeding inductively as in the proof of Lemma 4.4 we finally obtain that (67) holds generally. (b): We introduce W, B as before, and the polarization gives that $\mathbf{B}_i$ is skew: $\mathsf{B}_i[\mathbf{e}, \mathbf{f}] + \mathsf{B}_i[\mathbf{f}, \mathbf{e}] = 0$ for $\mathbf{e}, \mathbf{f}$ collections having the symmetry as stated for $\mathbf{e}$ in (b). Let us write $J = \alpha \cup \overline{J}$, $K = \beta \cup \overline{K}$ and let us choose $\mathbf{e}, \mathbf{f}$ in the special form

$$e^a_{\alpha \cup \overline{J}} = \xi^a_\alpha s^a_{\overline{J}}, \quad f^a_{\beta \cup \overline{K}} = \eta^a_\beta t^a_{\overline{K}} \quad \text{(no summation on } a\text{)}.$$

Then (64) gives

$$\sum_{a, b, \overline{J}, \overline{K}} W^{ab}_{\alpha \cup \overline{J}; \beta \cup \overline{K}} \xi^a_\alpha \eta^b_\beta s^a_{\overline{J} \cup j}, t^b_{\overline{K} \cup k} = \sum_{a, b, \overline{J}, \overline{K}} W^{ab}_{\alpha \cup \overline{J}; \beta \cup \overline{K}} \xi^a_\alpha \eta^b_\beta s^a_{\overline{J} \cup k}, t^b_{\overline{K} \cup j}.$$

Thus for fixed ξ, η the last identity has the same form as (65), and starting from this point the proof becomes identical to that of (b). □

LEMMA 4.6. *Let* $\eta^a_J = \eta^a_J(\mathbf{d}^{(r-1)})$ *be a system of homogeneous differential functions of order* $r - 1$ *with* $J \in \mathsf{M}^r$ *and with* η^a_J *symmetric in the last* $r - 1$ *indices of* J. *If*

$$\epsilon_{ijk} \frac{\partial \eta^a_J}{\partial d^{(r-1)b}_K} s^{(r-1)a}_{J \cup j} s^{(r-1)b}_{K \cup k} = 0 \quad \textit{for} \quad r = 1 \tag{68}$$

$$\epsilon_{ijk} \frac{\partial \eta^a_J}{\partial d^{(r-1)b}_K} d^{(r-1)a}_{J \cup j} d^{(r-1)b}_{K \cup k} = 0 \quad \textit{for} \quad r \geq 2 \tag{69}$$

for each $\mathbf{s}^{(r-1)a} \in \mathbf{S}^r$, $a = 1, 2, 3$, *or* $\mathbf{d}^{(r-1)a} \in \mathbf{C}^r_{\mathsf{L}}$, $a = 1, 2, 3$, *then there exists a homogeneous* R*-valued function* $f = f(\mathbf{d}^{(r-1)})$ *such that*

$$\eta^a_J = \frac{\partial f}{\partial d^{(r-1)a}_J}.$$

PROOF. The system $E^{ab}_{JK} := \partial \eta^a_J / \partial d^{(r-1)b}_K$ is symmetric by Lemma 4.5. □

5. THE REDUCTION OF ORDER

Throughout the section we assume that $\mathbf{P}$ is an elastic invariant line integrand of order $r \geq 1$. The main result (Proposition 5.7, below) says that it is possible to subtract from $\mathbf{P}$ a covariant vector $\mathbf{Q}$ of order r such that $\mathbf{P} - \mathbf{Q}$ is equivalent to an elastic invariant line integrand of order $r - 1$.

From the definition, $\operatorname{curl} \mathbf{P}$ is a weighted contravariant vector; moreover, it depends affinely on $\nabla^{r+1}\mathbf{d}$. By Proposition 3.3 there exist functions $\omega_{(1)a}, \omega_{(0)a}$, $a = 1, 2, 3$, such that

$$\operatorname{curl} \mathbf{P} = n\mathbf{l}_a \left\{ \omega_{(1)a}(E^{(r)})[W^{(r+1)}] + \omega_{(0)a}(E^{(r)}) \right\} \tag{70}$$

where $\omega_{(1)a}(E^{(r)})[\,\cdot\,]$ is a linear form. We may also write the right-hand side of (70) in the form (see the proof of Proposition 3.3)

$$\text{(71)} \qquad \operatorname{curl}\mathbf{P} = \mathbf{G}(E^{(r)}; \mathbf{d})[\mathbf{g}^{(r+1)}] + \mathbf{H}(\Delta^{(r)}).$$

The first lemma shows that P_i can be written as a sum of the term $\sigma_I^a s_{I,i}^{(r-1)a}$, whose curl depends only on $\Delta^{(r)}$, and a term A_i that depends on the highest gradient $\mathbf{d}^{(r)}$ only through $\mathbf{g}^{(r)}$.

Lemma 5.1. *We have*

$$\text{(72)} \qquad P_i = \sigma_I^a s_{I,i}^{(r-1)a} + A_i$$

where $\sigma^a_{i_1\ldots i_r}$ is completely symmetric in $i_1, \ldots, i_r$ and

$$\text{(73)} \qquad \sigma^a_{i_1\ldots i_r} = \sigma^a_{i_1\ldots i_r}(\Delta^{(r-1)}), \quad A_i = A_i(\Delta^{(r-1)}, \mathbf{g}^{(r)}).$$

As σ is symmetric, we can also write

$$P_i = \sigma_I^a d_{I,i}^{(r-1)a} + A_i.$$

Proof. From (71) we find

$$(\operatorname{curl}\mathbf{P})_i \overset{(r+1)}{=} \epsilon_{ijk}\left\{ \frac{\partial P_j}{\partial g_N^{(r)a}} g_{N,k}^{(r)a} + \frac{\partial P_j}{\partial s_I^{(r)a}} s_{I,k}^{(r)a} \right\} \overset{(r+1)}{=} G_i(E^{(r)}; \mathbf{d})[\mathbf{g}^{(r+1)}]$$

where we use the notation (11). As the derivatives of order $r+1$ are contained linearly,

$$\text{(74)} \qquad \epsilon_{ijk}\left\{ \frac{\partial P_j}{\partial g_N^{(r)a}} g_{N,k}^{(r)a} + \frac{\partial P_j}{\partial s_I^{(r)a}} s_{I,k}^{(r)a} \right\} = G_i(E^{(r)}; \mathbf{d})[\mathbf{g}^{(r+1)}].$$

Taking $\mathbf{d}^{(r+1)a}$ completely symmetric reduces the last equation to (52) of Lemma 4.4(*a*) and that lemma gives

$$P_i = \sigma_I^a s_{I\cup i}^{(r)a} + \beta_i \text{ where } \sigma_I^a = \sigma_I^a(\Delta^{(r-1)}, \mathbf{g}^{(r)}), \quad \beta_i = \beta_i(\Delta^{(r-1)}, \mathbf{g}^{(r)}),$$

and σ_I^a symmetric in the indices of I. Then

$$(\operatorname{curl}\mathbf{P})_i \overset{(r+1)}{=} \epsilon_{ijk}\left\{ \sigma_I^a s_{I\cup j,k}^{(r)a} + \frac{\partial \sigma_I^a}{\partial g_N^{(r)b}} s_{I\cup j}^{(r)a} g_{N\cup k}^{(r+1)b} + \frac{\partial \beta_j}{\partial g_N^{(r)b}} g_{N\cup k}^{(r+1)b} \right\}.$$

Next we eliminate $s_{I\cup j,k}^{(r)a}$ via (23) to obtain

$$(\operatorname{curl}\mathbf{P})_i \overset{(r+1)}{=} \epsilon_{ijk}\left\{ \sigma_I^a s_{I\cup j\cup k}^{(r+1)a} + \sigma_I^a C_{IM} g_M^{(r+1)a} + \frac{\partial \sigma_I^a}{\partial g_N^{(r)b}} s_{I\cup j}^{(r)a} g_{N\cup k}^{(r+1)b} + \frac{\partial \beta_j}{\partial g_N^{(r)b}} g_{N\cup k}^{(r+1)b} \right\}$$

where the constant matrix elements C_{IM} are determined by the form of (23). A comparison with (71) provides

$$\text{(75)} \qquad \epsilon_{ijk} \frac{\partial \sigma_I^a}{\partial g_N^{(r)b}} s_{I\cup j}^{(r)a} g_{N,k}^{(r)b} = 0$$

since in (71) the coefficient in front of $\mathbf{g}^{(r+1)}$ is independent of $\mathbf{s}^{(r)}$, in particular. By Remark 4.3(*b*), equation (75) provides $\partial\sigma_I^a/\partial g_N^{(r)b} = 0$; hence we have $(73)_1$. Finally, we use (23) to eliminate $s_{I\cup i}^{(r)a}$ in terms of $s_{I,i}^{(r-1)a}$ and $\mathbf{g}^{(r)}$ to obtain

$$\sigma_I^a s_{I\cup i}^{(r)a} + \beta_i = \sigma_I^a s_{I,i}^{(r-1)a} + A_i$$

where

$$A_i = \beta_i + \frac{1}{r+1}\sigma^a_{i_1\ldots i_r}\epsilon_{ii_1s}g^{(r)}_{si_2\ldots i_r}. \quad \square$$

Next we show that by subtracting an appropriate term from $\mathbf{A}$ in (72) one obtains a term $\overline{\mathbf{A}}$ that is a covariant vector. That covariant vector $\overline{\mathbf{A}}$ has a curl that differs from curl of $\mathbf{A}$ only by a lower order term by Lemma 5.3. Moreover, by Lemma 5.3, the difference $\mathbf{B} := \mathbf{P} - \overline{\mathbf{A}}$ is linear in $\mathbf{d}^{(r)}$ and the dependence on $\mathbf{d}^{(r)}$ is such that curl $\mathbf{B}$ depends only on $\Delta^{(r)}$. Let us now turn to the details of this step of the proof.

We write

$$\mathbf{A} = \tau_a\mathbf{d}^a, \quad \tau_a = \tau_a(\Delta^{(r-1)}, \mathbf{g}^{(r)})$$

and since by (26) there is one-to-one correspondence between $\mathbf{g}^{(r)}$ and $W^{(r)}$ at fixed $\Delta^{(r-1)}$, we may also write

$$\tau_a(\Delta^{(r-1)}, \mathbf{g}^{(r)}) = \overline{\tau}_a(\Delta^{(r-1)}, W^{(r)}).$$

Let furthermore $\overline{\mathbf{A}}$ be a differential function defined by

$$\overline{\mathbf{A}}(\Delta^{(r)}) = \overline{T}_a(E^{(r-1)}, W^{(r)})\mathbf{d}^a$$

where

$$\overline{T}_a(E^{(r-1)}, W^{(r)}) = \overline{\tau}_a(\mathsf{D}^{(r-1)}(E^{(r-1)}), W^{(r)}) \tag{76}$$

with $\mathsf{D}^{(r-1)}(E^{(r-1)})$ defined after (27). Then $\overline{\mathbf{A}}$ is a covariant vector, implying that curl $\overline{\mathbf{A}}$ is a weighted contravariant vector.

LEMMA 5.2. *We have*

$$\operatorname{curl}\mathbf{P} \overset{(r+1)}{=} \operatorname{curl}\mathbf{A} \overset{(r+1)}{=} \operatorname{curl}\overline{\mathbf{A}} \overset{(r+1)}{=} n\epsilon_{abc}\mathbf{l}_c \frac{\partial\overline{\tau}_a(\Delta^{(r-1)}, W^{(r)})}{\partial W_M^{(r)pq}} W_{M\cup b}^{(r+1)pq}, \tag{77}$$

and

$$\epsilon_{abc}\frac{\partial\overline{\tau}_a(\Delta^{(r-1)}, W^{(r)})}{\partial W_M^{(r)pq}} W_{M\cup b}^{(r+1)pq} = \omega_{(1)c}(\mathsf{E}^{(r-1)}(\Delta^{(r-1)}), W^{(r)})[W^{(r+1)}]. \tag{78}$$

PROOF. One finds that

$$(\operatorname{curl}\mathbf{A})_i \overset{(r+1)}{=} \epsilon_{ijk}\frac{\partial\overline{\tau}_a}{\partial W_M^{(r)pq}} d_j^a W_{M,k}^{(r)pq} \tag{79}$$

and

$$\epsilon_{ijk}d_j^a W_{M,k}^{(r)pq} = \epsilon_{ijk}d_j^a d_k^b \mathbf{l}_b\cdot\nabla W_M^{(r)pq} \overset{(r+1)}{=} (\mathbf{d}^a\times\mathbf{d}^b)_i W_{M\cup b}^{(r+1)pq} \overset{(r+1)}{=} n\epsilon_{abc}l_{ci}W_{M\cup b}^{(r+1)pq} \tag{80}$$

where we have used Remark 3.4. Combining (79) with (80) gives one equality in (77). Since the curl of the term $\sigma_I^a s_{I,i}^{(r-1)a}$ in (72) depends only on $\Delta^{(r)}$,

$$\operatorname{curl}\mathbf{P} \overset{(r+1)}{=} \operatorname{curl}\mathbf{A} \overset{(r+1)}{=} n\epsilon_{abc}\mathbf{l}_c \frac{\partial\tau_a}{\partial W_M^{(r)pq}} W_{M\cup b}^{(r+1)pq}. \tag{81}$$

Comparing with (70),

$$n\epsilon_{abc}\mathbf{l}_c \frac{\partial\tau_a}{\partial W_M^{(r)pq}} W_{M\cup b}^{(r+1)pq} \overset{(r+1)}{=} n\mathbf{l}_c\omega_{(1)c}(E^{(r)})[W^{(r+1)}]$$

and since $W^{(r+1)}$ is contained linearly, this gives (78). Finally, one finds that

$$\operatorname{curl}\overline{\mathbf{A}} \overset{(r+1)}{=} n\epsilon_{abc}\mathbf{l}_c \frac{\partial\tau_a(\mathsf{D}^{(r-1)}(E^{(r-1)}),\, W^{(r)})}{\partial W_M^{(r)pq}} W_{M\cup b}^{(r+1)pq} \overset{(r+1)}{=} n\mathbf{l}_c\omega_{(1)c}(E^{(r)})[W^{(r+1)}] \overset{(r+1)}{=} \operatorname{curl}\mathbf{A}$$

by (78) and (81). □

Lemma 5.3. *If*

$$\mathbf{B} := \mathbf{P} - \overline{\mathbf{A}} \tag{82}$$

then

$$B_i = \sigma_I^a s_{I,i}^{(r-1)a} + C_i \quad \textit{if} \quad r = 1 \tag{83}$$

and

$$B_i = \sigma_I^a s_{I,i}^{(r-1)a} + \gamma_J^a g_{J,i}^{(r-1)a} + C_i \quad \textit{if} \quad r \geq 2 \tag{84}$$

with σ *as in Lemma* 5.1 *and*

$$\gamma_J^a = \gamma_J^a(\Delta^{(r-1)}), \quad C_i = C_i(\Delta^{(r-1)}). \tag{85}$$

Moreover, $\operatorname{curl}\mathbf{B}$ *is a weighted contravariant vector.*

Proof. From (77) one finds that if

$$\mathbf{M} := \mathbf{A} - \overline{\mathbf{A}}, \tag{86}$$

then

$$\operatorname{curl}\mathbf{M} = F(\Delta^{(r)}). \tag{87}$$

By the definition, $\mathbf{M} = \mathbf{M}(\Delta^{(r)})$; it is also easy to see that $\mathbf{M}$ actually depends on $\nabla^r\mathbf{d}$ only through $\mathbf{g}^{(r)}$,

$$\mathbf{M} = \mathbf{M}(\Delta^{(r-1)}, \mathbf{g}^{(r)}). \tag{88}$$

Using (87) and (88), one finds that for each fixed $\Delta^{(r-1)}$, the function $\mathbf{g}^{(r)} \mapsto \mathbf{M}(\Delta^{(r-1)}, \mathbf{g}^{(r)})$ satisfies the hypothesis of Lemma 4.4(c) and hence

$$M_i = C_i \quad \text{if} \quad r = 1 \tag{89}$$

with C_i as in (85) and

$$M_i = \gamma_J^a\, g_{J,i}^{(r-1)a} + C_i \quad \text{if} \quad r \geq 2 \tag{90}$$

with γ^a_J, C_i as in (85). Thus collecting (72), (82), (86), and (90), we obtain (84). Moreover, we have

$$\operatorname{curl}\mathbf{B} = \operatorname{curl}\mathbf{P} - \operatorname{curl}\overline{\mathbf{A}}$$

and the last two terms are weighted contravariant vectors. □

The next lemma uses the fact that $\operatorname{curl}\mathbf{B}$ depends only on $\Delta^{(r)}$ to show that by subtracting and appropriate full gradient from $\mathbf{B}$ one obtains an expression that depends on $\Delta^{(r)}$ only through $\mathbf{g}^{(r)}$.

LEMMA 5.4. *If* $\mathbf{B}$ *is as in Lemma* 5.3 *then*

$$B_i = \nabla_i F + N_i \quad \textit{if} \quad r = 1 \tag{91}$$

and

$$B_i = \nabla_i F + m^a_M g^{(r-1)a}_{M,i} + N_i \quad \textit{if} \quad r \geq 2 \tag{92}$$

where

$$F = F(\Delta^{(r-1)}), \quad m^b_M = m^b_M(\Delta^{(r-2)}, \mathbf{g}^{(r-1)}), \quad N_i = N_i(\Delta^{(r-1)}).$$

PROOF. If we introduce θ^a_I via

$$\theta^a_I d^{(r)a}_{I\cup i} = \sigma^a_I s^{(r-1)a}_{I,i} + \gamma^a_N g^{(r)a}_{N\cup i} \quad \text{where} \quad \theta^a_I = \theta^a_I(\Delta^{(r-1)})$$

and where the term containing γ is omitted if $r = 1$ then

$$(\operatorname{curl}\mathbf{B})_i = \epsilon_{ijk}\left\{ \frac{\partial \theta^a_I}{\partial d^{(r-1)b}_J} d^{(r-1)a}_{I,j} d^{(r-1)b}_{J,k} + \frac{\partial C_j}{\partial d^{(r-1)a}_I} d^{(r-1)a}_{I,k} \right\} + \sigma_i(\Delta^{(r-1)}). \tag{93}$$

From this expression we learn that $\operatorname{curl}\mathbf{B}$ is independent of $\nabla^{r+1}\mathbf{d}$ and that it depends on $\nabla^r\mathbf{d}$ at most quadratically. Since $\operatorname{curl}\mathbf{B}$ is a weighted contravariant vector of order r we find, combining Proposition 3.3 with the quadratic form established above, that

$$\operatorname{curl}\mathbf{B} = n\mathbf{l}_a\left\{\omega_{(2)a}(E^{(r-1)})[W^{(r)}, W^{(r)}] + \omega_{(1)a}(E^{(r-1)})[W^{(r)}] + \omega_{(0)a}(E^{(r-1)})\right\}$$

where the forms $\omega_{(2)a}[\cdot\,,\,\cdot]$ and $\omega_{(1)a}[\cdot]$ are quadratic and linear, respectively. This in turn may be rewritten as (see (26))

$$\operatorname{curl}\mathbf{B} = \mathbf{H}(E^{(r-1)}, \mathbf{d})[\mathbf{g}^{(r)}, \mathbf{g}^{(r)}] + \mathbf{K}(E^{(r-1)}, \mathbf{d})[\mathbf{g}^{(r)}] + \mathbf{I}(E^{(r-1)}, \mathbf{d}).$$

Comparing with (93), we are led to

$$\epsilon_{ijk}\frac{\partial \theta^a_I}{\partial d^{(r-1)b}_J} d^{(r-1)a}_{I,j} d^{(r-1)b}_{J,k} = H_i(E^{(r-1)}, \mathbf{d})[\mathbf{g}^{(r)}, \mathbf{g}^{(r)}], \tag{94}$$

$$\epsilon_{ijk}\frac{\partial C_j}{\partial d^{(r-1)a}_I} d^{(r-1)a}_{I,k} = K_i(E^{(r-1)}, \mathbf{d})[\mathbf{g}^{(r)}], \quad \sigma_i(\Delta^{(r-1)}) = I_i(E^{(r-1)}, \mathbf{d}).$$

If $r = 1$ then taking $\nabla\mathbf{d}^a$ symmetric in (94) gives

$$\epsilon_{ijk}\frac{\partial \theta^a_I}{\partial d^{(r-1)b}_J} s^{(r-1)a}_{I,j} s^{(r-1)b}_{J,k} = 0$$

and Lemma 4.6 leads to the conclusion (91). Let now $r \geq 2$. Differentiating (94) with respect to $s_K^{(r-1)c}$ gives

(95) $$\epsilon_{ijk} \frac{\partial^2 \theta_I^a}{\partial s_K^{(r-1)c} \partial d_J^{(r-1)b}} d_{I,j}^{(r-1)a} d_{J,k}^{(r-1)b} = 0.$$

Fixing c, K and denoting

$$\eta_I^a = \frac{\partial \theta_I^a}{\partial s_K^{(r-1)c}}$$

we see that (95) may be rewritten as (69) and by Lemma 4.6 then for each c, K there exists a function $f_K^c = f_K^c(\Delta^{(r-1)})$ such that

(96) $$\frac{\partial \theta_I^a}{\partial s_K^{(r-1)c}} = \frac{\partial f_K^c}{\partial d_I^{(r-1)a}}.$$

From now on until the end of the proof we use the notation $S_K^c = s_K^{(r-1)c}$, $G_N^a = g_N^{(r-1)a}$. Equation (96) reads

(97) $$\frac{\partial \sigma_I^a}{\partial S_K^c} = \frac{\partial f_K^c}{\partial S_I^a}, \quad \frac{\partial \gamma_N^a}{\partial S_K^c} = \frac{\partial f_K^c}{\partial G_N^a}.$$

From that

$$\frac{\partial^2 \sigma_I^a}{\partial S_K^c \partial S_M^d} = \frac{\partial^2 f_K^c}{\partial S_I^a \partial S_M^d}, \quad \frac{\partial^2 \gamma_N^a}{\partial S_K^c \partial S_M^d} = \frac{\partial^2 f_K^c}{\partial G_N^a \partial S_M^d}$$

and the symmetry of the second partial derivatives of σ_I^a, γ_N^a with respect to S_K^c, S_M^d gives

$$\frac{\partial^2 f_K^c}{\partial S_I^a \partial S_M^d} = \frac{\partial^2 f_M^d}{\partial S_I^a \partial S_K^c}, \quad \frac{\partial f_K^c}{\partial G_N^a \partial S_M^d} = \frac{\partial f_M^d}{\partial G_N^a \partial S_K^c}$$

i.e.,

$$\frac{\partial}{\partial S_I^a} \left(\frac{\partial f_K^c}{\partial S_M^d} - \frac{\partial f_M^d}{\partial S_K^c} \right) = 0, \quad \frac{\partial}{\partial G_N^a} \left(\frac{\partial f_K^c}{\partial S_M^d} - \frac{\partial f_M^d}{\partial S_K^c} \right) = 0.$$

The integration provides

(98) $$\frac{\partial f_K^c}{\partial S_M^d} - \frac{\partial f_M^d}{\partial S_K^c} = L_{KM}^{cd}$$

where

$$L_{KM}^{cd} = L_{KM}^{cd}(\Delta^{(r-2)}) \text{ and } L_{KM}^{cd} = -L_{MK}^{dc}.$$

The integration of (98) implies that there exists a function $f = f(\Delta^{(r-1)})$ such that

$$f_I^a = \frac{\partial f}{\partial S_I^a} + \tfrac{1}{2} L_{IJ}^{ab} S_J^b$$

and (97) give

$$\frac{\partial}{\partial S_J^b}\left(\sigma_I^a - \frac{\partial f}{\partial S_I^a}\right) = -\tfrac{1}{2}L_{IJ}^{ab}, \quad \frac{\partial}{\partial S_J^b}\left(\gamma_N^a - \frac{\partial f}{\partial G_N^a}\right) = 0$$

from which

$$\sigma_I^a = \frac{\partial f}{\partial S_I^a} + \overline{\sigma}_I^a - \tfrac{1}{2}L_{IJ}^{ab}S_J^b, \quad \gamma_N^a = \frac{\partial f}{\partial G_N^a} + \overline{\gamma}_N^a$$

where

$$\overline{\sigma}_I^a = \overline{\sigma}_I^a(\Delta^{(r-2)}, \mathbf{g}^{(r-1)}) \quad \overline{\gamma}_N^a = \overline{\gamma}_N^a(\Delta^{(r-2)}, \mathbf{g}^{(r-1)})$$

are the initial conditions. Thus

$$\theta_I^a d_{I\cup i}^{(r)a} = \frac{\partial f}{\partial d_I^{(r-1)a}} d_{I,i}^{(r-1)a} + \overline{\sigma}_I^a S_{I,i}^a + \overline{\gamma}_N^a G_{N,i}^a - \tfrac{1}{2}L_{IJ}^{ab}S_J^b S_{I,i}^a$$

and (94) reads

$$\epsilon_{ijk}\left\{\frac{\partial \overline{\sigma}_I^a}{\partial G_N^b} S_{I,j}^a G_{N,k}^b + \frac{\partial \overline{\gamma}_N^a}{\partial G_M^a} G_{N,k}^a G_{M,k}^a - \tfrac{1}{2}L_{IJ}^{ab}S_{I,j}^a S_{J,k}^b\right\} = H_i(E^{(r-1)}, \mathbf{d})[\mathbf{g}^{(r)}, \mathbf{g}^{(r)}].$$

We eliminate $\nabla \mathbf{s}^{(r-1)}$ via (23) and learn from the structure of the resulting equation that

$$\epsilon_{ijk}\frac{\partial \overline{\sigma}_I^a}{\partial G_N^b} s_{I\cup j}^{(r)a} G_{N,k}^b = 0, \quad \epsilon_{ijk}L_{IJ}^{ab}s_{I\cup j}^{(r)a}s_{J\cup k}^{(r)b} = 0. \tag{99}$$

Combining $(99)_2$ with Lemma 4.5(*a*) we learn that the skew part of L_{IJ}^{ab} vanishes, and as this object itself is skew, we have finally $L_{IJ}^{ab} = 0$. Combining $(99)_1$ with Remark 4.3(*b*) we learn $\partial\overline{\sigma}_I^a/\partial G_N^b = 0$, *i.e.*, $\overline{\sigma}_I^a = \overline{\sigma}_I^a(\Delta^{(r-2)})$ and if we define F by

$$F = f + \overline{\sigma}_I^a(\Delta^{(r-2)})S_I^a$$

then

$$\sigma_I^a = \frac{\partial F}{\partial S_I^a}, \quad \gamma_N^a = \frac{\partial F}{\partial G_N^a} + \overline{\gamma}_N^a.$$

Equation (84) takes the form

$$B_i = \frac{\partial F}{\partial d_I^{(r-1)a}} d_{I,i}^{(r-1)a} + \gamma_M^a G_{M,i}^a + C_i = \nabla_i F + \overline{\gamma}_M^a G_{M,i}^a + C_i - \frac{\partial F}{\partial \Delta^{(r-2)}}\Delta_{,i}^{(r-2)}.$$

This gives (92) with

$$m_M^a = \overline{\gamma}_M^a, \quad N_i = C_i - \frac{\partial F}{\partial \Delta^{(r-2)}}\Delta_{,i}^{(r-2)}. \quad \square$$

Let us assume, till the statement of Proposition 5.7, that $r \geq 2$. If we define $\mathbf{V}$ by

$$V_i = m_M^a g_{M,i}^{(r-1)a} + N_i$$

then

$$\mathbf{B} = \mathbf{V} + \nabla F \tag{100}$$

and

$$\operatorname{curl}\mathbf{V} = \operatorname{curl}\mathbf{B}$$

and by the proof of Lemma 5.4 we have

$$\operatorname{curl}\mathbf{V} = n\mathbf{l}_a\Big\{\omega_{(2)a}(E^{(r-1)})[W^{(r)}, W^{(r)}] + \omega_{(1)a}(E^{(r-1)})[W^{(r)}] + \omega_{(0)a}(E^{(r-1)})\Big\}. \tag{101}$$

By (26) we may also write

$$V_i = d_i^a U_M^{pq} W_{M\cup a}^{(r)pq} + \overline{N}_i$$

where

$$U_M^{pq} = U_M^{pq}(\Delta^{(r-2)}, W^{(r-1)}), \quad \overline{N}_i = \overline{N}_i(\Delta^{(r-1)}).$$

Define $\mathbf{U}$, $\overline{\mathbf{U}}$ by

$$U_i = d_i^a U_M^{pq} W_{M\cup a}^{(r)pq}, \quad \overline{U}_i = d_i^a \overline{U}_M^{pq} W_{M\cup a}^{(r)pq}$$

where

$$\overline{U}_M^{pq} = \overline{U}_M^{pq}(E^{(r-1)}) = U_M^{pq}(\mathsf{D}^{(r-2)}(E^{(r-2)}), W^{(r-1)}).$$

Then $\overline{\mathbf{U}}$ is a covariant vector of order r.

The following lemma shows that $\mathbf{H} := \mathbf{V} - \overline{\mathbf{U}}$ has a curl of a special form and Lemma 5.6 shows that such differential functions themselves are of a special form.

Lemma 5.5. *If*

$$\mathbf{H} := \mathbf{V} - \overline{\mathbf{U}} \tag{102}$$

then $\operatorname{curl}\mathbf{H}$ *is a weighted contravariant vector of order* r *that depends on* $\nabla^r\mathbf{d}$ *affinely.*

Proof. Recall that $r \geq 2$. In the following calculation we use Remark 3.4 twice, once in the form

$$\mathbf{l}_b \cdot \nabla W_{M\cup a}^{(r)pq} = W_{M\cup a\cup b}^{(r+1)pq} + f_{M\cup a\cup b}^{(r)pq}(E^{(r-1)})[W^{(r)}] + g_{M\cup a\cup b}^{(r)pq}(E^{(r-1)})$$

where

$$f_{M\cup a\cup b}^{(r)pq}(E^{(r-1)})[W^{(r)}] + g_{M\cup a\cup b}^{(r)pq}(E^{(r-1)})$$

stands for the function $F_{M\cup a\cup b}^{(r)pq}(E^{(r)})$ occurring in (31) and where the fact that $F_{M\cup a\cup b}^{(r)pq}(E^{(r)})$ is affine in $W^{(r)}$ has been used. The second time we use Remark 3.4 in the form

$$\mathbf{l}_b \cdot \nabla W_N^{(r-1)mn} = W_{N\cup b}^{(r)mn} + F_{N\cup b}^{(r)mn}(E^{(r-1)}).$$

Also, the symmetry properties of $W^{(r)}$ are used at an appropriate stage, and the abbreviation

$$T_i := \epsilon_{ijk} d_{jk}^a U_M^{pq} W_{M\cup a}^{(r)pq}$$

is introduced. Then

$$(\operatorname{curl}\mathbf{U})_i = \epsilon_{ijk} d_j^a \left\{ \frac{\partial U_M^{pq}}{\partial W_N^{(r-1)mn}} W_{N,k}^{(r-1)mn} W_{M\cup a}^{(r)pq} + U_M^{pq} W_{M\cup a,k}^{(r)pq} \right\} + T_i =$$

$$= \epsilon_{ijk} d_j^a d_k^b \left\{ \frac{\partial U_M^{pq}}{\partial W_N^{(r-1)mn}} \mathbf{l}_b \cdot \nabla W_N^{(r-1)mn} W_{M\cup a}^{(r)pq} + U_M^{pq} \mathbf{l}_b \cdot \nabla W_{M\cup a}^{(r)pq} \right\} + T_i =$$

$$= n\epsilon_{abc} l_{ci} \left\{ \frac{\partial U_M^{pq}}{\partial W_N^{(r-1)mn}} \left(W_{N\cup b}^{(r)mn} + F_{Nb}^{(r-1)mn}(E^{(r-1)}) \right) W_{M\cup a}^{(r)pq} + \right.$$

$$\left. + U_M^{pq} \left(W_{M\cup a\cup b}^{(r+1)pq} + f_{M\cup a\cup b}^{(r)pq}(E^{(r-1)})[W^{(r)}] + g_{M\cup a\cup b}^{(r)pq}(E^{(r-1)}) \right) \right\} + T_i =$$

$$= n\epsilon_{abc} l_{ci} \left\{ \frac{\partial U_M^{pq}}{\partial W_N^{(r-1)mn}} W_{N\cup b}^{(r)mn} W_{M\cup a}^{(r)pq} + \frac{\partial U_M^{pq}}{\partial W_N^{(r-1)mn}} F_{Nb}^{(r-1)mn}(E^{(r-1)}) W_{M\cup a}^{(r)pq} + \right.$$

$$\left. + U_M^{pq} f_{M\cup a\cup b}^{(r)pq}(E^{(r-1)})[W^{(r)}] + h_{M\cup a\cup b}^{pq}(E^{(r-1)}) \right\} + T_i.$$

Thus

$$\operatorname{curl}\mathbf{U} = n\epsilon_{abc} \mathbf{l}_c \left\{ \frac{\partial U_M^{pq}}{\partial W_N^{(r-1)mn}} W_{N\cup b}^{(r)mn} W_{M\cup a}^{(r)pq} + \overline{\omega}_{(1)c}[W^{(r)}] + \overline{\omega}_{(0)c} \right\} + \mathbf{T}$$

where the partial derivatives of U_M^{pq} are calculated at $(\Delta^{(r-2)}, W^{(r-1)})$ and

$$\overline{\omega}_{(1)c}[W^{(r)}] = \overline{\omega}_{(1)c}(\Delta^{(r-2)}, W^{(r-1)})[W^{(r)}],$$

$$\overline{\omega}_{(0)c} = \overline{\omega}_{(0)c}(\Delta^{(r-2)}, W^{(r-1)})$$

are the terms determined by the form of the last three terms in the expression for curl $\mathbf{U}$ given above. Comparing with (101), using that curl $\overline{\mathbf{N}}$ depends affinely on $\nabla^r \mathbf{d}$, and equating the quadratic terms, one finds that

$$\epsilon_{abc} \frac{\partial U_M^{pq}(\Delta^{(r-1)}, W^{(r)})}{\partial W_N^{(r-1)mn}} W_{M\cup b}^{(r)pq} W_{N\cup c}^{(r)mn} = \omega_{(2)a}(E^{(r-1)})[W^{(r)}, W^{(r)}].$$

Furthermore, a similar calculation provides

$$\operatorname{curl}\overline{\mathbf{U}} = n\epsilon_{abc} \mathbf{l}_c \left\{ \frac{\partial U_M^{pq}}{\partial W_N^{(r-1)mn}} W_{N\cup b}^{(r)mn} W_{M\cup a}^{(r)pq} + \overline{\overline{\omega}}_{(1)c}[W^{(r)}] + \overline{\overline{\omega}}_{(0)c} \right\} + \overline{\mathbf{T}} \tag{103}$$

where the partial derivatives of U_M^{pq} are calculated at $(\mathsf{D}^{(r-2)}(E^{(r-2)}), W^{(r-1)})$,

$$\overline{\overline{\omega}}_{(1)c}[W^{(r)}] = \overline{\omega}_{(1)c}(\mathsf{D}^{(r-2)}(E^{(r-2)}), W^{(r-1)})[W^{(r)}],$$

$$\overline{\overline{\omega}}_{(0)c} = \overline{\omega}_{(0)c}(\mathsf{D}^{(r-2)}(E^{(r-2)}), W^{(r-1)}),$$

and

$$\overline{T}_i = \epsilon_{ijk} d_{jk}^a \overline{U}_M^{pq} W_{M\cup a}^{(r)pq}.$$

By (103) the quadratic terms in the expressions for curl $\mathbf{U}$ and curl $\overline{\mathbf{U}}$ agree and thus

$\operatorname{curl}\mathbf{U} - \operatorname{curl}\overline{\mathbf{U}}$ depends at most linearly on $\nabla^r\mathbf{d}$. Since also $\operatorname{curl}\overline{\mathbf{N}}$ depends at most linearly on $\nabla^r\mathbf{d}$, we have the assertion. □

Lemma 5.6. *Let* $\mathbf{H}$ *be a differential function of order* r *such that* $\operatorname{curl}\mathbf{H}$ *is a differential function of order* r *that depends on* $\nabla^r\mathbf{d}$ *affinely. Then*

$$\mathbf{H} = \nabla G + \mathbf{K} \tag{104}$$

where

$$G = G(\Delta^{(r-1)}), \quad \mathbf{K} = \mathbf{K}(\Delta^{(r-1)}). \tag{105}$$

Proof. One finds

$$(\operatorname{curl}\mathbf{H})_i \overset{(r+1)}{=} \epsilon_{ijk}\frac{\partial H_j}{\partial d_I^{(r)a}} d_{I,k}^{(r)a} \overset{(r+1)}{=} 0$$

where the last equality follows from the hypothesis of the lemma. Then Lemma 4.4(*b*) says that

$$H_i = \theta_I^a d_{I\cup i}^{(r)a} + \xi_i$$

where

$$\theta_I^a = \theta_I^a(\Delta^{(r-1)}), \quad \xi_i = \xi_i(\Delta^{(r-1)}).$$

From that

$$(\operatorname{curl}\mathbf{H})_i \overset{(r)}{=} \epsilon_{ijk}\left\{ \frac{\partial\theta_I^a}{\partial d_J^{(r-1)b}} d_{I\cup j}^{(r)a} d_{J\cup k}^{(r)b} + \frac{\partial\xi_j}{\partial d_J^{(r-1)b}} d_{J\cup k}^{(r)b}\right\} \overset{(r)}{=} 0.$$

This gives, among other things,

$$\epsilon_{ijk}\frac{\partial\theta_I^a}{\partial d_J^{(r-1)b}} d_{I\cup j}^{(r)a} d_{J\cup k}^{(r)b} = 0$$

and Lemma 4.6 gives that

$$\theta_I^a = \frac{\partial G}{\partial d_I^{(r-1)a}}$$

where G is as in $(105)_1$. Thus

$$H_i = \frac{\partial G}{\partial d_I^{(r-1)a}} d_{I\cup i}^{(r)a} + \xi_i = \nabla_i G + \xi_i - \frac{\partial G}{\partial \Delta^{(r-2)}}\Delta_{,i}^{(r-2)}.$$

Thus (104). □

Proposition 5.7. *If* $\mathbf{P}$ *is an invariant line integrand of order* r *then*

$$\mathbf{P} = \mathbf{Q} + \mathbf{R} + \nabla f \tag{106}$$

where $\mathbf{Q}$ *is a covariant vector of order* r, $\mathbf{R}$ *is an invariant line integrand of order* $r-1$ *and* f *is a differential function of order* $r-1$.

PROOF. If $r = 1$ then from (82) and (91) we see that (106) holds with

$$\mathbf{Q} = \overline{\mathbf{A}}\,, \quad \mathbf{R} = \mathbf{N}\,, \quad f = F$$

while if $r \geq 2$ then from (82), (100), (102), and (104) we find that (106) holds with

$$\mathbf{Q} = \overline{\mathbf{A}} + \overline{\mathbf{U}}\,, \quad \mathbf{R} = \mathbf{K}\,, \quad f = F + G. \quad \square$$

6. THE MAIN RESULT

Before the formulation of the theorem, recall that it is assumed that all differential functions are assumed to depend in an inifinitely differentiable way on the components of gradients of lattice vectors. However, using a mollification, one could extend the result to differential functions with lower degree of smoothness.

THEOREM 6.1. *Let* $\mathbf{P}$ *be an elastic invariant line integrand of order* r. *Then there exists an elastic covariant vector* $\mathbf{P}^*$ *of order* r *which is equivalent to* $\mathbf{P}$.

PROOF. By induction. For $r = 0$ Davini [1] proves that if $\mathbf{P}$ is an elastic invariant line integrand of order 0 then $\mathbf{P}$ must be of the form

$$\mathbf{P}(\mathbf{d}) = c_a \mathbf{d}^a + \mathbf{w}_0$$

where $c_a\,,\ a = 1\,,2\,,3\,,$ are constants and $\mathbf{w}_0 \in \mathsf{R}^3$. Thus if

$$\mathbf{P}^*(\mathbf{d}) = c_a \mathbf{d}^a$$

then $\mathbf{P}^*$ is a contravariant vector equivalent to $\mathbf{P}$. Let the assertion of the theorem be true for all orders $\leq r$. Let $\mathbf{P}$ be an elastic invariant line integrand of order $r + 1$. By Proposition 5.7 $\mathbf{P}$ is equivalent to a differential function $\overline{\mathbf{P}}$ of the form

$$\overline{\mathbf{P}} = \mathbf{Q} + \mathbf{R}$$

where $\mathbf{Q}$ is a covariant vector of order $r + 1$ and $\mathbf{R}$ is an invariant line integrand of order r. By the induction hypothesis, $\mathbf{R}$ is equivalent to a covariant vector $\mathbf{R}^*$ of order r. Hence $\mathbf{P}$ is equivalent to

$$\mathbf{P}^* = \mathbf{Q} + \mathbf{R}^*$$

and this is a covariant vector of order $r + 1$. $\square$

ACKNOWLEDGEMENTS

M. Š. gratefully acknowledges the support of EPSRC during his stay at the University of Nottingham, where the research was done. G. P. P. thanks the Isaac Newton Institute, and the EU through a TMR grant on «Phase transitions in crystals», for hospitality and support.

References

[1] C. Davini, *A proposal for a continuum theory of defective crystals*. Arch. Rational Mech. Anal., 96, 1986, 295-317.

[2] C. Davini, *Elastic invariants in crystal theory*. In: J. M. Ball (ed.), *Material instabilities in continuum mechanics and related mathematical problems*. Clarendon Press, Oxford 1988, 85-105.

[3] C. Davini - G. P. Parry, *On defect-preserving deformations in crystals*. Int. J. Plasticity, 5, 1989, 337-369.

[4] C. Davini - G. P. Parry, *A complete list of invariants for defective crystals*. Proc. R. Soc. Lond. A, 432, 1991, 341-365.

[5] C. Davini - G. P. Parry, *Uniform rearrangement in defective crystals*. Proc. R. Soc. Lond. A, 439, 1992, 247-256.

[6] I. Fonseca - G. P. Parry, *Equilibrium configurations of defective crystals*. Arch. Rational Mech. Anal., 120, 1992, 245-283.

[7] I. Fonseca - G. P. Parry, *On a class of invariant functionals*. Proc. R. Soc. Lond. A, 436, 1992, 317-329.

[8] P. J. Olver, *Conservation laws and null divergences*. Math. Proc. Camb. Phil. Soc., 94, 1983, 529-540.

[9] P. J. Olver, *Applications of Lie groups to partial differential equations*. Springer-Verlag, New York 1986.

[10] P. J. Olver, *Equivalence, invariants, and symmetry*. Cambridge University Press, Cambridge 1996.

[11] G. P. Parry, *Defects and rearrangements in crystals*. In: L. M. Brock (ed.), *Defects and anelasticity in the characterization of crystalline solids*. ASME, New York 1992, 117-132.

[12] G. P. Parry, *Elastic-plastic decompositions of defect-preserving changes of state in solid crystals*. In: M. D. P. Monteiro Marques - J. F. Rodrigues (eds.), *Trends in applications of mathematics to mechanics*. Longman, Harlow, Essex 1994, 154-163.

[13] G. P. Parry, *Material symmetry in solid crystals*. In: D. F. Parker - A. H. England (eds.), *IUTAM Symposium on Anisotropy, inhomogeneity and nonlinearity in solid mechanics*. Kluwer Academic Publishers, Amsterdam 1995, 229-234.

[14] G. P. Parry, *On defective crystalography*. Arch. Mech., 50, 1998, 567-575.

[15] G. P. Parry - M. Šilhavý, *Elastic invariants in the theory of defective crystals*. Proc. R. Soc. Lond. A, 455, 1999, 4333-4346.

Pervenuta il 12 novembre 1999,
in forma definitiva il 5 gennaio 2000.

G. P. Parry:
Division of Theoretical Mechanics
School of Mathematical Sciences
University of Nottingham
Nottingham NG7 2RD (Gran Bretagna)
Gareth.Parry@nottingham.ac.uk

M. Šilhavý:
Mathematical Institute of the AV ČR
Žitná 25
115 67 Prague 1 (Repubblica Ceca)
silhavy@math.cas.cz