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Discreteness of the spectrum for some differential operators with unbounded coefficients in $\mathbb{R}^n$

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Analisi matematica. — *Discreteness of the spectrum for some differential operators with unbounded coefficients in* $\mathbf{R}^n$. Nota (*) di GIORGIO METAFUNE e DIEGO PALLARA, presentata dal Socio G. Da Prato.

ABSTRACT. — We give sufficient conditions for the discreteness of the spectrum of differential operators of the form $Au = -\Delta u + \langle \nabla F, \nabla u \rangle$ in $L^2_\mu(\mathbf{R}^n)$ where $d\mu(x) = e^{-F(x)}\, dx$ and for Schrödinger operators in $L^2(\mathbf{R}^n)$. Our conditions are also necessary in the case of polynomial coefficients.

KEY WORDS: Singular differential operators; Discrete spectrum; Schrödinger operators.

RIASSUNTO. — *Proprietà di spettro discreto per operatori differenziali con coefficienti illimitati in* $\mathbf{R}^n$. In questa *Nota* si studiano operatori della forma $Au = -\Delta u + \langle \nabla F, \nabla u \rangle$ in $L^2_\mu(\mathbf{R}^n)$ con $d\mu(x) = e^{-F(x)}\, dx$, e operatori di Schrödinger in $L^2(\mathbf{R}^n)$. Si danno condizioni sufficienti affinché lo spettro di un tale operatore differenziale sia discreto. Le condizioni trovate sono anche necessarie nel caso di coefficienti polinomiali.

1. INTRODUCTION

In this paper we study the discreteness of the spectrum of two strictly related second order elliptic differential operators with unbounded coefficients on $\mathbf{R}^n$. These operators are

$$A = -\Delta + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \frac{\partial}{\partial x_i}\,, \qquad B = -\Delta + V\,,$$

with $F \in C^2(\mathbf{R}^n)$ and $V \in C(\mathbf{R}^n)$. B is the classical Schrödinger operator, whereas A is a special case of second order operators with (possibly) unbounded coefficients of the first order terms. These operators are of interest when dealing with diffusion processes on all of $\mathbf{R}^n$ in presence of a drift represented by the first order terms. Unlike the case of bounded coefficients, only recently has that of unbounded ones been studied, starting from the prototype of Ornstein-Uhlenbeck operators. Existence and regularity of the associated semigroups that describe the underlying processes have been studied both with stochastic (see [2]) and analytic tools (see [1, 10-13]).

To the operator A is canonically associated the measure $d\mu = e^{-F}dx$ on $\mathbf{R}^n$, which is the (unique) invariant measure of the associated Markov process, and therefore it is natural to study A in the Hilbert space $L^2(\mathbf{R}^n, d\mu)$, where it turns out to be self-adjoint and non-negative. Moreover, due to the gradient structure of the coefficients of the first order terms, the operator A is unitarily equivalent to B with $V = (1/4)|\nabla F|^2 - (1/2)\Delta F$. Hence, we can deduce properties of A on $L^2(\mathbf{R}^n, d\mu)$ from those of B on $L^2(\mathbf{R}^n, dx)$.

(*) Pervenuta in forma definitiva all'Accademia 3 settembre 1999.

As regards Schrödinger operators, various conditions on V are known (see [3, 4, 16, 17]) guaranteeing the discreteness of the spectrum, and also a characterisation (see [15, 7]) based on quantitative capacity estimates. We give here a new simple condition for positive potentials based on Sobolev embeddings, which turns out to be also necessary for polynomial potentials V, a case of interest in quantum mechanics (see [8, 18]). In particular, Theorem 3.5 confirms a conjecture of B. Simon's (see [18, §6, Remark 5]).

NOTATION. We use L^2 for $L^2(\mathbf{R}^n)$ with respect to the Lebesgue measure. Similarly, H^k stands for the usual Sobolev space $H^k(\mathbf{R}^n)$. By C_0^k, $(0 \le k \le +\infty)$ we denote the space of all C^k-functions with compact support in $\mathbf{R}^n$. The integration domain is always understood to be $\mathbf{R}^n$, if not otherwise stated. If E is a measurable subset of $\mathbf{R}^n$, we denote by $|E|$ its Lebesgue measure. We denote by $Q(c, d)$ the open cube of $\mathbf{R}^n$ with centre c and side $d > 0$.

2. REDUCTION TO A SCHRÖDINGER OPERATOR

We consider the differential operator on $\mathbf{R}^n$

$$Au = -\Delta u + \langle \nabla F, \nabla u \rangle = -e^F \operatorname{div}\left(e^{-F} \nabla u\right) ,$$

with $F \in C^2(\mathbf{R}^n)$, and study the compactness of its resolvent operator in the weighted Hilbert space

$$L^2_\mu = \left\{ u : \mathbf{R}^n \to \mathbf{C} : u \text{ measurable and } \int |u|^2 \, d\mu < +\infty \right\} , \tag{2.1}$$

where $d\mu(x) = e^{-F(x)} \, dx$, endowed with the inner product $(u, v)_\mu = \int u\bar{v} \, d\mu$. We introduce the Sobolev space

$$H^1_\mu = \left\{ u \in L^2_\mu : \nabla u \in L^2_\mu \right\} \tag{2.2}$$

endowed with the inner product $(u, v)_1 = (u, v)_\mu + (\nabla u, \nabla v)_\mu$ and we observe that C_0^∞ is dense both in L^2_μ and in H^1_μ.

We define the domain of A as follows

$$D(A) = \left\{ u \in H^1_\mu \cap H^2_{\text{loc}} : Au \in L^2_\mu \right\} \subset L^2_\mu ; \tag{2.3}$$

clearly $D(A)$ is dense in L^2_μ.

PROPOSITION 2.1. *The operator $(A, D(A))$ is self-adjoint and non-negative in L^2_μ.*

PROOF. The bilinear form $a(u, v) = (\nabla u, \nabla v)_\mu$, defined on $H^1_\mu \times H^1_\mu$, is (weakly) coercive on L^2_μ and defines a self-adjoint, non-negative operator $(L, D(L))$ on L^2_μ in the following way

$$D(L) = \left\{ u \in H^1_\mu : \exists f \in L^2_\mu \text{ such that } a(u, v) = (f, v)_\mu , \ \forall v \in H^1_\mu \right\} , \qquad Lu = f .$$

Let us prove that L coincides with A. By local elliptic regularity, $D(L) \subset H^2_{\text{loc}}$. If

$u \in D(L)$ and $v \in C_0^\infty$, integrating by parts the equality $a(u, v) = (f, v)_\mu$ we obtain

$$-\int \mathrm{div}\big(e^{-F}\nabla u\big)\bar{v}\,dx = \int f\bar{v}\,d\mu$$

and hence $f = Au$. This shows that $D(L) \subset D(A)$ and that $Lu = Au$ if $u \in D(L)$. Conversely, if $u \in D(A)$ and $f = Au$, the equality $a(u, v) = (f, v)_\mu$ clearly holds for every $v \in C_0^\infty$ (by integrating by parts) and, by density, for all $v \in H^1_\mu$. This concludes the proof. □

The analysis of the spectrum of A will be done by transforming it into a suitable Schrödinger operator $B = -\Delta + V$ on L^2, in the vein of [5] and [6]. We briefly recall the definition and the basic properties of these operators. We assume that V is real-valued, continuous on $\mathbf{R}^n$ and bounded from below and define B through the bilinear form

$$b(u, v) = \int \langle \nabla u, \nabla \bar{v}\rangle + Vu\bar{v}\,dx\ ,$$

$u, v \in \mathcal{H} = \Big\{u \in H^1 : |V|^{1/2}u \in L^2\Big\}$. More precisely, we define

$$D(B) = \big\{u \in \mathcal{H} : \exists f \in L^2 \text{ such that } b(u, v) = (f, v)\,,\ \forall v \in \mathcal{H}\big\}\ , \qquad Bu = f\,. \tag{2.4}$$

Arguing as in Proposition 2.1 it is easily checked that

$$D(B) = \big\{u \in \mathcal{H} \cap H^2_{\mathrm{loc}} : -\Delta u + Vu \in L^2\big\}\,;$$

moreover, C_0^∞ is a core of $(B, D(B))$, *i.e.*, B is essentially self-adjoint on C_0^∞ (see [7, Corollary VII.2.7]).

Under additional hypotheses on the function F, the operator A is similar to a suitable Schrödinger operator B. With the notation $\phi = e^{-F/2}$, we obtain that A is unitarily equivalent to B when

$$V = \frac{\Delta\phi}{\phi} = \frac{1}{4}|\nabla F|^2 - \frac{1}{2}\Delta F\,.$$

This is stated in the following proposition.

Proposition 2.2. *If the function $|\nabla F|^2 - 2\Delta F$ is bounded from below in $\mathbf{R}^n$, then the operator $(A, D(A))$ is unitarily equivalent to the Schrödinger operator $(B, D(B))$ with $V = (1/4)|\nabla F|^2 - (1/2)\Delta F$.*

Proof. Let $\phi = e^{-F/2}$ and $T : L^2_\mu \to L^2$ the unitary map defined by $Tf = \phi f$. We define the operator

$$Cu = TAT^{-1}u$$

for $u \in D(C) = T(D(A))$. Clearly C is unitarily equivalent to A and we show that $(B, D(B)) = (C, D(C))$, with the stated choice of V. Since $(B, D(B))$ is essentially self-adjoint on C_0^∞ it is sufficient to prove that $Bu = Cu$ for all $u \in C_0^\infty$. For, a straightforward computation gives for $u \in C_0^\infty$

$$Cu = -\Delta u + \langle \phi\nabla u, \phi^{-1}\nabla F - 2\nabla(\phi^{-1})\rangle + \big[\langle \nabla F, \phi\nabla(\phi^{-1})\rangle - \phi\Delta(\phi^{-1})\big]u\,.$$

Moreover, since $\phi^{-1} = e^{F/2}$, we obtain

$$\nabla(\phi^{-1}) = (1/2)e^{F/2}\nabla F, \qquad \Delta(\phi^{-1}) = (1/2)e^{F/2}\Delta F + 1/2\langle \nabla F, \nabla(\phi^{-1})\rangle \ ,$$

so that $\phi^{-1}\nabla F - 2\nabla(\phi^{-1}) = 0$ and

$$\langle \nabla F, \phi\nabla(\phi^{-1})\rangle - \phi\Delta(\phi^{-1}) = (\Delta\phi)/\phi = (1/4)|\nabla F|^2 - (1/2)\Delta F \, .$$

This gives immediately $Cu = Bu$ and concludes the proof. □

Remark 2.3. We observe that under the hypotheses of the above theorem, C_0^2 is a core also for $(A\, , D(A))$ since it is a core for $(B\, , D(B))$ and is invariant under the map T.

Remark 2.4. Since the quadratic form $(\nabla u\, , \nabla u)_\mu$ is non-negative, we obtain that $\int |\nabla u|^2 + V|u|^2\, dx$, with $V = \Delta\phi/\phi$, is non-negative, too. However, the potential V may be negative everywhere. For instance, take $n \geq 3$, $\alpha \in (1 - n/2\, , 0)$ and $F(x) = -2\alpha \log(1 + |x|^2)$; then $\phi(x) = e^{-F(x)/2} = (1 + |x|^2)^\alpha$ and $V(x) = \alpha(1 + |x|^2)^{-2}[(4\alpha - 4 + 2n)|x|^2 + 2n]$ is negative and bounded from below.

It may also happen that the quadratic form $\int |\nabla u|^2 + V|u|^2\, dx$ is non-negative on C_0^∞ with V unbounded from below. An example is $V(x\, , y) = y^4 + 4x^2y^2 - 2x$, coming from $F(x\, , y) = xy^2$ (see also item *a*) in Section 4).

3. Discreteness of the spectrum of Schrödinger operators

In this Section we study the compactness of the resolvent of the Schrödinger operator $-\Delta + V$, and give sufficient conditions if V is bounded from below, and a characterisation when V is a positive polynomial. We recall that a characterisation of the compactness of $(-\Delta + V)^{-1}$, for positive V, is due to A. M. Molcanov (see [15] and [7, Theorem VIII.4.1]), a result rather difficult to handle, since it involves explicit computations of capacities of arbitrary sets. V. Kondrat'ev and M. Shubin have generalised Molcanov's criterion to some Riemannian manifolds in [9], and have also deduced from it some simpler sufficient conditions, including our Theorem 3.1. However, it seems to be interesting to provide a direct proof of this result, based on Sobolev embeddings. Notice that Theorem 3.1 embodies the classical case in which the potential goes to $+\infty$ at infinity (see [16, Theorem XIII.67] and [3, Section 1.6] or [4, Section 8]): we show that the result is still true if $V \to +\infty$ in a measure-theoretic sense, as $|x| \to +\infty$.

If V is a potential and $M > 0$, we define $E_M = \{x \in \mathbf{R}^n : V(x) < M\}$. If $x \in \mathbf{R}^n$ we set $|x|_\infty = \max_{i=1,\dots,n}|x_i|$, so that $Q(c\, , d) = \{x :\ |x - c|_\infty < d\}$. .

Theorem 3.1. *Let $B = -\Delta + V$ be defined as in* (2.4) *and suppose that for every $M > 0$*

$$\lim_{|c| \to +\infty} |E_M \cap Q(c\, , 1)| = 0 \, . \tag{3.1}$$

Then B has compact resolvent in L^2.

Proof. We may suppose that $V \geq 0$. Since $D(B)$ is contained in H^1 and the embedding of $H^1(Q)$ into $L^2(Q)$ is compact for every cube Q, we have only to show that for every $\varepsilon > 0$ there is a cube Q such that $\int_{\mathbf{R}^n \setminus Q} |u|^2 < \varepsilon$ for every $u \in D(B)$ with $\int |u|^2 + |Bu|^2 \leq 1$.

Let $u \in D(B)$ as above and observe that $\int(|\nabla u|^2 + V|u|^2) \leq 1$. Fix $\varepsilon > 0$, set $M = \varepsilon^{-1}$ and take $R > 0$ such that if $|c|_\infty > R$, then $|E_M \cap Q(c, 1)| < \varepsilon$. Clearly we have

$$\int_{Q(c,1)\setminus E_M} |u|^2 \leq \varepsilon \int_{Q(c,1)\setminus E_M} V|u|^2 \leq \varepsilon \int_{Q(c,1)} V|u|^2 . \tag{3.2}$$

To estimate the integral over $Q(c, 1) \cap E_M$ for $|c|_\infty > R$, we take $p = (2n)/(n-2)$ if $n \geq 3$ and any $p > 2$ if $n = 1, 2$. By the Sobolev embedding we have

$$\left(\int_{Q(c,1)} |u|^p\right)^{1/p} \leq C \left(\int_{Q(c,1)} |u|^2 + |\nabla u|^2\right)^{1/2}$$

with C independent of c. Then we have

$$\int_{Q(c,1)\cap E_M} |u|^2 \leq |E_M \cap Q(c, 1)|^{1-2/p} \left(\int_{Q(c,1)\cap E_M)} |u|^p\right)^{2/p} \leq C\varepsilon^\gamma \int_{Q(c,1)} (|u|^2 + |\nabla u|^2) ,$$

with $\gamma = 1 - 2/p$. From the above inequality and (3.2) we deduce that

$$\int_{Q(c,1)} |u|^2 \leq \varepsilon \int_{Q(c,1)} V|u|^2 + C\varepsilon^\gamma \int_{Q(c,1)} (|u|^2 + |\nabla u|^2)$$

hence, summing over a partition of cubes with centres c satisfying $|c|_\infty \geq R$,

$$\int_{|x|_\infty \geq R} |u|^2 \leq \varepsilon \int_{|x|_\infty \geq R} V|u|^2 + C\varepsilon^\gamma \int_{|x|_\infty \geq R} (|u|^2 + |\nabla u|^2)$$

and finally

$$\int_{|x|_\infty \geq R} |u|^2 \leq \frac{\varepsilon + C\varepsilon^\gamma}{1 - C\varepsilon^\gamma}. \quad \square$$

Let us point out a particular case of the previous result.

Corollary 3.2. *If the potential $V > 0$ satisfies the condition*

$$\lim_{|c| \to +\infty} \int_{Q(c,1)} V^{-\alpha} = 0 ,$$

for some $\alpha > 0$, then B has compact resolvent.

Proof. In fact we have $E_M = \{x \in \mathbf{R}^n : V^{-\alpha}(x) > M^{-\alpha}\}$ and

$$|E_M \cap Q(c, 1)| \leq M^\alpha \int_{Q(c,1)} V^{-\alpha}. \quad \square$$

Notice that there is no need for choosing cubes of side 1: in fact it is easy to see that if (3.1) is satisfied for some $d > 0$, then it is satisfied for all d. We observe also that condition (3.1) is not necessary even for $n = 1$, as can be checked comparing it with Molcanov's theorem (see the next section).

We now consider positive polynomial potentials. Such potentials are analysed in [8] by a «volume counting» argument, which turns out to be equivalent to our condition (3.1). Once a level M has been fixed, Fefferman considers the number $N(M)$ of cubes of side $M^{-1/2}$ centred at $jM^{-1/2}$ ($j \in \mathbf{Z}^n$) contained in E_M. From [8, Theorem II.3], it follows that the spectrum of B is discrete if and only if $N(M)$ is finite for every $M > 0$. Instead, we consider the measure of the intersection of E_M with arbitrary cubes of fixed side. For potentials V of the form

$$V = f \circ p\,, \tag{3.3}$$

where p is a polynomial and $f : \mathbf{R} \to \mathbf{R}$ is a continuous function satisfying $f(t) \to +\infty$ as $|t| \to +\infty$, we show that condition (3.1) is also necessary for the discreteness of the spectrum of B. We need the following lemma, which shows that a control on the supremum of the polynomial p can be deduced from an estimate of the measure of its sublevels.

Lemma 3.3. *Let* $d \in \mathbf{N}$ *and* $\delta > 0$; *then there is constant* $C > 0$ *such that for every* $M > 0$

$$|\{x \in Q : |p(x)| \le M\}| \ge \delta \qquad \Longrightarrow \qquad \sup_{x \in Q} |p(x)| \le CM$$

for every polynomial p *of degree less than or equal to* d *and every unit cube* $Q \subset \mathbf{R}^n$.

Proof. It is sufficient to prove the existence of a constant C for the cube $Q = Q(0\,,1)$ since then the same constant works for every unit cube, by an elementary translation argument. The linear dependence of the upper bound on M readily follows replacing p with p/M, hence we take $M = 1$. If the statement is false for Q, there is a sequence (q_k) of polynomials of degree at most d such that $|\{x \in Q : |q_k(x)| \le 1\}| \ge \delta$ and $\|q_k\|_Q := \sup_{x\in Q} |q_k(x)| \to +\infty$. Then, the normalised polynomials $p_k = q_k/\|q_k\|_Q$ satisfy $\|p_k\|_Q = 1$ and $|\{x \in Q : |p_k(x)| \le \varepsilon_k\}| \ge \delta$, with $\varepsilon_k = 1/\|q_k\|_Q \to 0$, and (up to a subsequence) are uniformly convergent to a polynomial p with $\|p\|_Q = 1$. Setting $F_k = \{x \in Q : |q_k(x)| \le 1\}$ and $F = \cap_k \cup_{r \ge k} F_r$, we have $F \subset \{x \in Q : p(x) = 0\}$ and $|F| = \lim_k |\cup_{r\ge k} F_r| \ge \delta$, whence $|\{x \in Q : p(x) = 0\}| > 0$ and therefore $p \equiv 0$, in contrast with $\|p\|_Q = 1$. □

Proposition 3.4. *Let* V *be defined as in* (3.3). *If the operator* $B = -\Delta + V$ *has compact resolvent then* (3.1) *holds.*

Proof. If condition (3.1) does not hold, we can find $M,\ \delta > 0$, a sequence of pairwise disjoint cubes $Q(c_k\,, 1)$ and a positive number R such that

$$|E_M \cap Q(c_k\,, 1)| = |\{x \in Q(c_k\,, 1) : |p(x)| \le R\}| \ge \delta \qquad \forall\ k \in \mathbf{N}\,.$$

From the preceding lemma we infer that $\sup_{x\in Q(c_k,1)} |p(x)| \leq L$, hence the potential V is uniformly bounded on the sequence $Q(c_k, 1)$. Taking a non-vanishing $u \in C_0^\infty\big(Q(0,1)\big)$, the sequence $\big(u_k(x)\big) = \big(u(x-c_k)\big)$ is bounded in the graph norm and is not relatively compact in L^2, so that B cannot have compact resolvent. □

Finally, we show that the spectrum of $-\Delta + V$, $V = f \circ p$, is discrete if and only if the polynomial p is not independent of some variable, *i.e.* if and only if the only constant vector $c = (c_1, \dots, c_n)$ such that $\sum_i c_i \partial p/\partial x_i \equiv 0$ is $c = (0, \dots, 0)$.

THEOREM 3.5. *Let* $B = -\Delta + V$ *with* V *defined in* (3.3). *Then the resolvent of* B *is not compact if and only if there is a direction* $\omega \in \mathbf{R}^n$ *such that* $\dfrac{\partial p}{\partial \omega} \equiv 0$.

PROOF. If $\dfrac{\partial p}{\partial \omega} \equiv 0$ then V is bounded on a strip parallel to ω and the resolvent of B is not compact, by the argument of Proposition 3.4. Suppose now that B has not compact resolvent. Proposition 3.4 yields the existence of a sequence of unit cubes $Q(c_k, 1) \subset \mathbf{R}^n$, with $|c_k| \to +\infty$, on which $|p|$ is uniformly bounded. We write $c_k = t_k\omega_k$ with $|\omega_k| = 1$ and we may assume (up to a subsequence) that $\omega_k \to \omega$, as $k \to \infty$. We show, by induction on $n = \dim \mathbf{R}^n$, that p is constant along the direction ω. Without loss of generality, we may suppose that $\omega = (1, 0, \dots, 0)$. Of course the statement is true if $n = 1$.

First of all, we observe that for every $d \in \mathbf{N}$ and every multiindex α there is a constant C such that the inequality

$$\sup_Q |D^\alpha r| \leq C \sup_Q |r|$$

holds for every polynomial r of degree at most d and every unit cube Q.

We write

$$p(x) = \sum_{j=0}^{m} x_1^j q_j(x_2, \dots, x_n)$$

with q_j polynomials in the variables $x_2, \dots, x_n$ and $q_m \not\equiv 0$, and show that $m = 0$. Suppose, by contradiction, that $m > 0$; differentiating $(m-1)$-times with respect to x_1 we obtain that the polynomial $p_1(x) = m! x_1 q_m(x_2, \dots, x_n) + (m-1)! q_{m-1}(x_2, \dots, x_n)$ is uniformly bounded on the sequence $\big(Q(c_k, 1)\big)$. Since $q_m \not\equiv 0$, we can find a multiindex α such that $D^\alpha q_m$ is equal to a non-zero constant c. Then the polynomial

$$p_2(x) = D^\alpha p_1 = c_1 x_1 + r_1(x_2, \dots, x_n)\ ,$$

with $c_1 = cm!$, would be uniformly bounded on the sequence $\big(Q(c_k, 1)\big)$. Let us write $\omega_k = (\omega_k^1, \omega_k^2) \in \mathbf{R} \times \mathbf{R}^{n-1}$ with $\omega_k^1 \to 1$ and $\omega_k^2 \to 0$, as $k \to \infty$. Since $c_1 x_1$ is unbounded on $\big(Q(c_k, 1)\big)$, $r_1 \not\equiv 0$ and the sequence $(t_k \omega_k^2)$ is unbounded; moreover, the polynomial ∇r_1 is uniformly bounded on $\big(Q(c_k, 1)\big)$ (hence on the $(n-1)$-dimensional unit cubes centred at $t_k\omega_k^2$), as one can see by differentiating p_2 with respect the variables $x_2, \dots, x_n$. By the induction hypothesis we obtain that ∇r_1 is independent of some direction in $\mathbf{R}^{n-1}$, say that of x_2, and hence that $r_1(x_2, \dots, x_n) = c_2 x_2 + r_3(x_3, \dots, x_n)$.

Observe now that $c_1x_1 + c_2x_2$ is unbounded on $\big(Q(c_k\,,1)\big)$ since $\omega_k^1 \to 1$ and $\omega_k^2 \to 0$, as $k \to \infty$. We may therefore iterate the above procedure to obtain finally $p_2(x) = c_1x_1 + + \cdots c_nx_n$. However, this polynomial cannot be uniformly bounded on the sequence $\big(Q(c_k\,,1)\big)$ unless $c_1 = 0$ (by the same argument as above). Therefore $m = 0$ and p is independent of x_1. □

We end this section by considering briefly the case of potentials $V = V_+ - V_-$ not necessarily bounded from below. In order to regard V_- as a small perturbation of $-\Delta + V_+$ we assume that the condition

$$\lim_{|c|\to+\infty} \int_{Q(c,1)} V_-^p = 0 \tag{3.4}$$

holds for $p = n/2$ if $n \geq 3$ and for some $p > 1$ if $n = 1\,,2$.

We refer to [17] for a discussion of various conditions that allow to apply perturbation methods to general potentials and we only point out that (3.4) is a weak form of the classical condition $V_- \in L^{n/2}$.

Let $\varepsilon > 0$ and take $u \in C_0^\infty$; since $2p' = 2n/(n-2)$ $(n \geq 3)$, using Sobolev embedding, as in the proof of Theorem 3.1, we obtain from (3.4)

$$\int_{Q(c,1)} V_-|u|^2 \leq \Big(\int_{Q(c,1)} V_-^p\Big)^{1/p} \Big(\int_{Q(c,1)} |u|^{2p'}\Big)^{1/p'} \leq \varepsilon \Big(\int_{Q(c,1)} |\nabla u|^2 + (V_+ + 1)|u|^2\Big)\,,$$

for $|c|$ sufficiently large. Using the boundedness of V_- on compact sets of $\mathbf{R}^n$ we infer that there is a constant $C_\varepsilon > 0$ such that the inequality

$$\int V_-|u|^2 \leq \varepsilon\Big(\int |\nabla u|^2 + V_+|u|^2\Big) + C_\varepsilon \int |u|^2$$

holds for every $u \in C_0^\infty$. By density, the same inequality holds if $u \in D\big((-\Delta + + V_+)^{1/2}\big) = \Big\{u \in H^1 : |V_+|^{1/2}u \in L^2\Big\}$. The quadratic form

$$q(u) = \int |\nabla u|^2 + (V_+ - V_-)|u|^2$$

is therefore closed and bounded from below on the domain $D\big((-\Delta + V_+)^{1/2}\big)$ and defines a semibounded, self-adjoint operator $-\Delta + V$ (see [3, Theorem 1.8.2]. We generalise Theorem 3.1 in the following proposition.

Proposition 3.6. *Assume that conditions* (3.1) *and* (3.4) *hold; then the operator* $-\Delta + V$ *has compact resolvent.*

Proof. By Theorem 3.1, $-\Delta + V_+$ has compact resolvent and hence the embedding of $D\big((-\Delta + V_+)^{1/2}\big)$ (endowed with the graph norm) in L^2 is compact. Since the domain of the quadratic form of $-\Delta + V$ is $D\big((-\Delta + V_+)^{1/2}\big)$, the statement follows. □

4. Applications and examples

Let us present some concrete examples of application of the results of the previous sections.

a) Theorem 3.5 shows that the operators $-\Delta + x^2y^2$ and $-\Delta + \left[\sum_{i<j}(x_iy_j - -x_jy_i)^2\right]^{1/2}$ have discrete spectra in $L^2(\mathbf{R}^2)$ and $L^2(\mathbf{R}^{2n})$, respectively, even though the potentials do not tend to $+\infty$, as the variables go to ∞ (see [18] for different proofs). Another example to which Theorem 3.5 applies and which seems to be worth mentioning is $-\Delta + |y - x^2|$ in $L^2(\mathbf{R}^2)$.

Using Proposition 3.6, we can even construct polynomial potentials, unbounded from below, such that the corresponding Schrödinger operators have compact resolvents. In fact, we consider in $\mathbf{R}^2$ the polynomials $V(x, y) = x^{2k}y^2 - x$ with $k \geq 2$, and observe that (3.1) holds and that $\{V < 0\} = \{(x, y) \in \mathbf{R}^2 : x > 0,\ |y| < x^{-k+1/2}\}$. For $1 < p < k - 1/2$ the integral

$$\int_{-x^{-k+1/2}}^{x^{-k+1/2}} |V(x, y)|^p \, dy$$

converges to 0 as $x \to +\infty$ and from this condition (3.4) easily follows. Proposition 3.6 yields the compactness of the resolvent of $-\Delta + V$ in $L^2(\mathbf{R}^2)$.

b) We come back now to the operator

$$Au = -\Delta u + \langle \nabla F, \nabla u \rangle = -e^F \operatorname{div}\left(e^{-F}\nabla u\right)$$

of Section 2 and assume that the function $|\nabla F|^2 - 2\Delta F$ is bounded from below in $\mathbf{R}^n$. By Proposition 2.2 the operator $(A, D(A))$ is unitarily equivalent to the Schrödinger operator $(B, D(B))$ with $V = (1/4)|\nabla F|^2 - (1/2)\Delta F$ and hence we obtain the compactness of the resolvent of $(A, D(A))$ in L^2_μ if condition (3.1) is satisfied by V. We specialise our results in the polynomial case.

Proposition 4.1. *Let F be a polynomial such that $|\nabla F|^2 - 2\Delta F$ is bounded from below in $\mathbf{R}^n$. Then the resolvent of $(A, D(A))$ is not compact in L^2_μ if and only if there is $\omega \in \mathbf{R}^n$, $|\omega| = 1$, such that F can be written in the form $F(t\omega + z) = ct\omega + G(z)$, for all $t \in \mathbf{R}$ and $z \perp \omega$, where G is a polynomial in $(n-1)$ variables and $c \in \mathbf{R}$.*

Proof. We show that the stated representation of F holds if and only if $\dfrac{\partial V}{\partial \omega} \equiv 0$ and conclude, using Theorem 3.5.

If $F(t\omega + z) = c\omega + G(z)$, it is immediate that $\dfrac{\partial V}{\partial \omega} \equiv 0$. Suppose, conversely, that this last equality holds and assume that $\omega = (1, 0, \dots, 0)$. We write

$$F(x) = \sum_{j=0}^{m} x_1^j q_j(x_2, \dots, x_n)$$

with q_j polynomials in the variables $x_2, \dots, x_n$ and $q_m \not\equiv 0$. By assumption, the polynomial $(1/4)|\nabla F|^2 - (1/2)\Delta F$ does not depend on x_1. Comparing the coefficients of maximum degree of the variable x_1 in $(1/4)|\nabla F|^2$ and $(1/2)\Delta F$, one easily obtains that $m \leq 1$ and that q_1 is constant. $\square$

c) Let us point out some one-dimensional examples. In this case it is easy to state Molcanov's characterisation of compactness, which reads as follows: the operator $B = -D^2 + V$, $V \geq 0$, has compact resolvent in L^2 if and only if

$$\lim_{|c| \to +\infty} \int_c^{c+d} V(x)\, dx = +\infty\,, \qquad \forall d > 0\,. \tag{4.1}$$

Proposition 4.1 implies the discreteness of the spectrum of the operators $-D^2 + p(x)D$ in L^2_μ for every non-costant polynomial p. In particular, if $p(x) = x^{2k-1}$, $k \in \mathbf{N}$, the measure $d\,\mu(x) = \exp(-x^{2k}/2k)\, dx$ is finite and, if $k = 1$, we obtain the one-dimensional Ornstein-Uhlenbeck operator for which, however, the result is well-known.

Necessary and sufficient conditions for the compactness of the resolvent of the one-dimensional operators $-\alpha D^2 + \beta D$ have been proved also in [14] both in weighted L^2-spaces and in spaces of continuous functions, for general α, β. The methods in [14] are different and do not extend to the multidimensional case. For example, let $\alpha \equiv 1$ and F a primitive of β; if $e^{-F} \in L^1$, so that the measure $d\mu(x) = e^{-F(x)}\, dx$ is finite, the operator $-D^2 + \beta D$ has compact resolvent in L^2_μ if and only if

$$\lim_{x \to -\infty} \left(\int_{-\infty}^x e^{-F(t)}\, dt \right) \left(\int_0^x e^{F(t)}\, dt \right) = \lim_{x \to +\infty} \left(\int_x^{+\infty} e^{-F(t)}\, dt \right) \left(\int_0^x e^{F(t)}\, dt \right) = 0\,,$$

a condition that turns out to be equivalent to (4.1) with $V = (1/4)|F'|^2 - (1/2)F''$, even though this does not seem evident at first sight.

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