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On a question of M. Conder

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Teoria dei gruppi. — *On a question of M. Conder.* Nota (*) di M. Chiara Tamburini e Paola Zucca, presentata dal Socio C. De Concini.

Abstract. — We show that the special linear group $SL(3, \mathbb{Z})$, over the integers, is not $(2, 3)$-generated. This gives a negative answer to a question of M. Conder.

Key words: Linear Groups; Simple groups; (2, 3)-generation.

Riassunto. — *Su un problema di M. Conder.* Dimostriamo che il gruppo speciale lineare $SL(3, \mathbb{Z})$, sugli interi, non è $(2, 3)$-generato.

1. Introduction

Recently M. Conder raised the question whether the special linear group $SL(3, \mathbb{Z})$, over the integers, is $(2, 3)$-generated. It appears as Problem 14.49 in the 14° edition of the Kourovka Notebook [8]. The aim of this paper is to show that this question has a negative answer. Such an outcome is not so obvious as, for all primes p, the groups $SL(3, p)$ are $(2, 3)$-generated. This fact can easily be deduced from the result that $PSL(3, q)$ is $(2, 3)$-generated for all prime powers $q \neq 4$ (see [1, 3]).

Our motivation is founded on the vast literature concerning the $(2, 3)$-generation problem. So we find it appropriate to mention some relevant results in this area. We recall that a group G is said to be $(2, 3)$-generated if it can be generated by an involution and an element of order 3. Equivalently if it is a non-trivial epimorphic image of $PSL(2, \mathbb{Z})$, by a well-known result (see, for example, [10, p. 164]). Furthermore, for each natural number k, G is said to be $(2, 3, k)$-generated if it admits a (2,3)-generating pair (X, Y) such that XY has order k. First of all we mention a remarkable paper [4], where it is shown that, for all series of finite simple groups of Lie type (with the exception of $PSp(4, q)$, $q = 2^m$ or 3^m), a generic involution and a generic element of order 3 generate the group with probability 1. On the other hand, recent constructive results show that the family of $(2, 3, k)$-generated groups is very large. In fact, for each prime $k \geq 7$, there are $2^{\aleph_0}$ isomorphism classes of simple $(2, 3, k)$-generated groups (see [5, 7, 12]). In particular, most finite classical groups of large rank are $(2, 3, 7)$-generated [6].

It has been shown in [7, 11] that the special linear group $SL(n, \mathbb{Z})$ is $(2, 3)$-generated for all $n \geq 28$ and, indeed, that it is $(2, 3, 7)$-generated for all $n \geq 287$. On the other hand, if $n = 2, 4$, the groups $SL(n, \mathbb{Z})$ and $GL(n, \mathbb{Z})$ are not $(2, 3)$-generated. This assertion is trivial for $SL(2, \mathbb{Z})$, since the only involution is the central one. Considering the epimorphism $SL(2, \mathbb{Z}) \to GL(2, 2) \simeq Sym(3)$ it is easy to see

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that $GL(2, \mathbb{Z})$ contains a normal subgroup of index 4. It follows that $GL(2, \mathbb{Z})$ has an abelian quotient of order 4, and thus it cannot be $(2, 3)$-generated. Finally the groups $SL(4, \mathbb{Z})$ and $GL(4, \mathbb{Z})$ have $SL(4, 2) \simeq Alt(8)$ as an epimorphic image. And Miller [9], in 1901, showed that $Alt(8)$ is not $(2, 3)$-generated. For sake of completeness we also mention that $SL(n, \mathbb{Z})$ is not $(2, 3, 7)$-generated for all $n \leq 19$ and $n = 22$ (cf. [2]).

2. Proof of the result

As usual, we let $GL(3, \mathbb{Z})$ act on the right on the free abelian group $\mathbb{Z}^3$, consisting of row vectors, with canonical basis $\{e_1, e_2, e_3\}$. In the following p denotes a prime and $f_p : GL(3, \mathbb{Z}) \to GL(3, p)$ the obvious homomorphism. We will make repeated use of the fact that $\operatorname{Im} f_p$ contains $SL(3, p)$. In particular, for all primes p, $\operatorname{Im} f_p$ is absolutely irreducible (in both actions on row and column vectors) and it is not contained in the group of isometries of any non-zero bilinear form.

Theorem. *The groups* $GL(3, \mathbb{Z})$ *and* $SL(3, \mathbb{Z})$ *are not* $(2, 3)$*-generated.*

Proof. Assume, by contradiction, $GL_3(\mathbb{Z}) = \langle A, B \rangle$ with $A^2 = B^3 = I$. Clearly $\det A = -1$ and $\det B = 1$. In particular A fixes pointwise a 2-dimensional subspace W of $\mathbb{Q}^3$. It follows easily that, up to conjugation in $GL_3(\mathbb{Z})$, we may assume $W \cap \mathbb{Z}^3 = \langle e_1, e_2 \rangle$. Since B cannot be scalar, it admits the eighenvalue 1. So let $0 \neq w = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ be such that $wB = w$. The irreducibility of $\operatorname{Im} f_p$, for all primes p, implies $\lambda_3 = \pm 1$. It follows that $\{e_1, e_2, w\}$ is a basis of $\mathbb{Z}^3$. Hence, we may assume:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}$$

for suitable coprime integers a, b. Now let $z, t \in \mathbb{Z}$ be such that $az + bt = 1$ and consider the matrix $X = \operatorname{block\,diag}\left(\begin{pmatrix} z & -b \\ t & a \end{pmatrix}, 1\right)$. Conjugating A and B by X we get:

$$GL(3, \mathbb{Z}) = \langle A^X, B^X \rangle = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} , \begin{pmatrix} r & v & * \\ s & u & * \\ 0 & 0 & 1 \end{pmatrix} \right\rangle .$$

Clearly $v = \pm 1$, by the irreducibility of $\operatorname{Im} f_p$ in the dual action on column vectors, and $\begin{pmatrix} r & \pm 1 \\ s & u \end{pmatrix}$ has order 3 and trace -1. Setting $Y = \operatorname{block\,diag}\left(\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}, 1\right)$ and subsituting B with B^{-1} if necessary, one easily obtains:

$$GL(3, \mathbb{Z}) = \langle A^{XY}, B^{XY} \rangle = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} , \begin{pmatrix} 0 & 1 & x \\ -1 & -1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

where x and y are suitable coprime integers. Now consider the matrix

$$J = \begin{pmatrix} 2(2x + y) & 2(y - x) & 2x + y \\ 2(y - x) & (y + x - 4)(y - x) & y - x \\ 2x + y & y - x & 3 \end{pmatrix}$$

and set $\overline{A} = A^{XY}$, $\overline{B} = B^{XY}$. A direct calculation shows that $\overline{A}J\overline{A}^t = J$ and $\overline{B}J\overline{B}^t = J$, whenever $y(2x + y - 6) = 0$. This means that, if p divides $y(2x + y - 6)$, then $\mathrm{Im} f_p$ is a group of isometries with respect to the bilinear form induced by $f_p(J)$. It follows $f_p(J) = 0$, hence $p = 3$, $x \equiv y \pmod 3$. But, in this case, $\mathrm{Im} f_3$ would fix the subspace $\langle (1, -1, 0) \rangle$ of $\mathbb{Z}_3^3$, a contradiction. It follows $y = \pm 1$, $2x + y - 6 = \pm 1$ and we are left with 4 possibilities. Assume first $x = 3$, $y = 1$ or $x = 4$, $y = -1$. Then $\mathrm{Im} f_{13}$ would fix the subspace $\langle (1, -xy, 0) \rangle$ of $\mathbb{Z}_{13}^3$, against the irreducibilty. Finally, if $x = 2$, $y = 1$ or $x = 3$, $y = -1$, then $\mathrm{Im} f_7$ would fix the subspace $\langle (1, -xy, 0) \rangle$ of $\mathbb{Z}_7^3$, a final contradiction. We have thus shown that $\mathrm{GL}(3, \mathbb{Z})$ is not $(2, 3)$-generated.

Noting that $\mathrm{GL}(2m + 1, \mathbb{Z}) = \mathrm{SL}(2m + 1, \mathbb{Z}) \times \langle -I \rangle$ it follows immediately that $\mathrm{SL}(3, \mathbb{Z})$ is not $(2, 3)$-generated. In fact any $(2, 3)$-generating couple (C, D) of $\mathrm{SL}(2m+1, \mathbb{Z})$ gives rise to the $(2, 3)$-generating couple $(-C, D)$ of $\mathrm{GL}(2m+1, \mathbb{Z})$. □

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