Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Claudia Landi

Cohomology rings of Artin groups

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **11** (2000), n.1, p. 41–65.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2000_9_11_1_41_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2000.

Rend. Mat. Acc. Lincei
s. 9, v. 11:41-65 (2000)

Topologia. — *Cohomology rings of Artin groups.* Nota di Claudia Landi, presentata (*) dal Socio C. De Concini.

Abstract. — In this paper integer cohomology rings of Artin groups associated with exceptional groups are determined. Computations have been carried out by using an effective method for calculation of cup product in cellular cohomology which we introduce here. Actually, our method works in general for any finite regular complex with identifications, the regular complex being geometrically realized by a compact orientable manifold, possibly with boundary.

Key words: Artin group; Cup product; Regular complex; Identification; Dual block decomposition.

Riassunto. — *Anelli di coomologia dei gruppi di Artin.* In questo lavoro vengono determinati gli anelli di coomologia intera dei gruppi di Artin associati ai gruppi di Coxeter eccezionali. Una presentazione per tali anelli è ottenuta utilizzando un metodo effettivo, che introduciamo qui, per il calcolo del prodotto *cup* in coomologia cellulare. In generale mostriamo che tale metodo è applicabile ad ogni complesso cellulare finito e regolare, realizzato geometricamente da una varietà compatta e orientabile, eventualmente con bordo, su cui agisca una famiglia di identificazioni.

1. Introduction

In this paper we determine the integer cohomology rings of Artin groups associated with all the exceptional groups.

The study of the cohomology of Artin groups has been initiated in the seventies and pursued further by a number of authors who have computed the integer cohomology of braid groups for the infinite series A, C and D (see [1, 10, 11, 19]). The method used in these works is that of constructing a triangulation of the classifying space $Y/W = \left(\mathbb{C}^n \setminus \bigcup_{H\in\mathcal{A}} H\right)/W$, where W is the considered reflection group and $\mathcal{A}$ is the arrangement of the complexification of mirrors.

A cell complex with identifications having the same homotopy type as Y/W has been introduced in [16], allowing to produce an algebraic complex for the computation of the additive structure of cohomology of Artin groups with coefficients in rank-1 local systems (hence in particular integer cohomology groups associated with all the exceptional groups). Generalizations to any local system have been presented in [6]. For more recent developments in the investigation of cohomology of Artin groups we refer the reader to [7, 8].

Our aim is that of computing the multiplicative structure of the integer cohomology for the exceptional groups by using the classifying space suggested in [16]. To this end, since the additive structure is given in terms of cellular cohomology, it would be desirable to perform also cup products in cellular cohomology. We recall (see, *e.g.*,

(*) Nella seduta del 25 febbraio 2000.

[15]) that in cellular cohomology the cup product is defined as the composition of the cross product with the homomorphism induced by a cellular approximation to the diagonal map. To our knowledge, in the case of arbitrary cell complexes no general formulas are given for the calculation of a diagonal approximation (whereas for simplicial complexes one can use the well known Alexander-Whitney approximation map). Besides, also the construction of a diagonal approximation in each specific case may be not easy. Thus a method to calculate in practice cup product in cellular cohomology is needed.

Here we propose an effective method of computation of cup product for a particular class of finite regular cell complexes with identifications. More precisely, let X be a finite regular cell complex geometrically realized by a compact orientable manifold possibly with boundary. Let F be a family of identifications acting on the cells of X. The complex X/F is not necessarily realized by a manifold. Hence the classical method used for manifolds, based on the intersection of dual complexes, does not apply to X/F. However, by generalizing the standard method used for manifolds we have obtained a technique for the computation of cup product in the cellular cohomology of X/F which is still based on the intersection of complexes in general position. Our method leads to a combinatorial formula for the cup product in the cellular cohomology of X/F. This will be the subject of Section 4.

The classifying spaces of Artin groups introduced in [16] belong to the class of cell complexes to which our formula for cup product applies. Hence in particular we are able to compute the cohomology rings of Artin groups for all the exceptional groups. Calculations have been carried out partly by hand, partly by the use of a computer. The results of such computations are reported in Section 5.

We begin by recalling in Section 2 some basic facts about Artin groups and the main results from [16] and, in Section 3, the main properties of regular cell complexes with identifications.

2. Cohomology of Artin groups: definitions and basic results

In this section we shall introduce the spaces object of our work. For more details about reflection groups we refer the reader to [2] and [13]. As a reference about Artin groups, see [20].

Let (W, S) be a Coxeter system. It is well known that it is possible to realize W as a group generated by reflections in $\mathbb{R}^n$. W acts also on the complement $Y \subseteq \mathbb{C}^n$ of the complexification of the reflection hyperplanes of W. The orbit space Y/W is a space $K(\pi, 1)$ (see [9]). The fundamental group G_W of Y/W is called *Artin group* of type W (see, *e.g.*, [3]).

A simple realization of a space having the same homotopy type as Y/W has been introduced in [16]. One starts from the cell complex $Q \subseteq \mathbb{R}^n$ dual to the stratification $\mathcal{S}$ into facets associated to the reflection hyperplanes of W. Q can be constructed by choosing a point $v(F^j)$ inside each j-codimensional facet F^j of $\mathcal{S}$ and considering the simplexes

$$\sigma(F^{i_0}, \dots, F^{i_j}) \stackrel{\text{def}}{=} \left\{ \sum_{k=0}^{j} \lambda_k v(F^{i_k}) : \sum_{k=0}^{j} \lambda_k = 1 , \lambda_k \in [0, 1] \right\}$$

where $F^{i_k} > F^{i_{k+1}}$ for $k = 0, \dots, j-1$. The j-dimensional cell $e_j(\widetilde{F}^j)$ which is dual to $\widetilde{F}^j$ is defined as the union $\bigcup \sigma(F^0, \dots, F^{j-1}, \widetilde{F}^j)$ over all chains $F^0 > \dots > F^{j-1} > \widetilde{F}^j$. Now let us fix a chamber C_0 and call v_0 the vertex of Q contained in C_0. Let also $\mathcal{F}_0$ be the system of facets contained in the closure of C_0 and let $\mathcal{Q}_0$ be the set of cells in Q dual to a facet of $\mathcal{F}_0$. It turns out that for every facet F in $\mathcal{S}$ there is exactly one facet $F_0 \in \mathcal{F}_0$ in the same W- orbit as F. Dually for every cell e in Q there is exactly one cell $e^0 \in \mathcal{Q}_0$ in the same W-orbit as e. The elements $\gamma \in W$ such that $\gamma(e^0) = e$ describe a left-coset of the stabilizer W_{F_0} of the facet F_0 dual to e^0. There exists one and only one $\gamma_{(e)}$ of minimal length (in the Coxeter system associated to C_0) such that $\gamma_{(e)}^{-1}(e) \cap C_0 \neq \emptyset$. Then [16, Theorem 1.4] Y/W has the same homotopy type of the cell complex X_W obtained from Q by identifying two cells e, e' of Q if and only if they are in the same W-orbit, by using the homeomorphism induced by $\gamma_{(e')}(\gamma_{(e)})^{-1}$. The natural projection $Q \to X_W$ will be denoted by π_W.

It can be shown that when W is finite the cell complex X_W is a space of type $K(G_W, 1)$. Thus the cohomologies of X_W are by definition the cohomologies of G_W. Therefore, assuming that W is finite and essential and by using the above construction it is possible to compute the cohomology groups of Artin groups as follows. Each cell of X_W corresponds to a cell of $\mathcal{Q}_0$, therefore to a facet in $\mathcal{F}_0$. Let $H_1, \dots, H_n$ be the hyperplanes of C_0. Let also v_i with $i = 1, \dots, n$ be chosen points in $H_i \cap \overline{C_0}$. Each facet $F \in \mathcal{F}_0$ corresponds to a unique intersection $H_{i_1}, \dots, H_{i_k}$, $k = \operatorname{codim}(F)$, where the H_{i_j} are the hyperplanes containing F and $i_1 < i_2 < \cdots < i_k$. Therefore, each facet $F \in \mathcal{F}_0$ corresponds to a unique subset $\Gamma = \Gamma(F) \subseteq \{1, \dots, n\} = I_n$ and $\operatorname{card}(\Gamma) = \dim e(F)$. Hence in the following $e(F)$ with $F \in \mathcal{F}_0$ will be also denoted by $e(\Gamma)$ with $\Gamma = \Gamma(F)$, while its class in X_W will be denoted by $\underline{e}(\Gamma)$.

Let us give to the dual cell $e(F) \in \mathcal{Q}_0$ the orientation induced by the ordering $v_0, v_{i_1}, \dots, v_{i_k}$. Next, let us give an orientation to each cell $e \in Q$ by requiring that $\gamma_{(e)}$ is orientation preserving.

Finally, let us denote the parabolic subgroup of W generated by $\Gamma \subseteq I_n \cong S$ by W_Γ and, given two subsets Γ and Γ' of S such that $\Gamma \subseteq \Gamma'$, let us consider the subset of W

$$W_\Gamma^{\Gamma'} \stackrel{\text{def}}{=} \left\{ w \in W_{\Gamma'} : l(sw) > l(w) \ \forall s \in \Gamma \right\} ,$$

where l is the length function. In other words $W_\Gamma^{\Gamma'}$ is the set of representatives of minimal length in the left-coset of the parabolic subgroup W_Γ in $W_{\Gamma'}$.

Then, by identifying $\Gamma \subseteq I_n$ with the homomorphism dual to $\underline{e}(\Gamma)$, one has [16]:

Theorem 1. *Let* (C^*, δ^*) *be the algebraic complex defined by setting*

$$C^k \stackrel{\text{def}}{=} \left\{ \sum \nu_\Gamma \Gamma : \Gamma \subseteq I_n , \operatorname{card}(\Gamma) = k , \nu_\Gamma \in \mathbb{Z} \right\}$$

and

$$\delta^k(\Gamma) \stackrel{\text{def}}{=} \sum_{j\in I_n\setminus\Gamma} (-1)^{\sigma(j,\Gamma)+1} \sum_{h\in W_\Gamma^{\Gamma\cup\{j\}}} (-1)^{l(h)}(\Gamma\cup\{j\})$$

where $\sigma(j,\Gamma) \stackrel{\text{def}}{=} \operatorname{card}\{i\in\Gamma : i<j\}$. *Then* $H^*(X_W, \mathbb{Z}) \cong H^*(C^*)$.

By using this theorem in [16] the group structure of the integer cohomology of Artin groups for all the exceptional groups has been deduced. We report these results in table I for later use. The first column of table I contains the considered Coxeter group, the second column contains the integer cohomology groups for the corresponding Artin groups and the third column contains a list of representatives for the generators of these cohomologies. The elements of S, and hence the associated hyperplanes, are given the ordering corresponding to the standard one used in Coxeter diagrams (see [2]). For sake of simplicity we denote the element $\Gamma = \{i_1, \dots, i_k\} \subseteq I_n \cong S$ by the symbol $i_1 \dots i_k$.

3. Regular cell complexes: preliminaries

In this section we shall recall some standard facts about cell complexes. For more details we refer the reader to [5, 12, 14, 17].

If a cell complex X is geometrically realized by a space $|X|$ with skeletons $|X_n|$, for $n \geq -1$, then the *n-cells of* X, $n \geq 0$, will be the components of $|X_n| \setminus |X_{n-1}|$.

A cell complex X with finitely many cells is said to be *finite*. X is called *regular* if the closure of each n-cell is homeomorphic with the closed ball in Euclidean n-space. In what follows X will always denote a finite regular cell complex.

X will be said to be *oriented* if all its cells have been given an orientation. If $e' \subseteq e$ are oriented cells of X, we shall denote as usual their incidence number by $[e : e']$.

3.1. *Dual block decomposition.*

The cells of every finite regular complex X can be partially ordered according to the relation $\subseteq$ where $e \subseteq e'$ if e is a face of e'. Thus it is possible to construct a simplicial complex $\Delta(X)$ called *barycentric subdivision* of X, defined as follows: the vertices of $\Delta(X)$ are the cells of X and its simplices are the finite collections of cells of X such that the cells of each collection can be ordered by the relation $\subseteq$. In the following we shall denote the vertex of $\Delta(X)$ associated with a cell $e_p \in X$ by b_p.

It is well known that $\Delta(X)$ can be actually realized in $|X|$. Therefore it makes sense to say that a simplex of $\Delta(X)$ is included in a cell of X.

Let us now partially order the vertices of $\Delta(X)$ by decreasing dimension of the associated cells of X; this ordering produces an ordering on the vertices of each simplex of $\Delta(X)$.

Definition 1. For every cell $e \in X$ the union of all open simplices of $\Delta(X)$ such that the final vertex is associated to e is called *dual block to* e and is denoted by De.

TABLE I. – *Cohomologies of Artin groups associated with exceptional groups (integer coefficients) and representatives for their generators.*

$I_2(2s)$	$H^1 = \mathbb{Z}^2$ $H^2 = \mathbb{Z}$	$\alpha_1 = 1, \beta_1 = 2$ $\alpha_2 = 12$
$I_2(2s+1)$	$H^1 = \mathbb{Z}$ $H^2 = 0$	$\alpha_1 = 1 + 2$
H_3	$H^1 = \mathbb{Z}$ $H^2 = \mathbb{Z}$ $H^3 = \mathbb{Z}$	$\alpha_1 = 1 + 2 + 3$ $\alpha_2 = 13$ $\alpha_3 = 123$
H_4	$H^1 = \mathbb{Z}$ $H^2 = 0$ $H^3 = \mathbb{Z} \times \mathbb{Z}_2$ $H^4 = \mathbb{Z}$	$\alpha_1 = 1 + 2 + 3 + 4$ $\alpha_3 = 123, \beta_3 = 234$ $\alpha_4 = 1234$
F_4	$H^1 = \mathbb{Z}^2$ $H^2 = \mathbb{Z}^2$ $H^3 = \mathbb{Z}^2$ $H^4 = \mathbb{Z}$	$\alpha_1 = 1 + 2, \beta_1 = 3 + 4$ $\alpha_2 = 13 + 14 + 24, \beta_2 = 23$ $\alpha_3 = 123, \beta_3 = 234$ $\alpha_4 = 1234$
E_6	$H^1 = \mathbb{Z}$ $H^2 = 0$ $H^3 = \mathbb{Z}_2$ $H^4 = \mathbb{Z}_2$ $H^5 = \mathbb{Z}_2 \times \mathbb{Z}_3$ $H^6 = \mathbb{Z}_3$	$\alpha_1 = 1 + 2 + 3 + 4 + 5 + 6$ $\alpha_3 = 134 + 234 + 345 + 456 - 245$ $\alpha_4 = 1346 + 2346 - 1456 + 1245$ $\alpha_5 = 13456, \beta_5 = 12345 + 23456$ $\alpha_6 = 123456$
E_7	$H^1 = \mathbb{Z}$ $H^2 = 0$ $H^3 = \mathbb{Z}_2$ $H^4 = \mathbb{Z}_2^2$ $H^5 = \mathbb{Z}_2^2 \times \mathbb{Z}_3^2$ $H^6 = \mathbb{Z}_2 \times \mathbb{Z}_3^2 \times \mathbb{Z}$ $H^7 = \mathbb{Z}$	$\alpha_1 = 1 + 2 + 3 + 4 + 5 + 6 + 7$ $\alpha_3 = 134 + 234 + 345 + 456 - 245 + 567$ $\alpha_4 = 1346 + 2346 - 1456 + 1245 + 2347 +$ $1347 + 3457 - 3567 - 1567, \beta_4 = 2567 + 2457$ $\alpha_5 = 12345 + 23456, \beta_5 = 12457 + 12567 +$ $23567, \gamma_5 = 23567, \delta_5 = 13456 + 34567$ $\alpha_6 = 123457, \beta_6 = 123456, \gamma_6 =$ $124567, \delta_6 = 234567$ $\alpha_7 = 1234567$
E_8	$H^1 = \mathbb{Z}$ $H^2 = 0$ $H^3 = \mathbb{Z}_2$ $H^4 = \mathbb{Z}_2$ $H^5 = \mathbb{Z}_2^2 \times \mathbb{Z}_3$ $H^6 = \mathbb{Z}_2 \times \mathbb{Z}_3^2$ $H^7 = \mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}$ $H^8 = \mathbb{Z}$	$\alpha_1 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$ $\alpha_3 = 134 + 234 + 345 + 256 - 245 + 567 + 678$ $\alpha_4 = 1346 + 2346 - 1456 + 1245 + 2347 +$ $1347 + 3457 - 3567 - 1567 + 1348 + 2348 +$ $3458 + 4568 - 4678 - 2678 - 2567 - 3678 -$ $2458 - 2457 - 1678$ $\alpha_5 = 12345 + 23456, \beta_5 = 12457 + 12567 +$ $23567 + 12458 - 14568 + 14678 + 13468 +$ $12678 + 23678 + 23468, \gamma_5 = 45678 - 24567 +$ $13456 + 34567$ $\alpha_6 = 123457 + 123458 + 234568, \beta_6 =$ $123456, \gamma_6 = 134568 - 145678 + 124567$ $\alpha_7 = 2345678, \beta_7 = 1345678, \gamma_7 =$ $1234568, \delta_7 = 1234567$ $\alpha_8 = 12345678$

The collection of all dual blocks De with e varying in X is called *block decomposition dual to* X and we shall denote it by $\mathcal{D}(X)$.

Let us now assume that $|X|$ is an orientable n-dimensional compact and connected manifold, possibly with boundary. Let us assume that an orientation has been chosen on $|X|$. The cells of X are oriented in arbitrary fashion. Then it is possible to define an *orientation on the dual blocks of* X by the following rule: let e_j be a j-dimensional cell of X and De_j the dual block associated with it. Let $b_j, b_{j-1}, \ldots, b_0$ be the vertices of a simplicial cell σ_j of $\Delta(X)$, where b_j is the vertex associated to e_j. Thus $\sigma_j \subseteq e_j$. Let $b_n, b_{n-1}, \ldots, b_j$ be the vertices of a simplicial cell τ_{n-j} in De_j. The vertices $b_n, b_{n-1}, \ldots, b_j, b_{j-1}, \ldots, b_0$ are those of an n-simplex ρ_n of $\Delta(X)$. Thus we can orient σ_j by means of the orientation of e_j and ρ_n by means of the orientation of $|X|$: $\sigma_j = \xi(b_j, b_{j-1}, \ldots, b_0)$ and $\rho_n = \zeta(b_n, b_{n-1}, \ldots, b_j, \ldots, b_0)$ with $\xi, \zeta \in \{+1, -1\}$. We define the orientation of the simplex $\tau_{n-j} = \eta(b_n, b_{n-1}, \ldots, b_j)$ by choosing η satisfying the relation $\xi\eta = \zeta$. It can be seen (see, *e.g.*, [17]) that since $|X|$ is a manifold, this definition does not depend on the choice of σ_j and τ_{n-j}. Hence this gives an orientation of De_j.

We recall (see, *e.g.*,[14]) that if the n-cells of X are given the orientation of $|X|$ and e_j, $0 \le j \le n$, is in the interior of $|X|$, then De_j, is oriented by the above rule so that

$$De_j = \sum [e_n : e_{n-1}] \cdots [e_{j+1} : e_j](b_n, \ldots, b_j)$$

where the sum is running over all the chains of the kind $e_n \supseteq \ldots \supseteq e_j$ and b_j denotes the vertex of $\Delta(X)$ associated with e_j.

3.2. *Regular complexes with identifications.*

From a finite regular complex X one can realize a new cell complex with a different underlying space by identifying the faces of X (see, *e.g.*, [5]).

Definition 2. A *family F of identifications on X* is a collection of homeomorphisms between the closed cells of X satisfying the following properties:

i) For every $f : \bar{e}_j \to \bar{e}'_j \in F$, if e_i is a face of e_j, then $f_{|\bar{e}_i}$ carries $\bar{e}_i$ onto a closed face of $\bar{e}'_j$ of the same dimension.

ii) For each $e \in X$ the identity homeomorphism $\bar{e} \to \bar{e}$ lies in F.

iii) For each $f \in F$, f^{-1} is in F.

iv) If $f : \bar{e}_i \to \bar{e}'_i$ and $g : \bar{e}'_i \to \bar{e}''_i$ are in F then $g \circ f$ is in F.

v) If $f : \bar{e} \to \bar{e}$ is in F, f is the identity homeomorphism.

vi) If $f : \bar{e}_i \to \bar{e}'_i$ is in F and $\bar{e}_j \subseteq \bar{e}_i$ then $f_{|\bar{e}_j}$ is in F.

The family F determines an equivalence relation for points of $|X|$: $x \sim y$ if a map in F carries x to y. We shall denote by $|X/F|$ the space $|X|$ quotiented by the relation $\sim$ and by $p : |X| \longrightarrow |X/F|$ the natural projection. The space $|X/F|$ with the skeletons $p(|X_n|)$, $n \ge 0$ form a complex, in general not regular, that we shall denote by X/F. For every $e \in X$ we shall denote by $\underline{e}$ the cell $p(e) \in X/F$.

If the cells of X have been oriented then it is possible to define incidence relations for the cells of X/F in such a way that, denoting by $p_\#$ the natural homomorphism

$C_*(X, \mathbb{Z}) \to C_*(X/F, \mathbb{Z})$, $p_\# \partial = \partial p_\#$. Indeed, it is sufficient to set for every $\underline{e}, \underline{e}' \in X/F$, $[\underline{e} : \underline{e}'] = \sum_{e' \in \underline{e}'} [e : e']$, where e is any cell in $\underline{e}$.

Therefore, in the sequel we shall always assume that if a cell complex is obtained by means of identifications, $p_\#$ commutes with ∂.

3.3. *Subdivision of a regular cell complex.*

The barycentric subdivision is not the only way in which the space underlying a regular cell complex can be triangulated coherently with its cellular structure. Let us give a more general definition of subdivision.

Definition 3. We shall say that a simplicial complex $\widetilde{T}(X)$ is a *subdivision of a finite regular cell complex* X if for every sub-complex Y of X there exists a sub-complex $\widetilde{T}(Y)$ of $\widetilde{T}(X)$ that can be realized in $|Y|$.

In particular, $\widetilde{T}(X)$ can be realized in $|X|$.

If X is oriented then, for every oriented cell $e \in X$, we can give the simplices of $\widetilde{T}(e)$ of maximal dimension the orientation induced from that of e. This way it remains defined a homomorphism $\widetilde{T} : C_*(X, \mathbb{Z}) \to C_*(\widetilde{T}(X), \mathbb{Z})$ which, for every $p \geq 0$, takes a p-cell $e_p \in X$ into the formal sum of all the p-simplices of $\widetilde{T}(X)$ contained in e_p.

Definition 4. Let $\widetilde{T}(X)$ be a subdivision of X and let F be a family of identifications on X. We shall say that $\widetilde{T}(X)$ is *F-equivariant* if for every $e \in X$ and every $f : \overline{e} \to f(\overline{e})$ in F it holds that whenever $\sigma \in \widetilde{T}(X)$ and $\overline{\sigma} \subseteq \overline{e}$, $f_{|\overline{\sigma}}(\sigma) \in \widetilde{T}(X)$.

Given a family F of identification on a complex X and an F-equivariant subdivision $\widetilde{T}(X)$, F induces a family of identifications on $\widetilde{T}(X)$, still denoted by F, given by the restrictions of every $f : \overline{e} \to \overline{e}' \in F$ to the closure of each simplex of $\widetilde{T}(\overline{e})$. Therefore, from an F-equivariant subdivision of X, one obtains a new complex with identifications $\widetilde{T}(X)/F$ realized in $|X/F|$. Actually, $\widetilde{T}(X)/F$ is a pseudo-triangulation of $|X/F|$. We shall denote it by $T(X/F)$.

Let now $T : C_*(X/F, \mathbb{Z}) \to C_*(T(X/F), \mathbb{Z})$ be the homomorphism obtained by setting $T(p_\#(e)) = p_\#(\widetilde{T}(e))$ for every oriented $e \in X$. If $\widetilde{T}$ commutes with the boundary operator, since also $p_\#$ does, T will pass to cohomology. It is well known that T induces an isomorphism between the cellular cohomology rings of X/F and $T(X/F)$ (see, *e.g.*, [21]).

4. Effective computation of cup product in cellular cohomology

Let X be a finite oriented regular cell complex and let F be a family of identifications on X. Let us consider an F-equivariant subdivision of X, $\widetilde{T}(X)$ such that $\widetilde{T}$ commutes with the boundary operator. Thus we can construct the complex $T(X/F)$ as explained in Section 3.3.

Let us recall that $H^*(T(X/F), \mathbb{Z}) \cong H^*(X/F, \mathbb{Z})$ as rings. Nevertheless in general it is not obvious how to construct a set of representatives of generators of the cohomology groups of $T(X/F)$ from the knowledge of those of X/F. In other words it is not obvious how to explicitly give a chain map $\phi_T : C^*(X/F, \mathbb{Z}) \longrightarrow C^*(T(X/F), \mathbb{Z})$ inducing an isomorphism in cohomology. In the first part of this section we present a method to construct the map ϕ_T in the particular case when X is realized by an orientable compact manifold, possibly with boundary.

We point out that this is a major step in the direction of the computation of cup product, which is our motivation. Indeed, if such a map ϕ_T is given, then we can transform the generators of $H^*(X/F, \mathbb{Z})$ into those of $H^*(T(X/F), \mathbb{Z})$ so that we can apply the well known Alexander-Whitney formula for cup product on simplicial cells. This will be the subject of the second part of this section.

4.1. *Construction of* ϕ_T.

Let us now assume that X is realized by a compact n-dimensional orientable manifold possibly with boundary. We assume that $|X|$ has been oriented and we orient the n-cells of X accordingly with the orientation of $|X|$, while all the other cells can be oriented arbitrarily.

Definition 5. Let $\widetilde{T}(X)$ be a subdivision of X. We shall say that $\widetilde{T}(X)$ is *transverse to* $\mathcal{D}(X)$ if for every p-dimensional simplex $\sigma_p \in \widetilde{T}(X)$ and for every $(n-q)$-dimensional cell $e_{n-q} \in X$, with $0 \leq p, q \leq n$, the following conditions hold:

i) if $De_{n-q} \cap \sigma_p \neq \emptyset$ then $p + q \geq n$;

ii) when $p + q = n$, $De_{n-q} \cap \sigma_p$ consists at most of a finite set of points;

iii) when $p + q = n$, for each $x \in De_{n-q} \cap \sigma_p$ if x is in the interior of $|X|$, there exists a neighbourhood U of x in $|X|$ and a homeomorphism $h : U \to \mathbb{R}^q \times \mathbb{R}^p$ such that $h(x) = 0$, $h(U \cap \sigma_p) \subseteq \{0\} \times \mathbb{R}^p$ and $h(U \cap De_{n-q}) \subseteq \mathbb{R}^q \times \{0\}$. If x belongs to the boundary of $|X|$, there exists a neighbourhood U of x in $|X|$ and a homeomorphism $h : U \to \mathbb{R}^q_{\geq 0} \times \mathbb{R}^p$, where $\mathbb{R}^q_{\geq 0}$ denotes the half-space $\{(x_1, \ldots, x_q) \in \mathbb{R}^q : x_q \geq 0\}$, such that $h(x) = 0$, $h(U \cap \sigma_p) \subseteq \{0\} \times \mathbb{R}^p$ and $h(U \cap De_{n-q}) \subseteq \mathbb{R}^q_{\geq 0} \times \{0\}$.

Remark 1. In the case when $x \in \partial|X|$ and $x \in \sigma_p \cap De_p$, it must hold that $e_p \subseteq \partial|X|$ and, by condition *i)*, also $\sigma_p \subseteq \partial|X|$. Therefore, in such a case, transversality at x holds if and only if we have transversality in the closed manifold $\partial|X|$ between the same simplex σ_p and the dual block to e_p relative to the manifold $\partial|X|$.

Remark 2. It must be noticed that condition *i)* implies that if a q-simplex of $\widetilde{T}(X)$ intersects a dual block at a point, the intersection point belongs to the interior of the considered simplex of $\widetilde{T}(X)$ and to no other dual block.

From now on we shall assume that $\widetilde{T}(X)$ is transverse to $\mathcal{D}(X)$. As we have already seen in Section 3.1, the orientation on the cells of X induces an orientation on the blocks of $\mathcal{D}(X)$. Therefore it makes sense to give a notion of intersection index between cells of $\widetilde{T}(X)$ and blocks of $\mathcal{D}(X)$.

Definition 6. Let $\sigma_q = \xi(s_{i_0}, s_{i_1}, \ldots, s_{i_q})$ be an oriented q-simplex in $\widetilde{T}(X)$ and e_q be an oriented q-cell in X such that there exists a point $x \in De_q \cap \sigma_q$. Then there exists a simplex $\tau_{n-q} = \eta(b_n, b_{n-1}, \ldots, b_q)$ of De_q of dimension $n-q$ containing x. Let us consider the simplex $\rho_n = \zeta(b_n, b_{n-1}, \ldots, b_{q+1}, s_{i_0}, \ldots, s_{i_q})$ with the orientation induced by that of $|X|$. We shall define the *intersection index between* De_q *and* σ_q *at* x, denoted $(De_q \cdot \sigma_q)_x$ to be equal to $\xi\eta\zeta \in \{+1, -1\}$.

Finally, we define the *intersection index between* De_q *and* σ_q, denoted $(De_q \cdot \sigma_q)$, by setting $(De_q \cdot \sigma_q) = \sum_{x \in De_q \cap \sigma_q} (De_q \cdot \sigma_q)_x$ if any such x exists, $(De_q \cdot \sigma_q) = 0$ otherwise.

Remark 3. Because of the transversality between $\widetilde{T}$ and Δ, the simplex with vertices at $b_n, \ldots, b_{q+1}, s_{i_1}, \ldots, s_{i_q}$ is a non-degenerate n-dimensional simplex.

Moreover, this definition does not depend on the choice of the particular $(n-q)$-simplex of De_q containing x. Indeed, let $\tau_{n-q} = \eta(b_n, b_{n-1}, \ldots, b_q)$ be the chosen simplex and let us consider another simplex containing x, say τ'_{n-q}, which has all the vertices in common with τ_{n-q} except for one vertex, let us say b_n which is replaced by b'_n. Then $\tau'_{n-q} = -\eta(b'_n, b_{n-1}, \ldots, b_q)$ because τ_{n-q} and τ'_{n-q} must induce the same orientation in De_q. By choosing τ_{n-q} we obtain $\rho_n = \zeta(b_n, b_{n-1}, \ldots, b_{q+1}, s_{i_0}, \ldots, s_{i_q})$ while by choosing τ'_{n-q} we obtain $\rho'_n = -\zeta(b'_n, b_{n-1}, \ldots, b_{q+1}, s_{i_0}, \ldots, s_{i_q})$ whose sign is also given correctly because ρ_n and ρ'_n induce opposite orientations in the common $(n-1)$-face. As a consequence, the choice of τ'_{n-q} yields as intersection number at x the number $\xi(-\eta)(-\zeta) = \xi\eta\zeta$ which is the intersection number yielded by τ_{n-q}. By repeating this procedure of replacing a simplex by a neighbourhood one still containing x, we can get to any desired simplex containing x. In fact, given any two simplices τ'_{n-q} and τ''_{n-q} containing x, there exists a chain of neighbourhood simplices also containing x and joining τ'_{n-q} and τ''_{n-q}.

Finally, the definition does not depend on the ordering chosen for the vertices of σ_q. In fact, if we swap two vertices of σ_q, both ξ and ζ change sign so that the intersection index remains unchanged.

Let now F be a family of identifications on X and assume that $\widetilde{T}(X)$ and $\Delta(X)$ are F-equivariant. Moreover let us assume that $\widetilde{T}(X)$ is transverse to $\mathcal{D}(X)$ and that $\widetilde{T}$ is a chain map.

Lemma 1. *Let* e_q *be a* q*-cell of* X. *Let* σ_q *be a simplicial* q*-cell of* $\widetilde{T}(X)$. *Then, for every* $f \in F$ *whose domain contains* σ_q, *it holds that*

$$\left(De_q \cdot \sigma_q\right) = \left(Df(e_q) \cdot f(\sigma_q)\right).$$

Proof. We observe that if $De_q \cap \sigma_q \neq \emptyset$, then De_q intersects the domain of f. This implies that e_q is contained in the domain of f. Thus the claim follows easily from the F-equivariance of Δ and $\widetilde{T}$. $\square$

Lemma 2. *Let us assume that, for every* $q \geq 0$, *each* q*-simplex of* $\widetilde{T}(X)$ *intersects* $\cup_{e_q \in X} De_q$ *at most in one point. Let* $e_q, e_{q+1} \in X$, *with* $e_q \subseteq e_{q+1}$. *If there exist* $\sigma_q, \sigma_{q+1} \in \widetilde{T}(X)$, *with* $\sigma_q \subseteq \sigma_{q+1}$, *such that* $(De_{q+1} \cdot \sigma_{q+1}) \neq 0$ *and* $(De_q \cdot \sigma_q) \neq 0$, *then*

$$[\sigma_{q+1} : \sigma_q]\left(De_q \cdot \sigma_q\right) = [e_{q+1} : e_q]\left(De_{q+1} \cdot \sigma_{q+1}\right).$$

Proof. Let us denote by x the intersection point between De_{q+1} and σ_{q+1} and by x' the intersection point between De_q and σ_q. Let $\eta(b_n, \dots, b_{q+1})$ be the $(n-q-1)$-simplex of De_{q+1} containing x, where the orientation is the one induced from that of De_{q+1}. Let us first assume that the $(n-q)$-simplex of De_q containing x' is $\eta'(b_n, \dots, b_{q+1}, b_q)$, where the orientation is the one induced by that of De_q. It must hold $\eta' = [e_{q+1} : e_q]\eta$.

On the other hand, since $\sigma_q \subseteq \sigma_{q+1}$, if $\sigma_{q+1} = \xi(s_{i_0}, \dots, s_{i_{q+1}})$, then $\sigma_q = \xi'(s_{i_0}, \dots, \hat{s}_{i_j}, \dots, s_{i_{q+1}})$, where the hat as usual means the omission of the corresponding vertex, for some $j \in \{0, \dots, q+1\}$. It holds $\xi' = (-1)^j[\sigma_{q+1} : \sigma_q]\xi$.

Let us consider the n-simplices

$$\rho_n = \zeta(b_n, \dots, b_{q+2}, s_{i_0}, \dots, s_{i_j}, \dots, s_{i_{q+1}})$$

and

$$\rho'_n = \zeta'(b_n, \dots, b_{q+1}, s_{i_0}, \dots, \hat{s}_{i_j}, \dots, s_{i_{q+1}})$$

with the orientations induced by that of $|X|$. It is easy to see that ρ_n is oriented as $\zeta(b_n, \dots, b_{q+2}, b_{q+1}, s_{i_1}, \dots, s_{i_{q+1}})$. Hence,

$$\zeta(-1)^{n-(q+1)+1} = \zeta'(-1)^{n-(q+1)+j+1}$$

which yields $\zeta = (-1)^j\zeta'$.

Therefore, $\left(De_{q+1} \cdot \sigma_{q+1}\right) = \left(De_{q+1} \cdot \sigma_{q+1}\right)_x = \eta\xi\zeta$, while

$$\left(De_q \cdot \sigma_q\right) = \left(De_q \cdot \sigma_q\right)_{x'} = \eta'\xi'\zeta' = [e_{q+1} : e_q]\eta(-1)^j[\sigma_{q+1} : \sigma_q]\xi\zeta' = \\ = [e_{q+1} : e_q][\sigma_{q+1} : \sigma_q]\eta\xi\zeta.$$

In general, however, x' does not need to belong to the simplex $\eta'(b_n, \dots, b_{q+1}, b_q)$, but it can belong to another simplex of De_q. Nevertheless, we can always subdivide σ_{q+1} into two smaller simplices τ_{q+1} and τ'_{q+1} with $x \in \tau_{q+1}$ and so that a proper face of τ_{q+1}, say τ_q, intersects $\eta'(b_n, \dots, b_{q+1}, b_q) \subseteq De_q$ at just one point x'' (see fig. 1).

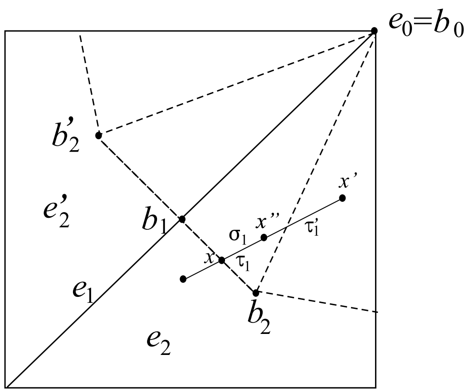

Fig. 1.

Let us give τ_{q+1} and τ'_{q+1} the orientation of σ_{q+1}. Since x' and x'' belong to different faces of τ'_{q+1}, its boundary intersects De_q at x' and at x'' with opposite signs. Hence $[\sigma_{q+1}:\sigma_q](De_q\cdot\sigma_q)_{x'}=[\tau_{q+1}:\tau_q](De_q\cdot\tau_q)_{x''}$. Now we can apply the above arguments to τ_{q+1} and obtain $[\tau_{q+1}:\tau_q](De_q\cdot\tau_q)_{x''}=[e_{q+1}:e_q](De_{q+1}\cdot\tau_{q+1})_x=$ $=[e_{q+1}:e_q](De_{q+1}\cdot\sigma_{q+1})$. □

Definition 7. Given a regular cell complex X and a family F of identifications on X, for every cell $\underline{e}$ in X/F let us fix an element $e^0\in\underline{e}$ and an orientation for e^0. For every other cell $e\in\underline{e}$ there exists a unique $f:e^0\to e\in\mathcal{F}$. Let us equip e with the orientation induced by that of e^0 by imposing that F preserves orientations. Then we shall say that the cells of X are *oriented coherently with* F.

We are now ready to define the homomorphism ϕ_T that will induce the desired isomorphism between the cohomology groups of X/F and those of $T(X/F)$. It will be sufficient to define ϕ_T on the set of homomorphisms $\underline{e}^*$ dual to cells $\underline{e}\in X/F$.

Theorem 2. *Let X be a finite regular cell complex realized by an orientable compact connected n-manifold (possibly with boundary). We assume that an orientation for $|X|$ has been chosen. Let F be a family of identifications defined on X and let X be oriented coherently with F. Moreover let $\widetilde{T}(X)$ be a simplicial subdivision of X transverse to the block decomposition dual to X. Finally, let us assume that both $\Delta(X)$ and $\widetilde{T}(X)$ are F-equivariant. For every $q=0,\ldots,n$ let*

$$\phi_T: C^q(X/F,\mathbb{Z})\longrightarrow C^q(T(X/F),\mathbb{Z})$$

be the homomorphism defined by linearly extending the equality

$$\langle\phi_T(\underline{e}_q^*),\underline{\sigma}_q\rangle\overset{\text{def}}{=}\sum_{e_q\in\underline{e}_q}(De_q\cdot\sigma_q),$$

where $\underline{e}_q$ is any q-cell in X/F, $\underline{\sigma}_q$ is any q-cell in $T(X/F)$ and σ_q is any element in $\underline{\sigma}_q$. Then ϕ_T is well defined and commutes with the coboundary operator.

Proof. That ϕ_T is independent of the choice of $\sigma_q\in\underline{e}_q$ follows from Lemma 1. So let us show that for every q-cell $\underline{e}_q$ in X/F and for every $(q+1)$-simplex $\underline{\sigma}_{q+1}$ in $T(X/F)$ it holds

$$\langle(\phi_T\circ\delta)(\underline{e}_q^*),\underline{\sigma}_{q+1}\rangle=\langle(\delta\circ\phi_T)(\underline{e}_q^*),\underline{\sigma}_{q+1}\rangle.$$

We have

$$\begin{aligned}\langle(\phi_T\circ\delta)(\underline{e}_q^*),\underline{\sigma}_{q+1}\rangle&=\left\langle\phi_T\left(\sum_{\underline{e}_{q+1}\in X/F}[\underline{e}_{q+1}:\underline{e}_q]\underline{e}_{q+1}^*\right),\underline{\sigma}_{q+1}\right\rangle=\\&=\sum_{\underline{e}_{q+1}\in X/F}[\underline{e}_{q+1}:\underline{e}_q]\langle\phi_T(\underline{e}_{q+1}^*),\underline{\sigma}_{q+1}\rangle=\\&=\sum_{\underline{e}_{q+1}\in X/F}[\underline{e}_{q+1}:\underline{e}_q]\sum_{e_{q+1}\in\underline{e}_{q+1}}(De_{q+1}\cdot\sigma_{q+1}),\end{aligned}$$

where σ_{q+1} is any element in $\underline{\sigma}_{q+1}$. On the other hand

$$\langle(\delta \circ \phi_T)(\underline{e}_q^*)\,,\underline{\sigma}_{q+1}\rangle = \langle(\phi_T(\underline{e}_q^*)\,,\partial\underline{\sigma}_{q+1}\rangle = \sum_{e_q \in \underline{e}_q} (De_q \cdot \partial\sigma_{q+1}).$$

Hence we must show that

$$\sum_{\underline{e}_{q+1} \in X/F} [\underline{e}_{q+1} : \underline{e}_q] \sum_{e_{q+1} \in \underline{e}_{q+1}} (De_{q+1} \cdot \sigma_{q+1}) = \sum_{e_q \in \underline{e}_q} (De_q \cdot \partial\sigma_{q+1}). \tag{1}$$

We can assume that $\widetilde{T}(X)$ satisfies the following properties:

(a) For every $q \geq 0$, each $\sigma_q \in \widetilde{T}(X)$ intersects $\cup_{e_q \in X} De_q$ at most in a point.

(b) For every $q \geq 0$ and for every $\sigma_q \in \widetilde{T}(X)$, if there exists $e_q \in X$ such that $De_q \cap \sigma_q \neq \emptyset$, then for every $e_{q-1} \in X$, with $e_{q-1} \subseteq e_q$, $De_{q-1} \cap \partial\sigma_q \neq \emptyset$.

In fact, if $\widetilde{T}(X)$ does not satisfy these conditions we can further subdivide it to obtain a new simplicial complex $\widetilde{T}(X)'$ on $|X|$ with the same properties as $\widetilde{T}(X)$ and these new ones. If equation (1) holds for $\widetilde{T}(X)'$, then it holds for $\widetilde{T}(X)$.

Let us now consider separately two cases that exhaust the list of possibilities.

Let us first assume that σ_{q+1} does not intersect $\cup_{e_{q+1} \in X} De_{q+1}$. Then the left-hand side of equation (1) vanishes. As for its right-hand side, if $(\cup_{e_q \in \underline{e}_q} De_q) \cap \partial\sigma_{q+1} = \emptyset$, then equation (1) is trivially verified. Otherwise, let us consider the case when $\partial\sigma_{q+1} \cap \cap \cup_{e_q \in \underline{e}_q} De_q \neq \emptyset$. It follows that there exists a cell $e_q \in \underline{e}_q$ such that $\partial\sigma_{q+1} \cap De_q \neq \emptyset$. Let us recall that we are assuming that σ_{q+1} does not intersect any De_{q+1} with $e_{q+1} \in X$. Hence $\sigma_{q+1} \cap De_q$ is a 1-cell and $\partial\sigma_{q+1} \cap De_q$ is its boundary. Therefore $(De_q \cdot \partial\sigma_{q+1}) = 0$.

Let us now assume that there exists a $(q+1)$-dimensional cell $\underline{e}_{q+1} \in X/F$ such that σ_{q+1} intersects $\cup_{e_{q+1} \in X} De_{q+1}$. Because of condition ($a$), there must exist only one cell $e_{q+1} \in X$ such that $De_{q+1} \cap \sigma_{q+1} \neq \emptyset$. Therefore equation (1) becomes

$$[\underline{e}_{q+1} : \underline{e}_q](De_{q+1} \cdot \sigma_{q+1}) = \sum_{e_q \in \underline{e}_q} (De_q \cdot \partial\sigma_{q+1}). \tag{2}$$

Since $De_{q+1} \cap \sigma_{q+1} \neq \emptyset$ then by condition (b) for every $e_q \subseteq e_{q+1}$ we have $De_q \cap \partial_j \sigma_{q+1} \neq \emptyset$, for some $j \in \{0\,,1\,,\ldots\,,q+1\}$. By transversality, such a j must be unique. Hence, by applying Lemma 2, for every $e_q \subseteq e_{q+1}$ it holds

$$(De_q \cdot \partial\sigma_{q+1}) = [e_{q+1} : e_q](De_{q+1} \cdot \sigma_{q+1}).$$

On the other hand, if $e_q \not\subseteq e_{q+1}$, then $[e_{q+1} : e_q] = 0$. Hence,

$$\sum_{e_q \in \underline{e}_q} (De_q \cdot \partial\sigma_{q+1}) = \sum_{e_q \in \underline{e}_q} [e_{q+1} : e_q](De_{q+1} \cdot \sigma_{q+1}) = [\underline{e}_{q+1} : \underline{e}_q](De_{q+1} \cdot \sigma_{q+1})\,,$$

thus proving equality (2). $\square$

As a consequence of the above proposition we get that ϕ_T induces a homomorphism $H^*(\phi_T)$ from $H^*(X/F, \mathbb{Z})$ to $H^*(T(X/F)\,,\mathbb{Z})$.

Lemma 3. *For every* $\underline{e}$, $\underline{e}' \in X/F$ *it holds that*

$$\langle \phi_T(\underline{e}^*)\,,\, T\underline{e}'\rangle = \langle \underline{e}^*\,,\, \underline{e}'\rangle.$$

Proof. By definition of ϕ_T it holds that $\langle \phi_T(\underline{e}^*)\,,\, T\underline{e}'\rangle = \sum_{e\in\underline{e}}(De\cdot \widetilde{T}e')$ for any $e'\in\underline{e}'$. But since

$$(De\cdot \widetilde{T}e') = \begin{cases} 1 & \text{if } e=e' \\ 0 & \text{otherwise} \end{cases}$$

we obtain that $\langle \phi_T(\underline{e}^*)\,,\, T\underline{e}'\rangle$ is equal to 1 if $\underline{e}=\underline{e}'$ and 0 otherwise. $\square$

Proposition 1. *$H^*(\phi_T)$ is the inverse of $H^*(T)$ and in particular it is a ring isomorphism.*

Proof. It is well known that $T: C_*(X/F,\mathbb{Z}) \to C_*(T(X/F),\mathbb{Z})$ induces a ring isomorphism $H^*(T)$ which takes $[\alpha]\in H^*(T(X/F),\mathbb{Z})$ to $[\alpha\circ T]\in H^*(X/F,\mathbb{Z})$ (see, *e.g.*, [21]). It is then sufficient to prove that $H^*(T)\circ H^*(\phi_T) = Id_{H^*(X/F,\mathbb{Z})}$. To this aim, let us consider any element $[\sum_k \alpha_k \underline{e}_q^{k*}] \in H^q(X/F,\mathbb{Z})$. Then

$$H^q(T)\circ H^q(\phi_T)\left(\left[\sum_k \alpha_k \underline{e}_q^{k*}\right]\right) = \left[\phi_T\left(\sum_k \alpha_k \underline{e}_q^{k*}\right)\circ T\right].$$

We must show that

$$\left[\phi_T\left(\sum_k \alpha_k \underline{e}_q^{k*}\right)\circ T\right] = \left[\sum_k \alpha_k \underline{e}_q^{k*}\right].$$

This turns out to be true because by Lemma 3 for any q-cell $\underline{e}_q \in X/F$ we have

$$\left\langle \phi_T\left(\sum_k \alpha_k \underline{e}_q^{k*}\right)\circ T,\, \underline{e}_q\right\rangle = \sum_k \alpha_k \langle \phi_T(\underline{e}_q^{k*})\,,\, T(\underline{e}_q)\rangle = \sum_k \alpha_k \langle \underline{e}_q^{k*}\,,\, \underline{e}_q\rangle. \quad \square$$

4.2. *Calculation of cup products.*

In this section we address the problem of computing a representative for $[a]\cup$ $\cup[b]\in H^{p+q}(X/F,\mathbb{Z})$, with $[a]\in H^p(X/F,\mathbb{Z})$ and $[b]\in H^q(X/F,\mathbb{Z})$, given a set of representatives for the generators of $H^*(X/F,\mathbb{Z})$.

Since the definition of cup product in cellular cohomology is purely abstract, we shall use the results of Section 4 to compute cup products in practice.

Let $\widetilde{T}$ a subdivision of the cells of X into simplices which is F-equivariant and transverse to the dual block decomposition of X. Since by Proposition 1 the map $H^*(\phi_T)$ is the ring isomorphism inverse to $H^*(T)$, it holds that

$$\begin{aligned}
[(\phi_T(a)\cup\phi_T(b))\circ T] &= H^{p+q}(T)([\phi_T(a)\cup\phi_T(b)]) = \\
&= H^{p+q}(T)\left(H^p(\phi_T)([a])\cup H^q(\phi_T)([b])\right) = \\
&= H^{p+q}(T)\circ H^{p+q}(\phi_T)([a]\cup[b]) = [a]\cup[b].
\end{aligned}$$

Hence $(\phi_T(a) \cup \phi_T(b)) \circ T$ is a representative for $[a] \cup [b]$. Therefore, if we are able to compute the value of $\phi_T(a) \cup \phi_T(b)$ at $T\underline{e}$ for any $(p+q)$-cell $\underline{e} \in X/F$, then we actually know a representative of $[a] \cup [b]$.

The computation of the value of $\phi_T(a) \cup \phi_T(b)$ at $T\underline{e}$ will depend on the particular subdivision $\widetilde{T}$ chosen, and of course the choice of the most suitable $\widetilde{T}$ will depend on the complex X/F under study. However, in general it may be useful to take the subdivision $\widetilde{T}$ to be a perturbation of the barycentric subdivision of X.

When we consider a perturbed barycentric subdivision we obtain correspondingly a perturbed dual block $\widetilde{D}e$ for every cell e of X.

Definition 8. For every pair of cells $e_q, e_r \in X$ with $e_q \subseteq e_r$ we shall define $D_{|e_r} e_q$ (resp. $\widetilde{D}_{|e_r} e_q$) to be the $(r-q)$-chain

$$\sum [e_r : e_{r-1}] \cdots [e_{q+1} : e_q](b_r, \ldots, b_q)$$

where the sum is running over all the chains $e_r \supseteq e_{r-1} \supseteq \cdots \supseteq e_q$ and b_j denotes the vertex of $\Delta(X)$ (resp. $\widetilde{\Delta}(X)$) associated with the cell e_j.

We point out that $D_{|e_r}(e_q)$ is the classical intersection chain between $\Delta(e_r)$ and $\Delta(De_q)$ (see, *e.g.*, [4]). In other words it is the dual block De_q relative to the manifold $|\bar{e}_r|$.

Proposition 2. *Let $\widetilde{T}(X)$ be a perturbation of the barycentric subdivision $\Delta(X)$. Let us assume that $\widetilde{T}(X)$ is transverse to the dual block decomposition induced by $\Delta(X)$ and that both $\widetilde{T}(X)$ and $\Delta(X)$ are F-equivariant. Then for every $\underline{e}_p, \underline{e}_q, \underline{e}_r \in X/F$ with $p + q = r$, it holds that*

$$\langle \phi_T(\underline{e}_p^*) \cup \phi_T(\underline{e}_q^*) \,,\, T\underline{e}_r \rangle = \sum_{e_p \in \underline{e}_p} \sum_{e_q \in \underline{e}_q} \left(D_{|e_r} e_p \cdot \widetilde{D}_{|e_r} e_q \right)$$

for any $e_r \in \underline{e}_r$.

Proof. Let us denote the vertex of $\widetilde{T}(X)$ associated with a cell $e_q \in X$ by $\widetilde{b}_q$. Then $\widetilde{T}(e_r) = \sum [e_r : e_{r-1}] \cdots [e_1 : e_0](\widetilde{b}_r, \ldots, \widetilde{b}_0)$ where the sum is running over all the chains of the kind $e_r \supseteq \ldots \supseteq e_0$. Hence if $e_r \in \underline{e}_r$,

$$\begin{aligned}
\langle \phi_T(\underline{e}_p^*) \cup \phi_T(\underline{e}_q^*) \,,\, T\underline{e}_r \rangle &= \langle \phi_T(\underline{e}_p^*) \cup \phi_T(\underline{e}_q^*) \,,\, p_\# \circ \widetilde{T}(e_r) \rangle = \\
&= \sum [e_r : e_{r-1}] \cdots [e_1 : e_0] \langle \phi_T(\underline{e}_p^*) \cup \phi_T(\underline{e}_q^*) \,,\, p_\#(\widetilde{b}_r, \ldots, \widetilde{b}_0) \rangle = \\
&= \sum [e_r : e_{r-1}] \cdots [e_1 : e_0] \langle \phi_T(\underline{e}_p^*) \,,\, p_\#(\widetilde{b}_r, \ldots, \widetilde{b}_q) \rangle \cdot \\
&\qquad \cdot \langle \phi_T(\underline{e}_q^*) \,,\, p_\#(\widetilde{b}_q, \ldots, \widetilde{b}_0) \rangle = \\
&= \left\langle \phi_T(\underline{e}_p^*) \,,\, \sum [e_r : e_{r-1}] \cdots [e_{q+1} : e_q] p_\#(\widetilde{b}_r, \ldots, \widetilde{b}_q) \right\rangle \cdot \\
&\qquad \cdot \langle \phi_T(\underline{e}_q^*) \,,\, \sum [e_q : e_{q-1}] \cdots [e_1 : e_0] p_\#(\widetilde{b}_q, \ldots, \widetilde{b}_0) \rangle = \\
&= \left\langle \phi_T(\underline{e}_p^*) \,,\, \sum [e_r : e_{r-1}] \cdots [e_{q+1} : e_q] p_\#(\widetilde{b}_r, \ldots, \widetilde{b}_q) \right\rangle \cdot \\
&\qquad \cdot \langle \phi_T(\underline{e}_q^*) \,,\, p_\# \circ \widetilde{T}(e_q) \rangle .
\end{aligned}$$

But by Lemma 3 we have that $\langle \phi_T(\underline{e}_q^*), p_\# \circ \widetilde{T}(e_q)\rangle = 0$ when $e_q \notin \underline{e}_q$ and it is equal to 1 otherwise. Therefore

$$\begin{aligned}\langle \phi_T(\underline{e}_p^*) \cup \phi_T(\underline{e}_q^*), \underline{e}_r\rangle &= \Big\langle \phi_T(\underline{e}_p^*), \sum_{e_q \in \underline{e}_q} [e_r : e_{r-1}] \cdots [e_{q+1} : e_q] p_\#(\widetilde{b}_r, \dots, \widetilde{b}_q)\Big\rangle = \\ &= \Big\langle \phi_T(\underline{e}_p^*), p_\#\Big(\sum_{e_q \in \underline{e}_q} \widetilde{D}_{|e_r} e_q\Big)\Big\rangle = \sum_{e_p \in \underline{e}_p} \sum_{e_q \in \underline{e}_q} (De_p \cdot \widetilde{D}_{|e_r} e_q) = \\ &= \sum_{e_p \in \underline{e}_p} \sum_{e_q \in \underline{e}_q} (D_{|e_r} e_p \cdot \widetilde{D}_{|e_r} e_q). \quad \square\end{aligned}$$

5. Ring structure in $H^*(X_W, \mathbb{Z})$

Our aim is now that of calculating the cohomology ring of Artin groups for all the exceptional groups by applying the results of Section 4.

We shall assume that a finite group W generated by a set of reflections S has been given. We shall take the regular complex X of Section 4 to be equal to the polyhedron Q described in Section 2. Here it is useful to construct Q as the convex hull of the orbit of $v_0 \in C_0$ and to choose the point $v(F)$, $F \in \mathcal{F}_0$, as the orthogonal projection of v_0 onto F. In this way all cells of Q become convex polyhedra in $\mathbb{R}^n$. We remark that $\Delta(Q)$ is exactly the complex obtained by barycentric subdivision of the facets.

The family of identifications F on X will be taken to be equal to the family of homeomorphisms $\{\gamma_{(e)}\}_{e \in Q}$ described in Section 2. Therefore $X/F = X_W$ and $p = \pi_W$. It is easy to see that $\Delta(Q)$ is F-equivariant. Moreover, for every cell $\underline{e}(\Gamma)$ in X_W, with $\Gamma \subseteq I_n$, the set $\cup_{e \in \underline{e}(\Gamma)} De$ is the union of all the facets of type Γ.

In Section 5.1 we shall deal with the problem of computing cup products of the kind $\cup : H^1(X_W, \mathbb{Z}) \otimes H^k(X_W, \mathbb{Z}) \to H^{k+1}(X_W, \mathbb{Z})$, for any $k \geq 0$. The general case will be treated later in Section 5.2. The reason for this dichotomy is that when the first cohomology group is involved it is not necessary to construct the homomorphism ϕ_T but it is sufficient to use Lemma 3 and cup products can be computed by hand. In the general case we shall need to explicitly construct ϕ_T and therefore it will be a more labourious task. Nevertheless, in both the cases, if $[a] \in H^p(X_W, \mathbb{Z})$ and $[b] \in H^q(X_W, \mathbb{Z})$ we shall compute the value of $(\phi_T(a) \cup \phi_T(b)) \circ T$, where T is induced by a suitable subdivision $\widetilde{T}(Q)$, at every $(p+q)$-cell of X_W in order to obtain a representative for $[a] \cup [b]$ as explained in Section 4.2. Let us recall that by using the results in [16] we know a list of representatives of the generators of $H^*(X_W, \mathbb{Z})$ and the coboundary matrices δ^* (see Section 2). Thus it is sufficient to compare (modulo coboundaries) the representative of $[a] \cup [b]$ we have found with those contained in such a list to obtain the expression of $[a] \cup [b]$ in terms of the generators of $H^{p+q}(X_W, \mathbb{Z})$.

5.1. $\cup : H^1(X_W, \mathbb{Z}) \otimes H^k(X_W, \mathbb{Z}) \to H^{k+1}(X_W, \mathbb{Z})$.

In order to compute cup products of the kind $\cup : H^1(X_W, \mathbb{Z}) \otimes H^k(X_W, \mathbb{Z}) \to$ $\to H^{k+1}(X_W, \mathbb{Z})$, $k \geq 0$, we shall consider the following subdivision of Q, $\widetilde{T}(Q)$.

We proceed by induction on the dimension of the cells of Q. The 0-cells of Q are left unchanged. Let us assume we have already subdivided every $(k-1)$-cell $e(F^{k-1})$, with $F^{k-1} \in \mathcal{F}$, into simplices to form a chain $\widetilde{T}e(F^{k-1})$. Then we subdivide every k-cell of $\mathcal{Q}_0$, say $e(\Gamma)$ with $\Gamma = \Gamma(F^k)$, $F^k \in \mathcal{F}_0$, into the following sum of simplices:

$$\widetilde{T}e(\Gamma) = \sum_{j \in \Gamma} (-1)^{\sigma(j,\Gamma)+1} \sum_{h \in W^{\Gamma}_{\Gamma \setminus \{j\}} \setminus \{Id\}} (-1)^{l(h)} v_0 h \widetilde{T}e(\Gamma \setminus \{j\}),$$

where $v_0 h\widetilde{T}e(\Gamma \setminus \{j\})$ denotes the cone from v_0 onto $h\widetilde{T}e(\Gamma \setminus \{j\})$. Moreover, we endow every other k-cell $e \in Q$ in the same W-orbit as $e(\Gamma)$, with the subdivision induced by $\gamma_{(e)}$. Of course $\widetilde{T}$ can be extended by linearity on $C_*(Q, \mathbb{Z})$.

$\widetilde{T}(Q)$ is F-equivariant by construction. Hence it induces a pseudo-triangulation $T(X_W)$ of X_W. As an example, in fig. 2 we show $T(X_W)$ in the case $W = I_2(4)$.

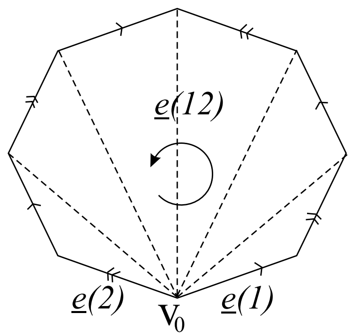

Fig. 2.

Moreover, it is always possible to slightly perturb $\widetilde{T}$ in order to make it transverse to $\mathcal{D}(Q)$.

Let us now see that it is a chain map.

LEMMA 4. *For every* $e(\Gamma) \in \mathcal{Q}_0$,

$$\partial e(\Gamma) = \sum_{j \in \Gamma} (-1)^{\sigma(j,\Gamma)+1} \sum_{h \in W^{\Gamma}_{\Gamma \setminus \{j\}}} (-1)^{l(h)} he(\Gamma \setminus \{j\}).$$

PROOF. Analogous to the proof of Theorem 1 in [16]. □

PROPOSITION 3. *For every* $F^k \in \mathcal{F}$,

$$\partial \widetilde{T}e(F^k) = \widetilde{T}\partial e(F^k).$$

PROOF. Because of the definition of $\widetilde{T}e(F^k)$, it will be sufficient to demonstrate the claim for the cells in $\mathcal{Q}_0$, i.e. for the cells dual to facets in $\mathcal{F}_0$.

We shall proceed by induction on the dimension of the cells $e(\Gamma)$ of $\mathcal{Q}_0$. The claim is certainly true when $\mathrm{card}(\Gamma) = 0$. Let us assume it to be true for $\mathrm{card}(\Gamma) = k-1$, with $k > 0$. It holds that

$$\partial \widetilde{T}e(\Gamma) = \sum_{j\in\Gamma}(-1)^{\sigma(j,\Gamma)+1} \sum_{h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}} (-1)^{l(h)}\partial v_0 h \widetilde{T}e(\Gamma\setminus\{j\}) =$$

$$= \sum_{j\in\Gamma}(-1)^{\sigma(j,\Gamma)+1} \sum_{h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}} (-1)^{l(h)} \left(h\widetilde{T}e(\Gamma\setminus\{j\}) - v_0\partial h\widetilde{T}e(\Gamma\setminus\{j\})\right).$$

Since the inductive hypothesis applies and by recalling the formula for the boundary of a cell of $\mathcal{Q}_0$ given in Lemma 4, this is equal to

$$\sum_{j\in\Gamma}(-1)^{\sigma(j,\Gamma)+1} \sum_{h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}} (-1)^{l(h)} \Bigg(h\widetilde{T}e(\Gamma\setminus\{j\}) -$$

$$- v_0 h \sum_{i\in\Gamma\setminus\{j\}} (-1)^{\sigma(i,\Gamma\setminus\{j\})+1} \sum_{g\in W^{\Gamma\setminus\{j\}}_{\Gamma\setminus\{j,i\}}} (-1)^{l(g)} g\widetilde{T}e(\Gamma\setminus\{j,i\})\Bigg) =$$

$$= \sum_{j\in\Gamma}(-1)^{\sigma(j,\Gamma)+1} \sum_{h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}} (-1)^{l(h)} h\widetilde{T}e(\Gamma\setminus\{j\}) -$$

$$- \sum_{\substack{j\in\Gamma\\ i\in\Gamma\setminus\{j\}}} \sum_{\substack{h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}\\ g\in W^{\Gamma\setminus\{j\}}_{\Gamma\setminus\{j,i\}}}} (-1)^{\sigma(j,\Gamma)+\sigma(i,\Gamma\setminus\{j\})+l(h)+l(g)} v_0 hg\widetilde{T}e(\Gamma\setminus\{j,i\}).$$

It is easy to see that the last term is equal to

$$\begin{aligned}
&\sum_{\substack{j\in\Gamma\\ i\in\Gamma\setminus\{j\}}} (-1)^{\sigma(j,\Gamma)+\sigma(i,\Gamma\setminus\{j\})} \sum_{\substack{h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\\ g\in W^{\Gamma\setminus\{j\}}_{\Gamma\setminus\{j,i\}}}} (-1)^{l(h)+l(g)} v_0 hg\widetilde{T}e(\Gamma\setminus\{j,i\}) - \\
&- \sum_{j\in\Gamma}\sum_{i\in\Gamma\setminus\{j\}} (-1)^{\sigma(j,\Gamma)+\sigma(i,\Gamma\setminus\{j\})} \sum_{g\in W^{\Gamma\setminus\{j\}}_{\Gamma\setminus\{j,i\}}} (-1)^{l(g)} v_0 g\widetilde{T}e(\Gamma\setminus\{j,i\}).
\end{aligned} \tag{3}$$

Now, the first term of (3) is equal to 0. In fact, we observe that for every $i, j \in \Gamma$ with $i \neq j$,

$$(-1)^{\sigma(j,\Gamma)+\sigma(i,\Gamma\setminus\{j\})} = -(-1)^{\sigma(i,\Gamma)+\sigma(j,\Gamma\setminus\{i\})}. \tag{4}$$

Moreover

$$W^{\Gamma}_{\Gamma\setminus\{j\}} \cdot W^{\Gamma\setminus\{j\}}_{\Gamma\setminus\{j,i\}} = W^{\Gamma}_{\Gamma\setminus\{i\}} \cdot W^{\Gamma\setminus\{i\}}_{\Gamma\setminus\{j,i\}},$$

where $W^{\Gamma}_{\Gamma\setminus\{j\}} \cdot W^{\Gamma\setminus\{j\}}_{\Gamma\setminus\{j,i\}}$ denotes the set whose elements are all the possible products hg with $h \in W^{\Gamma}_{\Gamma\setminus\{j\}}$ and $g \in W^{\Gamma\setminus\{j\}}_{\Gamma\setminus\{j,i\}}$. On the other hand, the second term of (3) is equal to

$$\sum_{\substack{j\in\Gamma\\ i\in\Gamma\setminus\{j\}}} (-1)^{\sigma(j,\Gamma)+\sigma(i,\Gamma\setminus\{j\})} \sum_{g\in W^{\Gamma\setminus\{j\}}_{\Gamma\setminus\{j,i\}}} (-1)^{l(g)} v_0 g \widetilde{T} e(\Gamma\setminus\{j,i\}) =$$

$$= \sum_{j\in\Gamma}(-1)^{\sigma(j,\Gamma)+1}\widetilde{T}e(\Gamma\setminus\{j\}) + \sum_{\substack{j\in\Gamma\\ i\in\Gamma\setminus\{j\}}} (-1)^{\sigma(j,\Gamma)+\sigma(i,\Gamma\setminus\{j\})} v_0 \widetilde{T} e(\Gamma\setminus\{i,j\}) =$$

$$= \sum_{j\in\Gamma}(-1)^{\sigma(j,\Gamma)+1}\widetilde{T}e(\Gamma\setminus\{j\}).$$

The last equality holds because of equation (4). Hence,

$$\partial\widetilde{T}e(\Gamma) = \sum_{j\in\Gamma}(-1)^{\sigma(j,\Gamma)+1} \sum_{h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}} (-1)^{l(h)} h\widetilde{T}e(\Gamma\setminus\{j\}) + \sum_{j\in\Gamma}(-1)^{\sigma(j,\Gamma)+1}\widetilde{T}e(\Gamma\setminus\{j\}) =$$

$$= \sum_{\substack{j\in\Gamma\\ h\in W^{\Gamma}_{\Gamma\setminus\{j\}}}} (-1)^{\sigma(j,\Gamma)+1+l(h)} h\widetilde{T}(e(\Gamma\setminus\{j\})) = \widetilde{T}(\partial e(\Gamma)). \quad \square$$

To sum up, all the conditions to apply the results of Section 4 are satisfied.

Lemma 5. *Let $h = s_{i_1}\cdots s_{i_r} \in W$, where s_{i_j} is the element of S corresponding to the reflection hyperplane H_{i_j} of C_0. Then the homology class of $\pi_W(v_0, hv_0)$ coincides with the homology class of $T\underline{e}(i_1)+\cdots+T\underline{e}(i_r)$.*

Proof. Let $\sigma = \sum_{j=1}^{r-1}\pi_W(v_0, s_{i_1}\cdots s_{i_j}v_0, s_{i_1}\cdots s_{i_{j+1}}v_0)$. We obtain

$$\partial\sigma = \sum_{j=1}^{r-1}\pi_W\big((s_{i_1}\cdots s_{i_j}v_0, s_{i_1}\cdots s_{i_{j+1}}v_0) - (v_0, s_{i_1}\cdots s_{i_{j+1}}v_0) + (v_0, s_{i_1}\cdots s_{i_j}v_0)\big) =$$

$$= -\pi_W(v_0, s_{i_1}\cdots s_{i_r}v_0) + \pi_W(v_0, s_{i_1}v_0) + \sum_{j=1}^{r-1}\pi_W(s_{i_1}\cdots s_{i_j}v_0, s_{i_1}\cdots s_{i_{j+1}}v_0) =$$

$$= -\pi_W(v_0, hv_0) + \pi_W\widetilde{T}e(i_1) + \sum_{j=1}^{r-1}\pi_W s_{i_1}\cdots s_{i_j}\widetilde{T}e(i_{j+1}) =$$

$$= -\pi_W(v_0, hv_0) + T\underline{e}(i_1) + \sum_{j=1}^{r-1} T\underline{e}(i_{j+1}).$$

Therefore, $\pi_W(v_0, hv_0)$ and $T\underline{e}(i_1)+\cdots+T\underline{e}(i_r)$ differ for a boundary, which yields the claim. $\square$

Proposition 4. *Let $[a]\in H^1(X_W, \mathbb{Z})$ and $[b]\in H^k(X_W, \mathbb{Z})$, $k\geq 0$. Let $\Gamma\subseteq I_n$ with*

$\mathrm{card}(\Gamma) = k+1$. *Then* $\langle \phi_T(a) \cup \phi_T(b)\,,\, T\underline{e}(\Gamma)\rangle$ *is equal to*

$$\sum_{j\in\Gamma}(-1)^{\sigma(j,\Gamma)+1} \sum_{h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}} (-1)^{l(h)} \sum_{\substack{i_1,\dots,i_{l(h)}:\\ h=s_{i_1}\cdots s_{i_{l(h)}}}} \langle a\,,\,\underline{e}(i_r)\rangle\langle b\,,\,\underline{e}(\Gamma\setminus\{j\})\rangle.$$

Proof. Let $\widetilde{a} = \phi_T(a)$ and $\widetilde{b} = \phi_T(b)$. If we denote $(-1)^{\sigma(j,\Gamma)+l(h)+1}$ by $\varepsilon_{j,\Gamma,h}$, we have that

$$\langle \widetilde{a}\cup\widetilde{b}\,,\,T\underline{e}(\Gamma)\rangle = \langle \widetilde{a}\cup\widetilde{b}\,,\,\pi_W \widetilde{T}e(\Gamma)\rangle =$$

$$= \langle \widetilde{a}\cup\widetilde{b}\,,\sum_{j\in\Gamma}\sum_{h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}} \varepsilon_{j,\Gamma,h}\pi_W v_0 h\widetilde{T}e(\Gamma\setminus\{j\})\rangle =$$

$$= \sum_{\substack{j\in\Gamma\\ h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}}} \varepsilon_{j,\Gamma,h}\langle \widetilde{a}\cup\widetilde{b}\,,\,\pi_W v_0 h \sum_{\substack{i\in\Gamma\setminus\{j\}\\ g\in W^{\Gamma\setminus\{j\}}_{\Gamma\setminus\{j,i\}}\setminus\{Id\}}} \varepsilon_{i,\Gamma\setminus\{j\},g} v_0 g\widetilde{T}e(\Gamma\setminus\{j,i\})\rangle =$$

$$= \sum_{\substack{j\in\Gamma\\ h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}}} \sum_{\substack{i\in\Gamma\setminus\{j\}\\ g\in W^{\Gamma\setminus\{j\}}_{\Gamma\setminus\{j,i\}}\setminus\{Id\}}} \varepsilon_{j,\Gamma,h}\varepsilon_{i,\Gamma\setminus\{j\},g}\langle \widetilde{a}\cup\widetilde{b}\,,\,\pi_W\left(v_0 h v_0 g\widetilde{T}e(\Gamma\setminus\{j,i\})\right)\rangle =$$

$$= \sum_{\substack{j\in\Gamma,\, i\in\Gamma\setminus\{j\}\\ h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}\\ g\in W^{\Gamma\setminus\{j\}}_{\Gamma\setminus\{j,i\}}\setminus\{Id\}}} \varepsilon_{j,\Gamma,h}\varepsilon_{i,\Gamma\setminus\{j\},g}\langle \widetilde{a}\,,\,\pi_W(v_0\,,\,hv_0)\rangle\langle \widetilde{b}\,,\,\pi_W h v_0 g\widetilde{T}e(\Gamma\setminus\{j,i\})\rangle =$$

$$= \sum_{\substack{j\in\Gamma\\ h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}}} \varepsilon_{j,\Gamma,h}\langle \widetilde{a}\,,\,\pi_W(v_0\,,\,hv_0)\rangle\langle \widetilde{b}\,,\,\pi_W h\widetilde{T}e(\Gamma\setminus\{j\})\rangle =$$

$$= \sum_{\substack{j\in\Gamma\\ h\in W^{\Gamma}_{\Gamma\setminus\{j\}}\setminus\{Id\}}} \varepsilon_{j,\Gamma,h}\langle \widetilde{a}\,,\,\pi_W(v_0\,,\,hv_0)\rangle\langle \widetilde{b}\,,\,T\underline{e}(\Gamma\setminus\{j\})\rangle.$$

Let us now observe that by Lemma 3,

$$\langle \widetilde{b}\,,\,T\underline{e}(\Gamma\setminus\{j\}\rangle = \langle b\,,\,\underline{e}(\Gamma\setminus\{j\})\rangle.$$

Moreover, by Lemma 5, if $h = s_{i_1}\cdots s_{i_{l(h)}}$ then $\pi_W(v_0\,,\,hv_0)$ is homologous to $T\underline{e}(i_1)+\cdots+T\underline{e}(i_{l(h)})$. Now, since $\widetilde{a}$ is a cocycle, it takes the same value at homologous chains. Therefore,

$$\langle \widetilde{a}\,,\,\pi_W(v_0\,,\,hv_0)\rangle = \langle \widetilde{a}\,,\,T\underline{e}(i_1)+\cdots+T\underline{e}(i_{l(h)})\rangle.$$

Again, since $\widetilde{a}=\phi_T(a)$ and by Lemma 3, $\langle \widetilde{a}\,,\,\pi_W(v_0\,,\,hv_0)\rangle = \langle a\,,\,\underline{e}(i_1)+\cdots+\underline{e}(i_{l(h)})\rangle$. □

By means of this proposition we can compute all cup products between H^1 and H^k for any $k\geq 0$ and for any reflection group.

EXAMPLE 1. We shall present such computation for the case $\cup : H^1(X_{H_3}, \mathbb{Z}) \otimes \otimes H^2(X_{H_3}, \mathbb{Z}) \to H^3(X_{H_3}, \mathbb{Z})$.

As it can be seen in table I,

$$H^1(X_{H_3}, \mathbb{Z}) \cong \mathbb{Z} = \langle [1+2+3] \rangle$$

and

$$H^2(X_{H_3}, \mathbb{Z}) \cong \mathbb{Z} = \langle [13] \rangle.$$

Since the only 3-cell is $\underline{e}(123)$ it will be sufficient to compute $\langle \phi_T(1+2+3) \cup \cup \phi_T(13), T\underline{e}(123) \rangle$ in order to obtain a representative of $[1+2+3] \cup [13]$. We have that

$$\langle \phi_T(1+2+3) \cup \phi_T(13), T\underline{e}(123) \rangle = (-1)^{\sigma(2,\{1,2,3\})+1} \sum_{h \in W^{\{1,2,3\}}_{\{1,3\}} \setminus \{Id\}} (-1)^{l(h)} \cdot$$

$$\cdot \langle \phi_T(1+2+3), \pi_W(v_0, hv_0) \rangle \cdot \langle \phi_T(13), \pi_W hTe(1,3) \rangle =$$

$$= \sum_{h \in W^{\{1,2,3\}}_{\{1,3\}} \setminus \{Id\}} (-1)^{l(h)} \langle \phi_T(1+2+3), \pi_W((v_0, hv_0)) \rangle.$$

Now we observe that the cocycle $1+2+3$ takes value 1 at any properly oriented 1-cell of X_W. Therefore,

$$\sum_{h \in W^{\{1,2,3\}}_{\{1,3\}} \setminus \{Id\}} (-1)^{l(h)} \langle \phi_T(1+2+3), \pi_W((v_0, hv_0)) \rangle = \sum_{h \in W^{\{1,2,3\}}_{\{1,3\}} \setminus \{Id\}} (-1)^{l(h)} l(h).$$

In general, if for any part H of W we denote as usual $H(q) = \sum_{h \in H} q^{l(h)}$, it holds that

$$\frac{d}{dq} H(q) = \sum_{h \in H} l(h) q^{l(h)-1} = \sum_{h \in H \setminus \{Id\}} l(h) q^{l(h)-1}.$$

In particular, by setting $q = -1$, we have

$$\sum_{h \in H \setminus \{Id\}} l(h)(-1)^{l(h)} = -\frac{d}{dq} H(q)_{|q=-1}.$$

Let us introduce the symbol $[n]$ to denote the *q-analog* $\frac{q^n - 1}{q-1}$. Since $H_3(q) = = [2][6][10]$ and $A_1(q) = [2]$ (see, *e.g.*, [13]), we obtain

$$\langle \phi_T(1+2+3) \cup \phi_T(13), T\underline{e}(123) \rangle = -\frac{d}{dq} \frac{W_{\{1,2,3\}}(q)}{W_{\{1,3\}}(q)}_{|q=-1} = -\frac{d}{dq} \frac{[2][6][10]}{[2][2]}_{|q=-1} = -15.$$

We have carried out similar computations for all the exceptional groups and the results are shown in table II together with all the products that can be easily deduced because of bilinearity.

Unfortunately, the method we have used up to now does not allow us to compute cup products between H^p and H^q with $p, q > 1$, for in general we are not able to find a suitable representative for the first p-face of $T\underline{e}(\Gamma)$ in terms of cells of X_W. Thus $\cup : H^2(X_{F_4}, \mathbb{Z}) \otimes H^2(X_{F_4}, \mathbb{Z}) \to H^4(X_{F_4}, \mathbb{Z})$, $\cup : H^3(X_{E_7}, \mathbb{Z}) \otimes H^3(X_{E_7}, \mathbb{Z}) \to$ $\to H^6(X_{E_7}\ \mathbb{Z})$ and $\cup : H^3(X_{E_8}, \mathbb{Z}) \otimes H^3(X_{E_8}, \mathbb{Z}) \to H^6(X_{E_8}, \mathbb{Z})$ still remain to be computed. In these cases we must explicitly compute $\phi_T(a)$ for $a \in C^*(X_W, \mathbb{Z})$.

5.2. $\cup : H^p(X_W, \mathbb{Z}) \otimes H^q(X_W, \mathbb{Z}) \to H^{p+q}(X_W, \mathbb{Z})$.

In order to deal with the general case, let us effectively construct ϕ_T by taking as $\widetilde{T}(Q)$ a perturbation of the barycentric subdivision of Q. More precisely, let us assume that all the reflection hyperplanes have been translated by a vector $v \in C_0$. This new system of hyperplanes still realizes W as a reflection group. Let us denote it by $\widetilde{W}$. Let $\widetilde{\mathcal{S}}$ be its stratification into facets. Let us observe that, by choosing v sufficiently small, the intersection between $\widetilde{F}^j \in \widetilde{\mathcal{S}}$ and $e(F^j)$ consists of one and only one point so that Q is a realization of the cell complex dual to $\widetilde{\mathcal{S}}$ as well.

In general, not every choice of v yields transversality between $\widetilde{T}(Q)$ and $\mathcal{D}(Q)$. However, the following proposition is useful to locate wrong choices for v.

Proposition 5. *Let F^p and F^q be facets in $\mathcal{S}$ of dimension p and q, respectively, with $p, q < n$. Let $\widetilde{F}^q$ be the facet of $\widetilde{\mathcal{S}}$ obtained by translating F^q by a vector $v \in C_0$. Let $\mathcal{V}(F^p, F^q)$ be the smallest vector space containing both F^p and F^q.*

If $\dim \mathcal{V}(F^p, F^q) < p + q$ then $F^p \cap \widetilde{F}^q = \emptyset$.

If $\dim \mathcal{V}(F^p, F^q) = p + q$ then $F^p \cap \widetilde{F}^q = \emptyset$ if and only if $v \notin \mathcal{V}(F^p, F^q)$.

Proof. If $\dim \mathcal{V}(F^p, F^q) < p + q$ then there exists a hyperplane H containing both F^p and F^q. Therefore $\widetilde{F}^q$ is contained in a hyperplane parallel to H. This hyperplane must be different from H because v is certainly different from 0. Hence $F^p \cap \widetilde{F}^q = \emptyset$.

Let us now assume that $\dim \mathcal{V}(F^p, F^q) = p + q$. Let $\{e_1, \dots, e_p\}$ be a basis of the smallest vector space containing F^p and let $\{f_1, \dots, f_q\}$ be a basis of the smallest vector space containing F^q. Then $\{e_1, \dots, e_p, f_1, \dots, f_q\}$ is a basis of $\mathcal{V}(F^p, F^q)$. Let $\{g_1, \dots, g_{n-p-q}\}$ be the completion of the above set to a basis of $\mathbb{R}^n$.

If $v \in \mathcal{V}(F^p, F^q)$ then $v = \sum_{i=1}^p h_i e_i + \sum_{i=1}^q k_i f_i$ for some h_i and k_i. Then $\sum_{i=1}^p h_i e_i \in F^p \cap \widetilde{F}^q$ and $F^p \cap \widetilde{F}^q \neq \emptyset$.

Viceversa, if there exists $a \in F^p \cap \widetilde{F}^q$ then $a = \sum_{i=1}^p s_i e_i$ for some s_i's in $\mathbb{R}$ and $a = a' + v$ for some $a' = \sum_{i=1}^q m_i f_i \in F^q$. Hence, if $v = \sum_{i=1}^p h_i e_i + \sum_{i=1}^q k_i f_i +$ $+ \sum_{i=1}^{n-p-q} l_i g_i$ it must hold that $m_i + k_i = 0$ for $i = 1, \dots, q$, $h_i = s_i$ for $i = 1, \dots, p$ and $l_i = 0$ for $i = 1, \dots, n-p-q$. Thus $v = -\sum_{i=1}^p m_i e_i + \sum_{i=1}^q s_i f_i$ that is to say that $v \in \mathcal{V}(F^p, F^q)$. $\square$

In the following we shall assume that v has been chosen so as to achieve transversality between $\widetilde{T}(Q)$ and $\mathcal{D}(Q)$ and that it is sufficiently close to 0 so that Q realizes also the cell complex dual to $\widetilde{\mathcal{S}}$. It is easy to see that $\widetilde{T}(Q)$ can be continuously deformed to make it F-equivariant while maintaining transversality.

Let now $\Gamma^1 = \{i_1, i_2, \dots, i_p\}$, $\Gamma^2 = \{j_1, j_2, \dots, j_q\}$ and $\mathrm{card}(\Gamma) = p + q$. We want to compute $\langle \phi_T(\Gamma^1) \cup \phi_T(\Gamma^2), T\underline{e}(\Gamma) \rangle$.

By Proposition 2 we have that

$$\langle \phi_T(\Gamma^1) \cup \phi_T(\Gamma^2)\,,\, T\underline{e}(\Gamma)\rangle = \sum_{e_p \in \underline{e}(\Gamma^1)} \sum_{e_q \in \underline{e}(\Gamma^2)} (D_{|e(\Gamma)} e_p \cdot \widetilde{D}_{|e(\Gamma)} e_q).$$

Thus, if either of Γ^1 and Γ^2 is not included in Γ, it must hold

$$\langle \phi_T(\Gamma^1) \cup \phi_T(\Gamma^2)\,,\, T\underline{e}(\Gamma)\rangle = 0\,.$$

So let us consider the case when $\Gamma^1, \Gamma^2 \subseteq \Gamma$. We observe that, by construction, $\bigcup_{e_p \in \underline{e}(\Gamma^1)} D_{|e(\Gamma)} e_p$ is the union of all the facets in $\mathcal{S}$ of type Γ^1 intersected with $e(\Gamma)$:

$$\bigcup_{e_p \in \underline{e}(\Gamma^1)} D_{|e(\Gamma)} e_p = \bigcup_{w \in W^{\Gamma}_{\Gamma_1}} w\,(F \cap e(\Gamma))\;,$$

where F is the only facet in $\mathcal{F}_0$ of type Γ^1. On the other hand, F is determined by the following conditions: it belongs to the intersection of hyperplanes $H_{i_1} \cap \cdots \cap H_{i_p}$ and for every $i \in I_n \setminus \Gamma^1$ it is on the same side of H_i as C_0.

We recall that if C_0 is a fixed chamber for W and if $H_1, \ldots, H_n$ are the hyperplanes of C_0, for every $i = 1, \ldots, n$, there exists one and only one root α_i associated with H_i and such that α_i is on the same side of H_i as C_0. The set $\{\alpha_1, \ldots, \alpha_n\}$ is called the *simple system of roots associated with the chamber* C_0 (see, *e.g.*, [2, 13]).

Therefore, if $\{\alpha_1, \ldots, \alpha_n\}$ is the simple system of roots associated with the chamber C_0, where α_i is associated with H_i, then F is determined by the p equalities $(\alpha_j, x) = 0$, $j \in \Gamma_1$, and by the $n-p$ inequalities $(\alpha_i, x) > 0$, $i \in I_n \setminus \Gamma^1$. Moreover, its intersection with $e(\Gamma)$ is determined by adding the equations of the affine $p+q$-subspace of $\mathbb{R}^n$ containing $e(\Gamma)$. Analogous observations hold for Γ^2.

All these facts lead to the following algorithm to compute $\phi_T(\Gamma^1) \cup \phi_T(\Gamma^2)$ at $T\underline{e}(\Gamma)$.

1. Choose a simple system of roots for W associated with C_0. Let it be $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$;
2. Order the roots in A according to the ordering fixed for the corresponding hyperplanes in Salvetti's construction;
3. Fix a vector v in the chamber corresponding to the simple system of roots such that $\widetilde{T}(Q)$ is transverse to $\mathcal{D}(Q)$.
4. Find a basis $f_1, \ldots, f_{n-(p+q)}$ for the subspace $\bigcap_{i \in \Gamma} H_i$;
5. Form the list of the elements of $W^{\Gamma}_{\Gamma^1}$ and the list of the elements of $W^{\Gamma}_{\Gamma^2}$ (for example, by using the results in [18]);
6. Set $\langle \phi_T(\Gamma^1) \cup \phi_T(\Gamma^2)\,,\, T\underline{e}(\Gamma)\rangle = 0$;
7. Pick an element w in $W^{\Gamma}_{\Gamma^1}$ and for every $i_l \in \Gamma^1$, with $l = 1, \ldots, p$, compute $w(\alpha_{i_l})$;
8. Pick an element w' in $W^{\Gamma}_{\Gamma^2}$ and for every $j_k \in \Gamma^2$, with $k = 1, \ldots, q$, compute $w'(\alpha_{j_k})$;

9. Consider the linear system with n equations and n unknowns

$$\begin{aligned}(w(\alpha_{i_1}),x)&=0\\ &\vdots\\ (w(\alpha_{i_p}),x)&=0\\ (w'(\alpha_{j_1}),x)&=(w'(\alpha_{j_1}),v)\\ &\vdots\\ (w'(\alpha_{j_q}),x)&=(w'(\alpha_{j_q}),v)\\ (f_1,x)&=(f_1,v_0)\\ &\vdots\\ (f_{n-(p+q)},x)&=(f_{n-(p+q)},v_0)\end{aligned}$$

where v_0 is the vertex of Q contained in C_0.
10. If the linear system in 9. has solution, say the point x_0, and for every $i\in\Gamma\setminus\Gamma^1$ it holds that $(w(\alpha_i),x_0)>0$ and for every $j\in\Gamma\setminus\Gamma^2$ it holds that $(w'(\alpha_j),x_0-v)>0$, then: if the determinant of the matrix associated with the linear system in 9. has the same sign as the determinant of the matrix associated with the ordered set of simple roots A, increase $\langle\phi_T(\Gamma^1)\cup\phi_T(\Gamma^2),\, T\underline{e}(\Gamma)\rangle$ by 1; otherwise decrease $\langle\phi_T(\Gamma^1)\cup\phi_T(\Gamma^2),\, T\underline{e}(\Gamma)\rangle$ by 1;
11. If in $W^{\Gamma}_{\Gamma^2}$ there is another element, go to 8; else, if in $W^{\Gamma}_{\Gamma^1}$ there is another element, go to 7; else, exit.

As an example, in fig. 3 we show, in the case $W=I_2(4)$, how to compute the cup product $1\cup 2$ on the cell $\underline{e}(12)$. The intersection points A, B, C, D give a contribution equal to $1, 1, 1, -1$ respectively. Therefore we obtain that this product is equal to 2, agreeing with what we have already shown in Section 5.1.

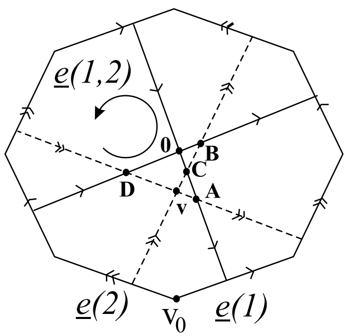

Fig. 3.

We have implemented the above algorithm on a computer to study the cases $\cup$:

$H^2(X_{F_4}, \mathbb{Z}) \otimes H^2(X_{F_4}, \mathbb{Z}) \to H^4(X_{F_4}, \mathbb{Z})$, $\cup : H^3(X_{E_7}, \mathbb{Z}) \otimes H^3(X_{E_7}, \mathbb{Z}) \to H^6(X_{E_7}, \mathbb{Z})$ and $\cup : H^3(X_{E_8}, \mathbb{Z}) \otimes H^3(X_{E_8}, \mathbb{Z}) \to H^6(X_{E_8}, \mathbb{Z})$.

In the case of F_4 it turns out that $\beta_2 \cup \beta_2 = 6\alpha_4$, $\alpha_2 \cup \alpha_2 = -24\alpha_4$ and $\alpha_2 \cup \beta_2 = 0$. Finally, both in the case of E_7 and in the case of E_8 we have obtained $\alpha_3 \cup \alpha_3 = 0$.

This results are reported in table II together with those already obtained in Section 5.1.

TABLE II. – *Cup products in the cohomologies of Artin groups for the exceptional groups (integer coefficients).*

$I_2(2s)$	$\alpha_1 \cup \alpha_1 = \beta_1 \cup \beta_1 = 0, \alpha_1 \cup \beta_1 = s\alpha_2$
$I_2(2s+1)$	$\alpha_1 \cup \alpha_1 = 0$
H_3	$\alpha_1 \cup \alpha_1 = 0$ $\alpha_1 \cup \alpha_2 = -15\alpha_3$
H_4	$\alpha_1 \cup \alpha_1 = 0$ $\alpha_1 \cup \alpha_3 = -45\alpha_4,\ \alpha_1 \cup \beta_3 = 0$
F_4	$\alpha_1 \cup \alpha_1 = 0,\ \beta_1 \cup \beta_1 = 0,\ \alpha_1 \cup \beta_1 = \alpha_2 + 2\beta_2$ $\alpha_1 \cup \alpha_2 = -4\alpha_3 - 2\beta_3,\ \alpha_1 \cup \beta_2 = 2\alpha_3 + \beta_3$ $\beta_1 \cup \alpha_2 = -2\alpha_3 - 4\beta_3,\ \beta_1 \cup \beta_2 = \alpha_3 + 2\beta_3$ $\alpha_1 \cup \alpha_3 = -4\alpha_4,\ \alpha_1 \cup \beta_3 = 8\alpha_4,\ \beta_1 \cup \alpha_3 = -8\alpha_4,\ \beta_1 \cup \beta_3 = 4\alpha_4$ $\beta_2 \cup \beta_2 = 6\alpha_4,\ \alpha_2 \cup \alpha_2 = -24\alpha_4,\ \alpha_2 \cup \beta_2 = 0$
E_6	$\alpha_1 \cup \alpha_1 = 0$ $\alpha_1 \cup \alpha_3 = \alpha_4$ $\alpha_1 \cup \alpha_4 = 0$ $\alpha_1 \cup \alpha_5 = 0,\ \alpha_1 \cup \beta_5 = 0$ $\alpha_3 \cup \alpha_3 = 0$
E_7	$\alpha_1 \cup \alpha_1 = 0$ $\alpha_1 \cup \alpha_3 = \alpha_4 + \beta_4$ $\alpha_1 \cup \alpha_4 = \beta_5,\ \alpha_1 \cup \beta_4 = \beta_5$ $\alpha_1 \cup \alpha_5 = \alpha_6,\ \alpha_1 \cup \beta_5 = 0,\ \alpha_1 \cup \gamma_5 = \gamma_6,\ \alpha_1 \cup \delta_5 = 0$ $\alpha_1 \cup \alpha_6 = 0,\ \alpha_1 \cup \beta_6 = 0,\ \alpha_1 \cup \gamma_6 = 0,\ \alpha_1 \cup \delta_6 = 63\alpha_7$ $\alpha_3 \cup \alpha_3 = 0$ $\alpha_3 \cup \alpha_4 = 0,\ \alpha_3 \cup \beta_4 = 0$
E_8	$\alpha_1 \cup \alpha_1 = 0$ $\alpha_1 \cup \alpha_3 = \alpha_4$ $\alpha_1 \cup \alpha_4 = 0$ $\alpha_1 \cup \alpha_5 = \alpha_6,\ \alpha_1 \cup \beta_5 = 0,\ \alpha_1 \cup \gamma_5 = \gamma_6$ $\alpha_1 \cup \alpha_6 = 0,\ \alpha_1 \cup \beta_6 = \gamma_7,\ \alpha_1 \cup \gamma_6 = 0$ $\alpha_1 \cup \delta_7 = -120\alpha_8$ $\alpha_3 \cup \alpha_3 = 0$ $\alpha_3 \cup \alpha_4 = 0$ $\alpha_3 \cup \alpha_5 = 0,\ \alpha_3 \cup \beta_5 = 0,\ \alpha_3 \cup \gamma_5 = 0$ $\alpha_4 \cup \alpha_4 = 0$

ACKNOWLEDGEMENTS

The author wishes to thank M. Salvetti for posing the problem and for the many helpful discussions.

References

[1] V. I. Arnol'd, *On some topological invariants of algebraic functions*. Trudy Mosk. Mat. Obshch., 21, 1970, 27-46.
[2] N. Bourbaki, *Groupes et algèbres de Lie*. Chap. 4-6. Masson-Paris 1981.
[3] E. Brieskorn, *Sur les groupes de tresses*. Sém. Bourb. (1971/72) Lect. Notes in Math., 317, 1973, 21-44.
[4] E. Čech, *Multiplication on a complex*. Annals of Mathematics, 37, 1936, 681-697.
[5] G. E. Cooke - R. L. Finney, *Homology of cell complexes*. Mathematical Notes, Princeton University Press 1967.
[6] C. De Concini - M. Salvetti, *Cohomology of Artin groups*. Math. Res. Letters, 3, 1996, 293-297.
[7] C. De Concini - C. Procesi - M. Salvetti, *Arithmetic properties of the cohomology of braid groups*. Topology, to appear.
[8] C. De Concini - C. Procesi - M. Salvetti - F. Stumbo, *Arithmetic properties of the cohomology of Artin groups*. Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat., to appear.
[9] P. Deligne, *Les immeubles des groupes de tresses généralisés*. Inv. Math., 17, 1972, 273-302.
[10] D. B. Fuks, *Cohomologies of the group COS mod 2*. Funkts Anal. Priloz., 4, 1970, 62-63.
[11] V. V. Goryunov, *Cohomology of groups of braids of series C and D and certain stratifications*. Funkts Anal. Priloz., 12, 1978, 138-140.
[12] P. J. Hilton - S. Wylie, *Homology theory*. Cambridge University Press 1960.
[13] J. E. Humphreys, *Reflection groups and Coxeter groups*. Cambridge University Press 1990.
[14] S. Lefschetz, *Topology*. Colloquium Publications, Volume XII. American Mathematical Society, New-York 1930.
[15] C. R. . Maunder, *Algebraic topology*. London: Van Nostrand Reinhold Company 1970.
[16] M. Salvetti, *The homotopy type of Artin groups*. Math. Res. Letters, 1, 1994, 565-577.
[17] H. Seifert - W. Threlfall, *A textbook of topology*. Academic Press 1980.
[18] F. Stumbo, *Minimal length coset representatives for parabolic subgroups in Coxeter groups*. preprint.
[19] F. V. Vainshtein, *Cohomologies of braid groups*. Funkts Anal. Priloz., 12, 1978, 72-73.
[20] V. A. Vassiliev, *Complements of discriminants of smooth maps*. Trans. of Math. Monog. AMS, 98, 1992.
[21] H. Whitney, *On products on a complex*. Annals of Mathematics, 39, 1938, 397-432.

Pervenuta il 19 ottobre 1999,
in forma definitiva l'11 febbraio 2000.

Dipartimento di Matematica
Università degli Studi di Pisa
Via Buonarroti, 2 - 56127 Pisa
landi@mail.dm.unipi.it