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Positive solutions for some quasilinear elliptic equations with natural growths

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Equazioni a derivate parziali. — *Positive solutions for some quasilinear elliptic equations with natural growths.* Nota di Lucio Boccardo, presentata (*) dal Socio S. Spagnolo.

Abstract. — We shall prove an existence result for a class of quasilinear elliptic equations with natural growth. The model problem is

$$\begin{cases} -\mathrm{div}((1+|u|^r)\nabla u) + |u|^{m-2}u|\nabla u|^2 = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Key words: Quasilinear elliptic equations; Natural growth coefficients; Euler-Lagrange equations.

Riassunto. — *Soluzioni positive per alcune equazioni ellittiche con crescite naturali.* È provato un teorema di esistenza di soluzioni per una classe di equazioni ellittiche quasi-lineari, con coefficienti a crescite naturali (come suggerito dal Calcolo delle variazioni). Il problema modello è il seguente

$$\begin{cases} -\mathrm{div}((1+|u|^r)\nabla u) + |u|^{m-2}u|\nabla u|^2 = f & \text{in } \Omega \\ u = 0 & \text{su } \partial\Omega. \end{cases}$$

1. Introduction

It is well known that the minimization in $W_0^{1,2}(\Omega)$ (Ω is a bounded domain in $\mathbb{R}^N$) of simple functionals like

$$I(v) = \frac{1}{2}\int_\Omega a(x, v)|\nabla v|^2 - \int_\Omega f(x)v(x),$$

where a is a bounded, smooth function and $f \in L^2(\Omega)$, leads to the following Euler-Lagrange equation

$$\text{(1)} \qquad \begin{cases} -\mathrm{div}(a(x, u)\nabla u) + \frac{1}{2}a'(x, u)|\nabla u|^2 = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(for a direct study of the existence of bounded solutions of boundary value problems of type (1), if $f \in L^q(\Omega)$, $q > \frac{N}{2}$, see [8]). Recall that the functional I is not Gateaux-differentiable. It is only differentiable along directions of $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ (see [11]). Moreover, if we consider

$$J(v) = \frac{1}{2}\int_\Omega (1+|v|^m)|\nabla v|^2 - \int_\Omega f(x)v(x), \qquad m > 1,$$

(*) Nella seduta del 12 novembre 1999.

the Euler-Lagrange equation is

$$\begin{cases} -\operatorname{div}((1+|u|^m)\nabla u) + \dfrac{m}{2}|u|^{m-2}u|\nabla u|^2 = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Remark that the direct study of Dirichlet problems similar to the previous one, like

$$u \in W_0^{1,2}(\Omega):\ -\operatorname{div}((1+|u|^m)\nabla u) + |u|^{r-2}u|\nabla u|^2 = f \in L^2(\Omega)$$

$(m > 0\,,\ r > 1)$ gives some difficulties, due to the fact that now the boundary value problems may not be the Euler-Lagrange equation of some functional and that, even if $m = 0$, u may be unbounded. The first difficulty is due to the fact that the principal part of the differential operator $-\operatorname{div}((1+|v|^m)\nabla v)$ is not well defined on the whole $W_0^{1,2}(\Omega)$. The second and main one is that the lower order term $|v|^{r-2}v|\nabla v|^2$ not only is not well defined on the whole $W_0^{1,2}(\Omega)$, but, even if $v \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$, $|v|^{r-2}v|\nabla v|^2$ does not belong to $W^{-1,2}(\Omega)$. However, the lower order term has the nice property that $v\cdot(|v|^{r-2}v|\nabla v|^2) \geq 0$; a generalization of this fact will be assumption (6), below.

In a more general setting, we will study here the Dirichlet problem

$$u \in W_0^{1,2}(\Omega):\ -\operatorname{div}(a(x,u)\nabla u) + g(x,u,\nabla u) = f\,. \tag{2}$$

On the right hand side f we assume that

$$f \in L^2(\Omega)\,, \tag{3}$$

$$f \geq 0\,. \tag{4}$$

Moreover $a(x,s) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $g(x,s,\xi) : \Omega \times \mathbb{R} \to \mathbb{R}$ are functions which are measurable with respect to x and continuous with respect to (s,ξ), such that

$$0 < \alpha \leq a(x,s) \leq \beta(s)\,, \tag{5}$$

$$g(x,s,\xi)\,s \geq 0\,, \tag{6}$$

$$|g(x,s,\xi)| \leq \gamma(s)(h(x) + |\xi|^2)\,, \tag{7}$$

where $\beta\,,\gamma$ are continuous, increasing (possibly unbounded) functions of a real variable and $h(x)$ is a given nonnegative function in $L^1(\Omega)$.

Contributions to the existence and nonexistence of solutions of (2), if the dependence on u of the principal part is bounded, can be found in [2, 7, 6, 9, 10, 12].

We refer to [5, 6, 15] for the existence of solutions of (2) if the right hand side belongs to $L^1(\Omega)$.

Other developments and general existence results are contained in [14].

The results of this paper have been presented in [3].

The results are quite easy to prove thanks to assumption (4), but the linearity with respect to the gradient of the principal part of the differential operator is never used. Moreover we want to underline that we cannot expect that the presence of the term

$\beta(u)$ can have a regularizing effect on the solution, because $a(x, s)$ is controlled by $\beta(s)$ only from above (see (5)); observe that, conversely, the assumption

$$0 < \alpha\beta(s) \le a(x, s) \le \beta(s) \,, \tag{8}$$

lies that $\beta(u)\nabla u \in L^2(\Omega)$, while, under our assumption (5), we will not even be able to prove that $\beta(u)\nabla u \in L^1(\Omega)$. Indeed (formally) the use of $B(u)$ as test function (where $B(s) = \int_0^s \beta(t)\,dt$) in (2) implies

$$c_0\left(\int_\Omega |B(u)|^{2^*}\right)^{\frac{2}{2^*}} \le \alpha\int_\Omega \beta(u)^2|\nabla u|^2 \le \int_\Omega f\,B(u) \le \|f\|_{L^{\frac{2N}{N+2}}(\Omega)}\|B(u)\|_{L^{2^*}(\Omega)} \,.$$

Observe also that the stronger assumption $f \in L^q(\Omega)$, $q > \frac{N}{2}$, implies thanks to (6) that u is bounded, so that the existence of bounded solutions follows from the general results of [8].

2. Approximation

The existence of a solution of the Dirichlet problem (2) will be proved by approximation. Our techniques will follow those of [2, 4].

Define

$$g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n}|g(x, s, \xi)|} \,,$$

and

$$a_n(x, s) = a(x, T_n(s)) \,,$$

where

$$T_n(s) = \begin{cases} s & \text{if } |s| \le n \\ \dfrac{s\,n}{|s|} & \text{if } |s| > n \,. \end{cases}$$

Consider the approximate Dirichlet problems

$$u_n \in W_0^{1,2}(\Omega) : -\mathrm{div}(a_n(x, u_n)\nabla u_n) + g_n(x, u_n, \nabla u_n) = f \,. \tag{9}$$

Thanks to the boundedness of $a_n(x, s)$ and $g_n(x, s, \xi)$, the existence of a solution u_n (which is positive thanks to (4) and (6)) of the boundary value problem (9) is a classical result of nonlinear elliptic equations (see [13]).

The assumptions (5), (6) and the use of u_n as test function in (9) imply the following lemma.

Lemma 2.1. *There exists a positive constant c_1 such that*

$$\int_\Omega a_n(x, u_n)\nabla u_n \nabla u_n \le c_1 \,, \tag{10}$$

$$\|u_n\|_{W_0^{1,2}(\Omega)} \le \frac{c_1}{\alpha} \,, \tag{11}$$

$$\int_\Omega u_n\, g_n(x, u_n, \nabla u_n) \le c_1 \,. \tag{12}$$

Thus there exist a positive function $u \in W_0^{1,2}(\Omega)$ and a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that u_n converges to u weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$.

This section is devoted to the proof of the strong convergence of u_n to u in $W_0^{1,2}(\Omega)$. Define

$$G_k(v) = v - T_k(v)\,.$$

The use of $G_k(u_n)$ as test functions in (9) imply, thanks to the fact that $f \in L^2(\Omega)$, the following lemma.

Lemma 2.2. *There exists a positive constant c_2 such that*

$$\int_{\{x\in\Omega: u_n(x)\geq k\}} |\nabla u_n|^2 \leq c_2 \int_{\{x\in\Omega: u_n(x)\geq k\}} |\,f\,|^2 \,. \tag{13}$$

Now we study the behaviour of the positive part of $u_n - T_h(u)$.

Lemma 2.3. *For any $\epsilon > 0$, there exists h_ϵ such that*

$$\limsup_{n\to\infty} \|[u_n - T_{h_\epsilon}(u)]^+\|_{W_0^{1,2}(\Omega)} \leq 2\,\epsilon\,,$$

and

$$\|u - T_{h_\epsilon}(u)\|_{W_0^{1,2}(\Omega)} \leq \epsilon.$$

Proof. Since u is positive, on the subset $\{x \in \Omega : k \leq u_n(x) - T_h(u(x))\}$, it is $u_n(x) \geq k$. Therefore

$$\int_\Omega |\nabla G_k[u_n - T_h(u)]^+|^2 = \int_{\{x\in\Omega: k\leq u_n(x) - T_h(u(x))\}} |\nabla[u_n - T_h(u)]|^2 \leq$$
$$\leq 2 \int_{\{x\in\Omega: k\leq u_n(x)\}} |\nabla u_n|^2 + 2 \int_{\{x\in\Omega: k\leq u_n(x)\}} |\nabla u|^2\,.$$

Thus, thanks to Lemma 2.2, we get the following inequality

$$\int_\Omega |\nabla G_k[u_n - T_h(u)]^+|^2 \leq \int_{\{x\in\Omega: k\leq u_n(x)\}} \{c_2|f|^2 + |\nabla u|^2\}\,. \tag{14}$$

The previous inequality implies, since the measure of the set $\{u_n(x) \geq k\}$ tends to zero as k tends to infinity, uniformly in n, that, if we fix $\epsilon > 0$, there exists $k_\epsilon > 0$ such that, for every n in $\mathbb{N}$, and for every $h > 0$,

$$\int_\Omega |\nabla G_{k_\epsilon}[u_n - T_h(u)]^+|^2 \leq \epsilon\,. \tag{15}$$

The use of $T_{k_\epsilon}[u_n - T_h(u)]^+$ (for any $h > 0$) as test function in (9) and the assumption (6) imply that

$$\int_\Omega |\nabla T_{k_\epsilon}[u_n - T_h(u)]^+|^2 \leq c_3 \int_\Omega f\, T_{k_\epsilon}(G_h(u)) + \epsilon_n\,, \tag{16}$$

for some positive constant c_3, where $\epsilon_n \to 0$.

Now we choose h_ϵ such that (for $n > n_\epsilon$)

$$\int_\Omega |\nabla T_{k_\epsilon}[u_n - T_{h_\epsilon}(u)]^+|^2 , \quad \|u - T_{h_\epsilon}(u)\|_{W_0^{1,2}(\Omega)} \leq \epsilon .$$

The fact that $\|u - T_{h_\epsilon}(u)\|_{W_0^{1,2}(\Omega)} \leq \epsilon$ follows from the fact that u belongs to $W_0^{1,2}(\Omega)$. $\square$

Now we study the behaviour of the negative part of $u_n - T_{h_\epsilon}(u)$.
Define

$$\varphi_\lambda(s) = s e^{\lambda s^2} , \quad \lambda = \lambda(h_\epsilon) = \frac{\gamma(h_\epsilon)^2}{\alpha^2} .$$

Even if the principal part of the differential operator is unbounded with respect to u the following lemma, proved in [2], still holds. Remark that, since u is positive, $\{x \in \Omega : u_n(x) - T_h(u(x)) \leq 0\} = \{x \in \Omega : 0 \leq u_n(x) \leq T_h(u(x))\}$.

LEMMA 2.4. *The use of* $\varphi_\lambda([u_n - T_h(u)]^-)$ *as test function in* (9) *implies that*

$$\lim_{n\to\infty} \int_\Omega |\nabla [u_n - T_h(u)]^-|^2 = 0 , \tag{17}$$

for any $h > 0$.

Thus we have the following result.

PROPOSITION 2.5. *The use of Proposition* 2.3 *and Lemma* 2.4 *implies that the sequence* $\{u_n\}$ *converges strongly to* u *in* $W_0^{1,2}(\Omega)$.

PROOF. We have

$$\|u_n - u\|_{W_0^{1,2}(\Omega)} \leq$$
$$\leq \|[u_n - T_h(u)]^+\|_{W_0^{1,2}(\Omega)} + \|[u_n - T_h(u)]^-\|_{W_0^{1,2}(\Omega)} + \|T_h(u) - u\|_{W_0^{1,2}(\Omega)} .$$
$\square$

3. EXISTENCE

We have proved that

$$u_n \to u \quad \text{strongly in } W_0^{1,2}(\Omega) .$$

Thus (again for some subsequence) we have that

$$\nabla u_n(x) \to \nabla u(x) , \text{ almost everywhere in } \Omega. \tag{18}$$

In order to pass to the limit in (9) we need the L^1 compactness of the sequence $g_n(x, u_n, \nabla u_n)$ proved (see again [2]) in the following lemma, by means of Proposition 2.5 and (12).

LEMMA 3.1. *The sequence* $g_n(x, u_n, \nabla u_n)$ *converges in* $L^1(\Omega)$ *to* $g(x, u, \nabla u)$.

Observe that, even if we have both (11) and the fact that u_n converges strongly to u in $W_0^{1,2}(\Omega)$, we are not able to say that the sequence $a_n(x, u_n)\nabla u_n$ converges in $L^1(\Omega)$ to $a(x, u)\nabla u$. So, in order to pass to limit in (9), we will use the approach of [1].

We use in (9) $T_k[u_n - \varphi]$ as test function, where $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. We can then pass to the limit, thanks to Proposition 2.5 and Lemma 3.1, and we obtain our main result.

THEOREM 3.2. *There exists a solution* u *of* (2) *in the following sense*

$$\left\{\begin{array}{l} u \in W_0^{1,2}(\Omega), \quad g(x, u, \nabla u) \in L^1(\Omega), \\ \displaystyle\int_\Omega a(x, u)\nabla u \nabla T_k[u - \varphi] + \int_\Omega g(x, u, \nabla u)\, T_k[u - \varphi] = \int_\Omega f\, T_k[u - \varphi] \\ \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \quad \forall k > 0. \end{array}\right. \tag{19}$$

REMARK 3.3. We point out that, in the previous equality, any term is well defined: in the second and the third integral $g(x, u, \nabla u)$, $f \in L^1(\Omega)$ and $T_k[u - \varphi] \in L^\infty(\Omega)$; in the first $\nabla T_k[u - \varphi]$ is not zero on the subset $\{x \in \Omega : \varphi(x) - k \leq u(x) \leq \varphi(x) + k\}$, that is in a subset where u (and also $a(x, u)$) is bounded.

We repeat (see Introduction) that under our assumption (5), we are not able to prove that $a(x, u)\nabla u \in L^1(\Omega)$, so that the usual definition of weak solution

$$\int_\Omega a(x, u)\nabla u \nabla \varphi + \int_\Omega g(x, u, \nabla u)\, \varphi = \int_\Omega f\, \varphi, \quad \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega),$$

does not make sense and thus the previous definition of solution is useful.

COROLLARY 3.4. *Choosing* $\varphi = 0$ *in* (19), *letting* k *tend to infinity, and using Fatou Lemma implies that*

$$a(x, u)\nabla u \nabla u \in L^1(\Omega), \tag{20}$$

$$u g(x, u, \nabla u) \in L^1(\Omega). \tag{21}$$

REMARK 3.5. If $a(x, u)\nabla u \in L^2(\Omega)$ (see Introduction and (8)), we take in (19) $\varphi = G_h(u) - \psi$, $\psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, and we pass to limit (for $h \to \infty$), thanks to Lebesgue Theorem and (20). Thus if $a(x, u)\nabla u \in L^2(\Omega)$ we deduce the existence of usual weak solutions

$$\int_\Omega a(x, u)\nabla u \nabla \psi + \int_\Omega g(x, u, \nabla u)\, \psi = \int_\Omega f\, \psi, \quad \forall \psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

4. $L^1(\Omega)$ RIGHT HAND SIDE

Our approach can also be adapted to the case of L^1 right hand side, using some techniques of [5], but we need a slightly stronger assumption on $g(x, s, \xi)$.

In this section, we assume again (4), (5), (7) and the following (22), (23).

There exist $\sigma \geq 0$, $\lambda > 0$ such that

$$|g(x, s, \xi)| \geq \lambda |\xi|^2 \tag{22}$$

when $|s| \geq \sigma$, and

$$0 \leq f \in L^1(\Omega)\,. \tag{23}$$

Consider now the approximate Dirichlet problems

$$u_n \in W_0^{1,2}(\Omega) : -\mathrm{div}(a_n(x, u_n)\nabla u_n) + g_n(x, u_n, \nabla u_n) = f_n\,, \tag{24}$$

where $a_n(x, s)$, $g_n(x, s, \xi)$ are defined in Section 2 and f_n is a sequence of smooth functions converging to f in $L^1(\Omega)$.

The use in (24) of the test function $T_k(u_n)$ yields for any $k > 0$ (see [5], if necessary),

$$\int_\Omega |\nabla T_k(u_n)|^2 \leq \int_\Omega f_n\, T_k(u_n) \leq c_4 k\,,$$

$$k \int_{\{x\in\Omega: u_n(x) > k\}} g_n(x, u_n, \nabla u_n) \leq \int_\Omega f_n\, T_k(u_n) \leq c_4 k\,.$$

In particular, the choice $k = \sigma$ implies that

$$\int_{\{x\in\Omega: u_n(x) > \sigma\}} |\nabla u_n|^2 + \int_{\{x\in\Omega: u_n(x) > \sigma\}} |\nabla u_n|^2 \leq c_5\,.$$

Thus again there exist a positive function $u \in W_0^{1,2}(\Omega)$ and a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that u_n converges to u weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$.

The use of $T_1(G_k(u_n))$ as test function in (24) implies that, for any $k > 0$,

$$\int_{\{x\in\Omega: k+1 \leq u_n(x)\}} g_n(x, u_n, \nabla u_n) \leq \int_{\{x\in\Omega: k \leq u_n(x)\}} |f|\,,$$

so that the choice $k \geq \sigma$ gives the following lemma.

Lemma 4.1.

$$\int_{\{x\in\Omega: k+1 \leq u_n(x)\}} |\nabla u_n|^2 \leq \int_{\{x\in\Omega: k \leq u_n(x)\}} |f|\,. \tag{25}$$

Thanks to the use of Lemma 4.1 instead of Lemma 2.2, Lemma 2.3 still holds. Lemma 2.4 holds even if the right hand side belongs to $L^1(\Omega)$. Remark that the test function of Lemma 2.4 is uniformly bounded in $L^\infty(\Omega)$ by $\varphi_\lambda(2h)$. Thus, also in this case, the sequence $\{u_n\}$ strongly converges to u in $W_0^{1,2}(\Omega)$ and (again for some subsequence) we have (18).

In order to pass to the limit in (24), we also need the following lemma.

Lemma 4.2. *The sequence $g_n(x, u_n, \nabla u_n)$ converges in $L^1(\Omega)$ to $g(x, u, \nabla u)$.*

PROOF. We prove that $g_n(x, u_n, \nabla u_n)$ is uniformly equiintegrable. For any measurable subset E of Ω and for any $m \in \mathbb{R}^+$ we have

$$\int_E g_n(x, u_n, \nabla u_n) = \int_{\{x\in E: 0\le u_n(x)<m\}} g_n(x, u_n, \nabla u_n) + \int_{\{x\in\Omega: m\le u_n(x)\}} g_n(x, u_n, \nabla u_n) \le$$
$$\le \int_E \gamma(m)[h(x) + |\nabla u_n|^2] + \int_{\{x\in\Omega: m\le u_n(x)\}} |f| \,,$$

which proves the uniform equiintegrability of $g_n(x, u_n, \nabla u_n)$. In view of (18) we thus have $g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u)$ strongly in $L^1(\Omega)$. □

So it is now easy to pass to the limit in (24) to obtain that u is a solution. We use again $T_k[u_n - \varphi]$ as test function in (24), where $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. So it is now easy to pass to the limit in (24) to obtain that u is a solution and we obtain the following result.

THEOREM 4.3. *Assume* (4), (5), (7), (22), (23). *There exists a solution u of* (2) *in the following sense*

$$(26) \qquad \begin{cases} u \in W_0^{1,2}(\Omega)\,, \quad g(x, u, \nabla u) \in L^1(\Omega)\,, \\ \displaystyle\int_\Omega a(x, u)\nabla u \nabla T_k[u-\varphi] + \int_\Omega g(x, u, \nabla u)\, T_k[u-\varphi] = \int_\Omega f\, T_k[u-\varphi] \\ \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,, \quad \forall k > 0\,. \end{cases}$$

REMARK 4.4. With respect to Theorem 3.2 we cannot say that in this case $a(x, u)\nabla u \nabla u \in L^1(\Omega)$, nor that $ug(x, u, \nabla u) \in L^1(\Omega)$.

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