José M. Isidro, Jean-Pierre Vigué

On the product property of the Carathéodory pseudodistance


Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2000_9_11_1_21_0>

Abstract. — We prove that, for certain domains $D$, continuous product of domains $D_\omega$, the Carathéodory pseudodistance satisfies the following product property

$$C_D(f, g) = \sup_\omega C_{D_\omega}(f(\omega), g(\omega)).$$

Key words: Carathéodory pseudodistance; Product domains; Product property.

Riassunto. — Proprietà del prodotto della pseudodistanza di Carathéodory. Si prova che per alcuni domini $D$, che sono prodotti continui di domini $D_\omega$, la pseudodistanza di Carathéodory soddisfa la seguente proprietà:

$$C_D(f, g) = \sup_\omega C_{D_\omega}(f(\omega), g(\omega)).$$

1. Introduction

Let $\Omega$ and $E$ respectively be a completely regular topological space and a complex Banach space with open unit ball $B(0)$. Let $E := C_b(\Omega, E)$ be the Banach space of all continuous bounded $E$-valued functions $f: \Omega \to E$, endowed with the pointwise operations and the norm of the supremum. Whenever $E$ is a complex Banach space and $D \subset E$ is a domain, we let $C_D$ denote the Carathéodory distance in $D$.

Recall [5, Definition 1.5] a domain $D \subset C(\Omega, E)$ is the continuous $\Omega$-product of the family $(D_\omega)_{\omega \in \Omega}$ of bounded domains in $E$ if the following two conditions hold: $D$ is the interior of

$$\{ f \in C(\Omega, E) : f(\omega) \in D_\omega, \ (\omega \in \Omega) \}, \quad D_\omega = \{ f(\omega) : f \in D \}, \ (\omega \in \Omega).$$

In that case $D$ consists of continuous sections of $D_\omega := \{ (\omega, x) \in \Omega \times E : \omega \in \Omega, x \in D_\omega \}$ with respect to the fibration $p: D_\omega \to \Omega$ given by $(\omega, x) \mapsto \omega$. Let $\Omega$, $E$ and $\| \cdot \|_\omega$, $(\omega \in \Omega)$, respectively be a compact topological space, a complex Banach space and a family of norms in $E$ with open unit balls $D_\omega$, and let $D := \{ f \in C(\Omega, E) : f(\omega) \in \in D_\omega, \ (\omega \in \Omega) \}$ be a bounded domain in $C(\Omega, E)$. Then $D$ is the continuous $\Omega$-product of the family $(D_\omega)_{\omega \in \Omega}$ if and only if there are constants $0 < m \leq M < \infty$ such that $m \| \cdot \| \leq \| \cdot \|_\omega \leq M \| \cdot \|$ for all $\omega \in \Omega$ and the function $N(\omega, x) := \| x \|_\omega$ is upper semicontinuous on $\Omega \times E$.

Definition 1.1. Let $(D_\omega)_{\omega \in \Omega}$ be a family of domains $D_\omega \subset E$ whose $\Omega$-product $D$ is domain in $C_b(\Omega, E)$.

(1) We say that the continuous product property (the CPP for short) holds for $D$ if

(*) Pervenuta in forma definitiva all’Accademia il 6 ottobre 1999.
the Carathéodory distance $C_D$ satisfies
\[(1) \quad C_D(f, g) = \sup_{\omega \in \Omega} C_{D_\omega}[f(\omega), g(\omega)], \quad f, g \in D.\]

(2) We say that the CPP holds for the space $C_b(\Omega, E)$ if whenever $(D_\omega)_{\omega \in \Omega}$ is a family whose $\Omega$-product $D$ is a domain in $C_b(\Omega, E)$, the CPP holds for $D$.

In general no information is available about how $D_\omega$ depends on $\omega \in \Omega$. If all domains $D_\omega$ coincide (say with $D \subset E$) then $s \mapsto C_D[f(\omega), g(\omega)]$ is continuous, hence the supremum in (1) is attainable whenever $\Omega$ is compact. In the general case, the evaluation $e_\omega : C_b(\Omega, E) \to E$ is a holomorphic map, hence it is a contraction for the Carathéodory distances and so
\[(2) \quad \sup_{\omega \in \Omega} C_{D_\omega}[f(\omega), g(\omega)] \leq C_D(f, g),\]
holds, hence the CPP for the domain $D$ is equivalent to
\[(3) \quad C_D(f, g) \leq \sup_{\omega \in \Omega} C_{D_\omega}[f(\omega), g(\omega)], \quad f, g \in D.\]

In [3], Jarnicki and Pflug have proved that (3) holds whenever $\Omega$ is finite and $E$ is finite dimensional. The general case seems to be very difficult, and we prove this property in the following cases:

(a) $\Omega$ is a finite set and $E$ is a Banach space.

(b) $D$ is contained in a space of sequences converging to zero at infinity.

(c) $\Omega$ is an infinite set with the discrete topology and we consider an infinite product of copies of the same domain $D \subset \mathbb{C}^n$, with an additional hypothesis on $D$.

2. Finite products in complex Banach spaces

We get the following result

**Proposition 2.1.** Let $A$ and $B$ be domains in the Banach spaces $E$ and $F$ respectively. Then

\[C_{A \times B}[(a, b), (a', b')] = \max\{C_A(a, a'), C_B(b, b')\}\]
holds for all pairs $a, a' \in A$ and $b, b' \in B$.

**Proof.** For a domain $D$ in a complex Banach space $X$ and a pair of points $x, x' \in D$, we let $F(X, x, x')$ denote the family of all vector subspaces $Z \subset X$ such that $\dim Z < \infty$ and $x, x' \in Z$. By [1, Th. 2.1] we have
\[(4) \quad C_D(x, x') = \inf_{Z \in F(X, x, x')} C_{D \cap Z}(x, x').\]

Let $\epsilon > 0$ be given. By (4) there are subspaces $X \in F(E, a, a')$ and $Y \in F(F, b, b')$ such that
\[C_A(a, a') + \epsilon > C_{A \cap X}(a, a'), \quad C_B(b, b') + \epsilon > C_{B \cap Y}(b, b').\]
Obviously we have \( F(E \times F, (a, b), (a', b')) \supseteq F(F, b, b') \). Therefore by [3, Th. 1.1]
\[
C_{A \times B}([a, b], (a', b')) \leq \inf_{X,Y} C_{(A \cap X) \times (B \cap Y)}([a, b], (a', b')) = \\
= \inf_{X,Y} \max\{ C_{A \cap X}(a, a'), C_{B \cap Y}(b, b') \} \leq \\
\leq \max\{ C_A(a, a') + \epsilon, C_B(b, b') + \epsilon \} = \max\{ C_A(a, a'), C_B(b, b') \} + \epsilon.
\]
Since this is valid for all \( \epsilon > 0 \), the result follows from (3). \( \square \)

3. Space of continuous sections converging to zero at infinity

Let \( \Omega \) be a a locally compact space and let \( C_0(\Omega, E) \) be the Banach space of continuous maps \( f: \Omega \to E \) converging to 0 at infinity. First, we prove the following proposition

**Proposition 3.1.** Let \( \Omega, E \) and \( D \) respectively be a locally compact space, a complex Banach space and a domain \( D \subset E \) such that \( 0 \in D \). Let \( K \) be a compact set in \( C_0(\Omega, E) \). Then
\[
\lim_{\omega \to \infty} C_D[f(\omega), g(\omega)] = 0
\]
holds uniformly for \( f, g \in K \). In particular, for \( f, g \in C_0(\Omega, E) \) with \( f(\Omega), g(\Omega) \subset D \), the function \( d: \omega \mapsto C_D[f(\omega), g(\omega)] \) satisfies \( d \in C_0(\Omega, \mathbb{R}) \).

**Proof.** Let \( f, g \in C_0(\Omega, E) \) satisfy \( f(\Omega), g(\Omega) \subset D \). The evaluations and the Carathéodory distance are continuous functions, hence so is \( d: \omega \mapsto C_D[f(\omega), g(\omega)] \). Thus we only have to prove that
\[
\lim_{\omega \to \infty} C_D[f(\omega), g(\omega)] = 0
\]
holds uniformly for \( f, g \in K \). Let \( \epsilon > 0 \) be given. For a suitable \( \rho > 0 \), the ball \( B_\rho(0) := \{ x \in E : \| x \| < \rho \} \) clearly satisfies \( B_\rho(0) \subset D \), hence by [2, Th. IV.2.] there is a constant \( M \) such that
\[
C_D(z, w) \leq M \| z - w \|, \quad z, w \in B_\rho(0).
\]
Let \( \epsilon' := \min\{ \frac{1}{2}, \frac{\epsilon}{2M} \} \). Since \( K \) is a compact subset \( C_0(\Omega, E) \), there is a compact set \( S \subset \Omega \) such that \( \| h(\omega) \| \leq \epsilon' \) for all \( \omega \in \Omega \setminus S \) and all \( h \in K \). Therefore
\[
C_D[f(\omega), g(\omega)] \leq M \| f(\omega) - g(\omega) \| \leq M (\| f(\omega) \| + \| g(\omega) \| ) \leq 2M \epsilon' = \epsilon, \quad s \in \Omega \setminus S
\]
which completes the proof. \( \square \)

For every compact subset \( K \subset \Omega \) we let \( S(K, E) \) denote the (possibly non closed) normed subspace of \( C_0(\Omega, E) \) consisting of the functions \( f \) such that \( \text{supp}(f) \subset K^\circ \).

**Proposition 3.2.** Let \( \Omega, E \) and \( D \) respectively be a locally compact \( \sigma \)-compact topological space, a complex Banach space and a star-like domain \( D \subset E \). Let \( D_0 \subset C_0(\Omega, E) \) denote the \( c_0(\Omega) \)-power of \( D \). If the CPP holds in \( S(K, E) \) for every compact set \( K \subset \Omega \), then the CPP holds for \( D_0 \).
Proof. For every compact subset $K \subset \Omega$ we define $\mathbb{D}(K, E)$ by

$$\mathbb{D}(K, E) := \mathbb{D}_0 \cap S(K, E).$$

Clearly $\mathbb{D}(K, E)$ is a domain in $S(K, E)$ since it an open star-like (hence connected) subset of $S(K, E)$. Also if $K$ and $L$ are compact subsets of $\Omega$ such that $K \subset L^\circ$, then we have the inclusions

$$\mathbb{D}(K, E) \hookrightarrow \mathbb{D}(L, E) \hookrightarrow \mathbb{D}_0.$$

Lemma 3.3. There are a sequence $(S_n)_{n \in \mathbb{N}}$ of compact subsets of $\Omega$ such that $S_n \subset S_{n+1}^\circ$ for all $n \in \mathbb{N}$ and $\Omega = \bigcup_{n \in \mathbb{N}} S_n$ and a sequence of functions $(\varphi_n)_{n \in \mathbb{N}}$ in $C_0(\Omega, E)$ such that $\varphi_n|_{S_n} \equiv 1$ and $\varphi_n|_{\Omega \setminus S_{n+1}} \equiv 0$ such that the following statement holds: For every $h \in C_0(\Omega, E)$ we have $h = \lim_{n \to \infty} h\varphi_n$ in the space $C_0(\Omega, E)$.

Proof. Combining the $\sigma$-compactness of $\Omega$ and Urysohn’s lemma we can easily construct sequences $(S_n)_{n \in \mathbb{N}}$ and $(\varphi_n)_{n \in \mathbb{N}}$ meeting the properties required in the first sentence of the lemma.

Let $h \in C_0(\Omega, E)$ and $\epsilon > 0$ be given. Then there is a compact set $K \subset \Omega$ such that $\sup_{\omega \in \Omega \setminus K} \|h(\omega)\| \leq \epsilon$, and for $n \in \mathbb{N}$ large enough we have $K \subset S_n$. Therefore

$$\|h - h\varphi_n\| = \sup_{\omega \in \Omega \setminus K} \|h(\omega) - h(\omega)\varphi_n(\omega)\| \leq \sup_{\omega \in \Omega \setminus K} \|h(1 - \varphi_n)\| \leq 2\epsilon$$

which shows that $\lim_{n \to \infty} h\varphi_n = h$ in the space $C_0(\Omega, E)$. □

Now we prove the proposition. Take sequences $(S_n)_{n \in \mathbb{N}}$ and $(\varphi_n)_{n \in \mathbb{N}}$ in accordance with (3.3). Note that the products $f\varphi_n$, $g\varphi_n$ belong to $\mathbb{D}_0$ due to the star-likeness. Since the Carathéodory distance in $\mathbb{D}_0$ is continuous, we have

$$C_{\mathbb{D}_0}(f, g) = \lim_{n \to \infty} C_{\mathbb{D}_0}(f\varphi_n, g\varphi_n), \quad f, g \in \mathbb{D}_0. \quad (5)$$

To simplify the notation, write $\mathbb{D}_n$ instead of $\mathbb{D}(S_n, E)$. Consider the maps $\mathbb{D}_n \xrightarrow{\varphi_n} \mathbb{D}_{n+1} \xrightarrow{i} \mathbb{D}_0$, where the arrows are the operator of multiplication by $\varphi_n$ and the canonical inclusion respectively. Note that $\text{supp}(\varphi_n h) \subset S_n$ so that $\varphi_n h \in \mathbb{D}_{n+1}$ for all $h \in \mathbb{D}_0$. By the contractive property of $\mathbb{D}_{n+1} \xrightarrow{i} \mathbb{D}_0$

$$C_{\mathbb{D}_0}(f\varphi_n, g\varphi_n) \leq C_{\mathbb{D}_{n+1}}(f\varphi_n, g\varphi_n), \quad n \in \mathbb{N}. \quad (6)$$

By taking upper limits and using (5) we get $C_{\mathbb{D}_0}(f, g) \leq \limsup_{n \in \mathbb{N}} C_{\mathbb{D}_{n+1}}(f\varphi_n, g\varphi_n)$. We shall prove that $\limsup_{n \in \mathbb{N}} C_{\mathbb{D}_{n+1}}(f\varphi_n, g\varphi_n) \leq \sup_{\omega \in \Omega} C_D[f(\omega), g(\omega)]$ from which the result follows. By assumption the CCP holds for every $S(K, E)$. Hence for every fixed $n \in \mathbb{N}$ we have

$$C_{\mathbb{D}_{n+1}}(f\varphi_n, g\varphi_n) = \max_{\omega \in \Omega} C_D[(f\varphi_n)(\omega), (g\varphi_n)(\omega)] = \max_{\omega \in S_{n+1}} C_D[(f\varphi_n)(\omega), (g\varphi_n)(\omega)]$$

$$= \max_{\omega \in S_n} \left[ C_D[(f\varphi_n)(\omega), (g\varphi_n)(\omega)] \right] \sup_{\omega \in S_{n+1} \setminus S_n} C_D[(f\varphi_n)(\omega), (g\varphi_n)(\omega)].$$

For $\omega \in S_n$ we have $\varphi_n(\omega) = 1$, therefore

$$\sup_{\omega \in S_n} C_D[(f\varphi_n)(\omega), (g\varphi_n)(\omega)] = \sup_{\omega \in S_n} C_D[f(\omega), g(\omega)] \leq \sup_{\omega \in \Omega} C_D[f(\omega), g(\omega)], \quad n \in \mathbb{N}.$$
On the other hand, the set \( K := \{ h \} \cup \{ \varphi_n ; n \in \mathbb{N} \} \) is compact for every \( h \in C_0(\Omega, E) \), hence (3.1) applies. Let \( \epsilon > 0 \) be given. There is a compact subset \( K \subset \Omega \) such that
\[
C_D((f \varphi_n)(\omega), (g \varphi_n)(\omega)) < \epsilon, \quad \omega \in \Omega \setminus K, \ n \in \mathbb{N}.
\]
For \( n \) large enough (say \( n \geq n_0 \)) we have \( K \subset S_n \subset S_{n+1} \subset \Omega \), therefore \( S_{n+1} \setminus S_n \subset \Omega \setminus K \) and so
\[
\sup_{\omega \in S_{n+1} \setminus S_n} C_D((f \varphi_n)(\omega), (g \varphi_n)(\omega)) \leq \epsilon, \quad n \geq n_0.
\]
Replacing this in (7) we get \( C_{D_{n+1}}(f \varphi_n, g \varphi_n) \leq \max \{ \sup_{\omega \in \Omega} C_D(f(\omega), g(\omega)), \ \epsilon \} \) for \( n \geq n_0 \). Since \( \epsilon \) was arbitrary, \( \limsup \sup_{n \in I} C_{D_{n+1}}(f \varphi_n, g \varphi_n) \leq \sup_{\omega \in \Omega} C_D(f(\omega), g(\omega)) \) which completes the proof. \( \square \)

**Example 3.4.** Take \( \Omega := \mathbb{N} \) with the discrete topology, and let \( D \) be a balanced domain in \( E \). Then \( D_0 \), the \( c_0(\mathbb{N}) \)-power of \( D \), is a balanced domain in \( c_0(\mathbb{N}, E) \) and it is easy to see that the assumptions in (3.2) are satisfied. Hence the CPP holds for \( C_D \).

## 4. Infinite product of a finite dimensional domain

Let \( I \) and \( E \) respectively be a set of indices and a normed space. As usually, we let \( \ell^\infty(I, E) \) be the vector space of all bounded sequences \((x_i)_{i \in I}\) with the supremum norm \( \| (x_i)_{i \in I} \| := \sup_{i \in I} \| x_i \| \). In this case, we can prove the following theorem

**Theorem 4.1.** Let \( E \) be a finite dimensional vector space with a norm. Let \( D \) be a bounded domain in \( E \) such that, for every \( r > 0 \) and for every \( a \in D \), the ball \( B_C(a, r) \) for the Carathéodory distance is relatively compact in \( D \). Let \( D := \Pi_{i \in I} D_i \), where \( D_i \) is a copy of \( D \). More precisely,
\[
D := \{ (x_i)_{i \in I} ; x_i \in D \text{ and } \exists \eta > 0 \text{ such that } \forall i \in I \ d(x_i, \partial D) > 0 \}.
\]

Then
\[
C_D((x_i)_{i \in I}, (y_i)_{i \in I}) = \sup_{i \in I} C_D(x_i, y_i).
\]

**Proof.** The inequality \( \geq \) is trivial. Let \( \epsilon > 0 \). We have to prove that
\[
C_D((x_i)_{i \in I}, (y_i)_{i \in I}) \leq \sup_{i \in I} C_D(x_i, y_i) + \epsilon.
\]

First we get the following lemma

**Lemma 4.2.** Let \( a := (a, \ldots, a, \ldots) \) and \( b := (b, \ldots, b, \ldots) \) be constant sequences equal to \( a \) (resp. to \( b \)) in \( D \). Then \( C_D(a, b) = C_D(a, b) \)

**Proof.** Clear because there exists an inverse mapping \( D \to D \). \( \square \)

**Lemma 4.2.** Let \( c := (c, \ldots, c, \ldots) \in D \). Let \( B_C(c, r) \) be a ball for the Carathéodory distance in \( D \). Let \( a = (a, \ldots, a, \ldots) \) and \( b = (b, \ldots, b, \ldots) \) be two points in \( B_C(c, r) \). Then for all \( \epsilon > 0 \) there is an \( \eta > 0 \) such that, if \( (a_i)_{i \in I} \) and \( (b_i)_{i \in I} \) satisfy \( \| a_i - a \| < \eta \) and \( \| b_i - b \| < \eta \) for all \( i \in I \), then we have
\[
C_D((a_i)_{i \in I}, (b_i)_{i \in I}) \leq C_D(a, b) + \epsilon.
\]
Proof. By the triangle inequality, we get
\[ C_D((a_i)_{i \in I}, (b_i)_{i \in I}) \leq C_D((a_i)_{i \in I}, a) + C_D(a, b) + C_D(b, (b_i)_{i \in I}). \]
But there is some \( r_0, 0 < r_0 < r_1 \), such that for all \( d = (d, \ldots, d, \ldots) \in B_C(c, r) \) we have \( B(d, r_0) \subset D \subset B(d, r_1) \), and it is easy to prove the existence of \( \eta > 0 \) such that
\[ \|a_i - a\| < \eta \quad \forall i \in I \implies C_D(a, a) < \frac{\epsilon}{2}. \]
This implies the result. \( \square \)

Now we can end the proof of the theorem. For every \((a, b) \in B_C(c, r)^2 \) the ball \( B(a, \eta) \times B(b, \eta) \) covers \( \overline{B_C(c, r)^2} \) which is compact. We can extract a finite cover
\[ \overline{B_C(c, r)^2} \subset \bigcup_{j=1, \ldots, n} \bigtimes_{k=1, \ldots, m} B(d_j, \eta) \times B(e_k, \eta). \]
This enables us to define a partition \( I = \bigcup_{j=1, \ldots, n} \bigtimes_{k=1, \ldots, m} I_{j,k} \) with the property that, for all \( i \in I_{j,k} \) we have \( |x_i - d_j| < \eta \) and \( |y_i - e_k| < \eta \). Of course,
\[ C_D((x_i)_{i \in I}, (y_i)_{i \in I}) = \sup_{j,k} C_{D_{j,k}}((x_i)_{i \in I_{j,k}}, (y_i)_{i \in I_{j,k}}) \]
(where \( D_{j,k} \) is the product of copies of \( D \) over \( I_{j,k} \)) by the finite product property. But
\[ C_{D_{j,k}}((x_i)_{i \in I_{j,k}}, (y_i)_{i \in I_{j,k}}) \leq C_D(d_j, e_k) + \epsilon \]
and this proves the result. \( \square \)

References