Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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Existence and regularity of solutions of the $\bar{\delta}$-system on wedges of $\mathbb{C}^N$

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **10** (1999), n.4, p. 271–278.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1999_9_10_4_271_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1999.

Rend. Mat. Acc. Lincei
s. 9, v. 10:271-278 (1999)

Funzioni di variabile complessa. — *Existence and regularity of solutions of the $\bar\partial$-system on wedges of* $\mathbb{C}^N$. Nota di GIUSEPPE ZAMPIERI, presentata (*) dal Corrisp. C. De Concini.

ABSTRACT. — For a wedge W of $\mathbb{C}^N$, we introduce two conditions of weak q-pseudoconvexity, and prove that they entail solvability of the $\bar\partial$-system for forms of degree $\geq q+1$ with coefficients in $C^\infty(W)$ and $C^\infty(\overline{W})$ respectively. Existence and regularity for $\bar\partial$ in W is treated by Hörmander [5, 6] (and also by Zampieri [9, 11] in case of piecewise smooth boundaries). Regularity in $\overline{W}$ is treated by Henkin [4] (strong q-pseudoconvexity by the method of the integral representation), Dufresnoy [3] (full pseudoconvexity), Michel [8] (constant number of negative eigenvalues), and Zampieri [10] (more general q-pseudoconvexity and wedge type domains). This is an announcement of our papers [10, 11]; it contains refinements both in statements and proofs and, mainly, a parallel treatement of regularity in W and $\overline{W}$. All our techniques strongly rely on the method of L^2 estimates by Hörmander [5, 6].

KEY WORDS: L^2 estimates; Cauchy-Riemann system; C.R. structures.

RIASSUNTO. — *Esistenza e regolarità delle soluzioni del sistema $\bar\partial$ in «wedges» di* $\mathbb{C}^N$. Si introducono due condizioni di q-pseudoconvessità debole per un «wedge» di $\mathbb{C}^N$, e si dimostra che esse sono sufficienti per la risolubilità del sistema $\bar\partial$ per forme di grado $\geq q+1$ a coefficienti in $C^\infty(W)$ e $C^\infty(\overline{W})$ rispettivamente. Esistenza e regolarità in W per il $\bar\partial$ sono trattate da Hörmander [5, 6] (e anche da Zampieri [9, 11] per bordi C^2 a tratti). Regolarità in $\overline{W}$ è trattata da Henkin [4] (q-pseudoconvessità forte con il metodo della rappresentazione integrale), Dufresnoy [3] (pseudoconvessità «completa»), Michel [8] (costanza del numero di autovalori negativi) e Zampieri [10] (q-pseudoconvessità più generale e domini di tipo «wedge»). Questa è una nota preliminare agli articoli [10, 11]; contiene miglioramenti negli enunciati e nelle dimostrazioni e, soprattutto, una trattazione parallela della regolarità in W e $\overline{W}$. Tutte le tecniche qui impiegate si basano profondamente sul metodo delle stime L^2 introdotto da Hörmander in [5, 6].

Let W be a wedge of $\mathbb{C}^N$ defined, in a neighborhood of a point $z_o \in \partial W$ by $r_j < 0$, $j = 1, \dots, l$ with $\partial r_1 \wedge \cdots \wedge \partial r_l \neq 0$. We shall use the following notations: M_j will denote the hypersurfaces $\{r_j\} = 0$, $\widehat{M}_j$ the «faces» $M_j \cap \partial W$, R the union of the «wedges» $\{r_j = 0, r_i = 0 \text{ for } i \neq j\}$. $\bar\partial\partial r_j$ (resp. $\bar\partial\partial r_j|_{\partial r_j^\perp}$) will denote the Levi form of the function r_j (resp. of the hypersurface M_j) where $\partial r_j^\perp$ denotes the plane orthogonal to ∂r_j *i.e.* the complex tangent plane to M_j. We shall formulate two different conditions of weak q-pseudoconvexity. For an orthonormal system of $(1,0)$-forms $\omega' = \{\omega_1 \dots \omega_q\}$ on ∂W at z_o whose dual tangent derivations $\partial_{\omega'}$ verify $\mathrm{Span}\{\partial_{\omega'}\}|_{\widehat{M}_i} \subset T^{1,0}M_i\ \forall i$, which are $C^0(\partial W) \cap C^2(\partial W \setminus R)$ with bounded first and second derivatives, and for an orthonormal completion ω'' (possibly different on each

(*) Nella seduta del 14 maggio 1999.

M_i), we have

(1) $$\bar\partial\partial r_i(z)(\bar v'', v') = 0, \quad \bar\partial\partial r_i(z)(\bar v'', v') \geq 0, \\ \forall v = (v', v'') \in \partial r_j^{\perp} \; (= \mathbb{C}^{N-1}) \quad \forall z \in \widehat{M}_i \cap U.$$

(Here $\{U\}$ denotes a system of neighborhoods of z_o). We shall also deal with a slight improvement of (1):

(2) We have (1) and $\bar\partial\partial r_i(z)(\bar v'', v'') \geq \bar\partial\partial r_i(z)(\bar v', v')$ $\forall z \in \widehat{M}_i$ $\forall |v'|=1, |v''|=1$.

Note that (1) means that Span$\{\partial_{\omega'}\}$ is engendred by a system of eigenvectors (the first q in case of (2)) which contains all negative ones.

Remark 1. It shall be clear from our proofs that we can allow a «thin» set R' of C^0-discontinuity for the coefficients of the forms $\{\omega_i'\}$ *i.e.* a set verifying codimension$_{\partial W} R' \geq 2$.

Let us discuss our conditions (1) and (2) by means of some examples. We assume that W is a half-space $\{r < 0\}$ with C^4-boundary $M = \{r = 0\}$, denote by $\mu_1(z) \leq \mu_2(z) \dots$ the ordered eigenvalues of the Levi form $L_M(z) := \bar\partial\partial r(z)|_{\partial r(z)^{\perp}}$ and let $s^+(z)$, $s^-(z)$, $s^0(z)$ be the numbers of its positive, negative and null eigenvalues respectively. With these notations it is clear that a sufficient condition for (2) is

(3) $$\mu_q(z) < \mu_{q+1}(z) \text{ and } \mu_{q+1}(z) \geq 0 \;\; \forall z \in M.$$

As for (3) three many cases are given.

(*a*) $q = N - 1 - s^+(z_o)$ (strong q-pseudoconvexity). In this case (3) clearly holds.

(*b*) $q = s^-(z_o)$. The first of (3) holds but the second generally fails.

(*c*) $q \equiv s^-(z)$ $\forall z$. Then (3) clearly holds.

We can also easily exhibit an example, in $\mathbb{C}^4$ in which (1) holds (for $q = 2$) but (2) fails:

$$W = \{z \in \mathbb{C}^4 : x_1 > -|z_2|^2 + x_2|z_3|^2\}$$
$$\text{Span}\{\partial_{\omega'}\} = \text{ the projection of Span}\{\partial_{z_2}, \partial_{z_3}\} \text{ on } T^{1,0}M.$$

We come back to the general case of a wedge W and aim to rephrase (1) and (2) into properties for an exhaustion function of W. We choose complex coordinates $z = x + \sqrt{-1}y$ in $\mathbb{C}^N$ and represent ∂W as a graph $x_1 = h(y_1, z_2, \bar z_2, \dots)$, W as $x_1 > h$ and each M_i as $x_1 = h_i$. We put $r = -x_1 + h$, $\delta = -r$ and define $\phi = -\log\delta + \lambda|z|^2$ (λ a large constant to be fixed in the following). Let $S := R + \mathbb{R}_{x_1} = \{z \in W : h_i = h_j \text{ for } i \neq j\}$. S is a manifold with boundary whose conormal n_S at generic points verifies

$$n_S = \frac{\partial(h_i - h_j)}{|\partial(h_i - h_j)|} \left(= \mathcal{J}(\frac{\partial r}{|\partial r|})\right) \text{ the «jump» of } \frac{\partial r}{|\partial r|} \text{ between the «}i \text{ and } j \text{ sides of } S\text{».}$$

We shall deal with vectors $(w_J)_J$ $(J = (j_1 \dots j_k))$ with alternate complex coefficients. We also extend the forms $\{\omega'\}$ of (1), (2) from ∂W to W in a neighborhood of z_o by prescribing a constant value on the fibers of the projection $W \to \partial W$ $z \mapsto z^*$ along the

x_1-axis and complete to a full system of forms ω', ω'' in $T^{1,0}X|_W$. (Here the forms ω'' have rank $N-q$ and not $N-1-q$ as was the case of (1), (2)). We shall denote by ϕ_{ji} the matrix of $\overline{\partial}\partial\phi$ in the basis ω', ω''.

THEOREM 2. (*i*) *Let* (1) *hold. Then for* $k \geq q+1$, *we have* (*in a neighborhood of* z_o *on* W):

$$\sum_{|K|=k-1}{}' \sum_{i \text{ or } j \geq q+1} \phi_{ji}\overline{w}_{jK} w_{iK} \geq \lambda |w|^2 \ \ \forall\, z \in W \setminus S,\ \forall (w_J). \tag{4}$$

(*ii*) *Let* (2) *hold. Then for* $k \geq q+1$, *we have*:

$$\sum_{|K|=k-1}{}' \sum_{ij=1,\dots N} \phi_{ji}\overline{w}_{jK} w_{iK} - \sum_{|J|=k}{}' \sum_{j\leq q} \phi_{jj}|w_J|^2 \geq \lambda |w|^2 \ \ \forall\, z \in W \setminus S,\ \forall (w_J). \tag{5}$$

PROOF. We begin by the proof of (*ii*) which is more involved. We first observe that

$$\overline{\partial}\partial r(z)|_{\partial r(z)^\perp} = \overline{\partial}\partial r(z^*)|_{\partial r(z^*)^\perp} \ \ \forall\, z \in W, \tag{6}$$

where $z \mapsto z^*$ is the projection along $\mathbb{R}_{x_1}$. Let $\lambda_1 \leq \lambda_2 \leq \dots$ and $\mu_1 \leq \mu_2 \leq \dots$ denote the eigenvalues of $\overline{\partial}\partial\phi$ and $\overline{\partial}\partial r|_{\partial r^\perp}$ respectively. Note that

$$\overline{\partial}\partial\phi = \delta^{-1}\overline{\partial}\partial r + \delta^{-2}\overline{\partial} r \wedge \partial r + \lambda\, d\overline{z} \wedge dz\,. \tag{7}$$

In particular by (6), (7) the eigenvalues of $\overline{\partial}\partial\phi(z)|_{\partial r(z)^\perp}$ are $\delta^{-1}\mu_j(z^*) + \lambda$. Also, by (7) if ∂^τ and ∂^ν denote the derivatives of type $(1,0)$ normal to (resp. parallel to) ∂r on $W \setminus S$, then we have for a suitable c:

$$\overline{\partial}\partial\phi \geq \delta^{-1}\overline{\partial}^\tau\partial^\tau r - c\, d\overline{z}^\tau \wedge dz^\tau + \lambda\, d\overline{z} \wedge dz\,. \tag{8}$$

It follows

$$\sum_{j=1,\dots,k} \lambda_j(z) \geq \delta^{-1} \sum_{j=1,\dots,k} \mu_j(z^*) + (\lambda - c)k\ , \tag{9}$$

and

$$\sum_{j=1,\dots,q} \phi_{jj}(z) = \delta^{-1} \sum_{j=1,\dots,q} \mu_j(z^*) + \lambda q\,. \tag{10}$$

In conclusion the left hand side (I) of (5) verifies (for $|w| = 1$)

$$(I) \geq \left(\sum_{j=1,\dots,k} \lambda_j(z) - \sum_{j=1,\dots,q} \phi_{jj}(z) \right) \geq \delta^{-1} \sum_{j=q+1,\dots,k} \mu_j(z^*) + \lambda(k-q) - ck \geq$$

$$\geq \lambda' \quad \text{(for suitable } \lambda \text{ and for a new } \lambda'\text{)}.$$

(*i*): Let us put $w'_K = (w_{jK})_{j\leq q}$, $w''_K = (w_{jK})_{j \geq q+1}$. Then (8) implies, on account of (6) and (1)

$$\sum_{i \text{ or } j\geq q+1} \phi_{ji}\overline{w}_{jK} w_{iK} \geq -c(|w'_K|^2 + |w''_K|^2) + \lambda |w''_K|^2. \tag{11}$$

Observe now that any J with $J = k \geq q + 1$ can be written as $J = jK$ for some $j \geq q + 1$; hence $\sum'_{|K|=k-1} |w''_K|^2 \geq c'|w|^2$. It follows that if we take summation of (11) over K we get (4) for a new λ. □

THEOREM 3. (*i*) *Let W be a wedge of $\mathbb{C}^N$ at z_o which satisfies* (1). *Then for any $\overline{\partial}$-closed form f of degree $k \geq q + 1$ with $C^\infty(W \cap U)$-coefficients, there is a solution u of degree $k - 1$ with $C^\infty(W \cap U)$-coefficients to the equation $\overline{\partial} u = f$.*

(*ii*) *Let W satisfy* (2) *at z_o. Then the same statement as above holds for forms with $C^\infty(\overline{W \cap U})$-coefficients.*

PROOF OF THEOREM 3 (*ii*). We start by (*ii*) which is more difficult. We denote by $L^2_\phi(W)$ (ϕ a real positive function) the space of square integrable functions on W in the measure $e^{-\phi}\, dV$ (dV being the Euclidean element of volume). We denote by $||\cdot||_\phi$ the norm in the above space. We denote by $L^2_\phi(W)^k$ the space of $(0\,, k)$-forms $f = \sum'_{|J|=k} f_J \overline{\omega}_J$ with coefficients f_J in $L^2_\phi(W)$. (Here $\sum'$ denotes summation over ordered indices, $\{\omega_j\}$ denotes a basis of $(1\,, 0)$ forms, and finally $\overline{\omega}_J = \overline{\omega}_{j_1} \wedge \cdots \wedge \overline{\omega}_{j_k}$. Also the forms ω_j's are supposed to fulfill all assumptions in Theorem 2 and in particular have bounded first and second derivatives in $\overline{W} \cap U$). We denote by (ϕ_{ji}) the matrix of the Hermitian form $\overline{\partial}\partial\phi$ in the chosen basis. If ψ is another real positive function, which shall be fixed according to our future need, we shall deal with the complex of closed densely defined operators

$$L^2_{\phi-2\psi}(W)^{k-1} \overset{\overline{\partial}}{\to} L^2_{\phi-\psi}(W)^k \overset{\overline{\partial}}{\to} L^2_\phi(W)^{k+1}. \tag{12}$$

We denote by $\overline{\partial}^*$ the adjoint of $\overline{\partial}$ and also define the operator $\delta_{\omega_j}(\cdot) = e^\phi \partial_{\omega_j}(e^{-\phi}\cdot)$. We have

$$\begin{aligned} \overline{\partial}^* f &= -\sum'_{|K|=k-1} \sum_{j=1,\dots,N} e^{-\psi} \delta_{\omega_j}(f_{jK})\overline{\omega}_K - \sum'_{|K|=k-1} \sum_{j=1,\dots,N} e^{-\psi} f_{jK} \partial_{\omega_j}\psi \overline{\omega}_K + e^{-\psi} R_f \\ \overline{\partial} f &= \sum'_{|J|=k} \sum_{j=1,\dots,N} \overline{\partial}_{\omega_j}(f_J)\overline{\omega}_j \wedge \overline{\omega}_J + R_f\,, \end{aligned} \tag{13}$$

where R_f are errors which involve products of the f_J's by derivatives of coefficients of the ω_j's. By means of (13) we then get the following estimate which generalizes [6, (4.2.8)]

$$\begin{aligned} &\sum'_{|K|=k-1} \sum_{ij=1,\dots,N} \int_W e^{-\phi}(\delta_{\omega_i}(f_{iK})\overline{\delta_{\omega_j}(f_{jK})} - \overline{\partial}_{\omega_j}(f_{iK})\overline{\overline{\partial}_{\omega_i}(f_{jK})})\, dV + \sum'_{|J|=k} \sum_{j=1,\dots,N} \int_W e^{-\phi} \\ &\qquad |\overline{\partial}_{\omega_j}(f_J)|^2\, dV \leq 3||\overline{\partial}^* f||^2_{\phi-2\psi} + 2||\overline{\partial} f||^2_\phi + c||f||^2_\phi + 3|||\partial\psi|f||^2_\phi\,, \end{aligned} \tag{14}$$

where c depends on the sup-norm of the derivatives of the coefficients of the ω_j's over the support of f. Since these are bounded in $\overline{W} \cap U$ (maybe with a smaller $U' \subset\subset U$), then there is an uniform c $\forall f \in C_c^\infty(W \cap U)^k$. We have the commutation relations

$$(15)\qquad [\delta_{\omega_i}, \overline{\partial}_{\omega_j}] = \partial_{\overline{\omega}_j}\partial_{\omega_i}\phi + \sum_h c_{ji}^h \partial_{\omega_h} - \sum_h \overline{c}_{ij}^h \partial_{\overline{\omega}_h} = \phi_{ji} + \sum_h c_{ji}^h \delta_{\omega_h} - \sum_h \overline{c}_{ij}^h \partial_{\overline{\omega}_h},$$

for suitable functions c_{ji}^h. We apply (15) to all terms in the first sum on the left of (14) and to the terms with $j \leq q$ in the second. We obtain, if f belongs to $C_c^\infty(W)^k$:

$$\sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \cdot + \sum_{|J|=k}' \sum_{j=1,\dots,N} \cdot =$$

$$= \left(\sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \int_\Omega e^{-\phi}\phi_{ji} f_{iK}\overline{f}_{jK}\, dV - \sum_{|J|=k}' \sum_{j\leq q} \int_\Omega e^{-\phi}\phi_{jj}|f_J|^2\, dV \right) +$$

$$(16)\qquad + \left(\sum_{|J|=k}' \sum_{j\leq q} ||\delta_{\omega_j} f_J||_\phi^2 + \sum_{|J|=k}' \sum_{j\geq q+1} ||\partial_{\overline{\omega}_j} f_J||_\phi^2 \right) +$$

$$+ \left(\sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \int_S e^{-\phi} J(\partial_{\omega_i}\phi)\overline{n}_j f_{iK}\overline{f}_{jK}\, dS - \sum_{|J|=k}' \sum_{j\leq q} \int_S e^{-\phi} J(\partial_{\omega_j}\phi)\overline{n}_j |f_J|^2\, dS \right) +$$

$$+ \text{Error},$$

where the error term has the estimate

$$(17)\qquad |\text{Error}| \leq \left(\sum_{|J|=k}' \sum_{j\leq q} ||\delta_{\omega_j} f_J||^2 + \sum_{|J|=k}' \sum_{j\geq q+1} ||\overline{\partial}_{\omega_j} f_J||^2 \right) + c||f||_\phi^2,$$

(where c depends now also on the second derivatives of the coefficients of the ω_i's). Note that $n' = 0$ whence $\sum_{|J|=k}' \sum_{j\leq q} \int_S \cdot\, dS = 0$. Also, since $n = \frac{\mathcal{J}(\partial\phi)}{|\mathcal{J}(\partial\phi)|}$, then $\sum_{ij=1,\dots,N} \mathcal{J}(\partial_{\omega_i}\phi)\overline{n}_j f_{iK}\overline{f}_{jK}$ is a square; hence the third term on the right of (16) is positive (thus negligeable). Assume that ϕ satisfies (5) of Theorem 2 on the whole W. Then by (16) we have the estimate:

$$(18)\qquad \sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \cdot + \sum_{|J|=k}' \sum_{j=1,\dots,N} \cdot \geq \lambda||f||_\phi^2 - c||f||_\phi^2.$$

By plugging together (14) and (18) we get with a new c

$$(19)\qquad \lambda||f||_\phi^2 \leq 3||\overline{\partial}^* f||_{\phi-2\psi}^2 + 2||\overline{\partial} f||_\phi^2 + c||f||_\phi^2 + 3|||\partial\psi|f||_\phi^2 \quad \forall f \in C_c^\infty(W)^k.$$

We fix now a compact subset $K \subset\subset W$, and choose ψ according to [6, Lemma 4.1.3]; (in particular we can choose $\psi|_K \equiv 0$). This ensures density of C_c^∞ into L^2-forms; hence now (19) holds for L^2 instead of C_c^∞ forms. We assume w.l.o.g. that $K = \{\phi \leq n\}$; we replace the above ϕ by

$$\chi(\phi) + (3 + c)|z|^2,$$

where χ is a positive convex function of a real argument t which satisfies:

$$\begin{cases} \chi(t) \equiv 0\,, & \text{for } t \le n \\ \dot\chi(t) \ge \sup\limits_{\{z:\phi(z)\le t\}} \dfrac{3(|\partial\psi|^2 + e^{\psi} - 1)}{\lambda}\,, & \text{for } t \ge n. \end{cases}$$

Under this choice of ϕ and ψ we conclude, for $k \ge q+1$,

$$||f||^2_{\phi-\psi} \le ||\overline\partial^* f||^2_{\phi-2\psi} + ||\overline\partial f||^2_{\phi} \quad \forall f \in D^k_{\overline\partial} \cap D^k_{\overline\partial^*}\,, \tag{20}$$

where $D^k_{\overline\partial}$ and $D^k_{\overline\partial^*}$ are the domains in $L^2_\phi(W)^k$ of $\overline\partial$ and $\overline\partial^*$ respectively. Moreover for any compact subset $K \subset\subset \Omega$, we may choose $\psi|_K \equiv 0$ and $\phi|_K \equiv c|z|^2$ where we still write c instead of $3+c$. Let us point out that the estimate (20), with the additional condition $\phi|_K \equiv c|z|^2$, will be the main ingredient of our proof. Let us recall that it was obtained by assuming that ϕ satisfies (5) on the whole $W \setminus S$.

End of proof of Theorem 3 (*ii*). We come back to our wedge W which satisfies (2). We suppose W be locally defined by $-x_1 + h < 0$ and then set $W_\nu = \{-x_1 + h < \frac{\eta^{2^\nu}}{2}\}$ for $0 < \eta < \frac{1}{2}$. Let U_ν (resp. U) be the sphere with center z_o and radius $\rho + \frac{\eta^{2^\nu}}{2}$ (resp. ρ) with ρ small. By an easy variant of Theorem 2 (*ii*), the functions $\phi := -\log(-r + \frac{\eta^{2^\nu}}{2}) + \lambda|z|^2 + \log\big(-|z-z_o|^2 + (\rho + \frac{\eta^{2^\nu}}{2})^2\big)$ will be exhaustion functions for the domains $W_\nu \cap U_\nu$ which satisfy (5) (globally). Thus (20) holds on each $W_\nu \cap U_\nu$. This easily implies that for $k \ge q+1$ and for any form $f \in L^2_{c|z|^2}(W_\nu \cap U_\nu)^k$ with $\overline\partial f = 0$, there exists $u \in L^2_{c|z|^2}(W_\nu \cap U_\nu)^{k-1}$ such that

$$(\overline\partial u = f\ ,\ \overline\partial^* u = 0) \quad ||u||^2_{c|z|^2} \le ||f||^2_{c|z|^2}\,. \tag{21}$$

We note now that

$$\{z \in \mathbb{C}^N : \text{dist}(z, W) < \eta^{2^{\nu+1}}\} \subset W_\nu \subset \left\{z \in \mathbb{C}^N : \text{dist}(z, W) < \frac{\eta^{2^\nu}}{2}\right\}, \tag{22}$$

(in a neighborhood of z_o). According to [3] we can show that (21) implies, by the aid of (22), that for $k \ge q+1$, for $f_\nu \in C^\infty(\overline{W_\nu \cap U_\nu})^k$ with $\overline\partial f_\nu = 0$, there is $u_\nu \in C^\infty(\overline{W_{\nu+1} \cap U_{\nu+1}})^{k-1}$ such that

$$(\overline\partial u_\nu = f_\nu\ ,\ \overline\partial^* u_\nu = 0) \quad ||u_\nu||_{(s+1)} \le \frac{M_s}{\eta^{2^{\nu+1}(s+1)}} ||f_\nu||_{(s)}\,, \tag{23}$$

(where $||u_\nu||_{(s+1)}$ (resp. $||f_\nu||_{(s)}$) are the norms in the Sobolev spaces $H^{s+1}(W_{\nu+1} \cap U_{\nu+1})$ (resp. $H^s(W_\nu \cap U_\nu)$).

We are ready to conclude. Let $f \in C^\infty(\overline{W} \cap U_1)^k$ satisfy $\overline\partial f = 0$. Extend f to $\tilde f$ such that

$$||\overline\partial \tilde f||_{(s)} \le M_{rs}\eta^{r2^\nu} \text{ on } W_\nu \cap U_\nu \text{ for any } r,\ s \text{ and for suitable } M_{rs}.$$

However $\tilde f$ is no more $\overline\partial$-closed. To overcome this problem we take a solution h_ν on

$W_{\nu+1} \cap U_{\nu+1}$ of

$$\begin{cases} \overline{\partial} h_\nu = \overline{\partial} \tilde{f} \\ ||h_\nu||_{(s+1)} \leq M_s (\eta^{2^{\nu+1}})^{-s-1} ||\overline{\partial}\tilde{f}||_{(s)} \,, \end{cases}$$

provided by (23). Now $\overline{\partial}(\tilde{f} - h_\nu) = 0$. We then solve on W_2 the equation $\overline{\partial} g_1 = \tilde{f} - h_1$, and, inductively on $W_{\nu+2} \cap U_{\nu+2}$:

$$\overline{\partial} g_{\nu+1} = h_\nu - h_{\nu+1} \,,$$

with the estimates

$$||h_{\nu+1}||_{(s+2)} \leq M_{s+1} (\eta^{2^{\nu+2}})^{-(s+2)} ||h_\nu - h_{\nu+1}||_{(s+1)} \leq M'_s (\eta^{2^{\nu+2}})^{-2s-3} M_{rs} \eta^{r2^\nu} \leq$$
$$\leq M'_{rs} \frac{1}{2^\nu} \quad (r,\ \nu \text{ large }).$$

Therefore $\sum_{\nu=1}^{\infty} g_\nu$ converges in $C^\infty(\overline{W} \cap U)$ and solves on $\overline{W} \cap U$ the equation:

$$\overline{\partial}\left(\sum_{\nu=1}^{\infty} g_\nu\right) = \tilde{f} - \lim_\nu h_\nu = \tilde{f} \,. \quad \square$$

PROOF OF THEOREM 3 (i). We shall prove that if there is an exhaustion function ϕ which satisfies (4) globally on $W \setminus S$, then an estimate of type (20) will still hold. But in this case we shall have $\phi|_K = c|z|^2$, $c = c_K$; *i.e.* c will be no more uniform on compact subsets of $\overline{W}$. However this suffices for $C^\infty(W)$ regularity of $\overline{\partial}$ [5, 6]. We recall (14) and decompose the term in the left side as

$$\sum_K{}' \sum_{ij} \cdot + \sum_J{}' \sum_j \cdot = \sum_K{}' \sum_{i \text{ or } j \geq q+1} \cdot + (1-\epsilon)\left(\sum_K{}' \sum_{i,j \leq q} \cdot + \sum_J{}' \sum_{j \leq q} \cdot\right) +$$
$$+ \epsilon \sum_K{}' \sum_{i,j\leq q} \cdot + \left(\epsilon \sum_J{}' \sum_{j \leq q} \cdot + \sum_J{}' \sum_{j \geq q+1} \cdot\right).$$

We apply (15) to the first term in the right

$$\sum_K{}' \sum_{i \text{ or } j\geq q+1} \cdot = \sum_K{}' \sum_{i \text{ or } j \geq q+1} \int_W e^{-\phi} \phi_{ji} \overline{f}_{jK} f_{iK}\, dV +$$
$$+ \sum_K{}' \sum_{i \text{ or } j \geq q+1} \int_S e^{-\phi} \mathcal{J}(\partial_{\omega_i}\phi) \overline{n}_j f_{iK} \overline{f}_{jK}\, dS + \text{Error}\,.$$

Note that the projection of $n = \frac{\mathcal{J}(\partial\phi)}{|\mathcal{J}(\partial\phi)|}$ on the plane of $\text{Span}\{\partial_{\omega'}\}$ is 0. Hence the term which involves $\int_S \cdot$ is a square. On the other hand if ϕ satisfies (4) we have

$$\sum_K{}' \sum_{i \text{ or } j \geq q+1} \int_W e^{-\phi} \phi_{ji} \overline{f}_{jK} f_{iK}\, dV \geq \lambda ||f||^2_\phi \,.$$

We remark now that $(1-\epsilon)(\cdot)$ equals $||\overline{\partial}'^* f||^2_\phi + ||\overline{\partial}' f||^2_\phi$ up to a term $|||\partial\psi| f||^2_\phi$ + Error. Also if ν is an upper bound for the $|\phi_{ji}|$ for $i, j \leq q$, then $\epsilon(\cdot) \geq -\epsilon\nu ||f||^2_\phi$ +

$+\,\epsilon\sum_J{}'\sum_{ij\le q}||\bar\partial_{\omega_j}f_j||^2_\phi$ + Error. Collecting all together:

$$(\lambda-\epsilon\nu)||f||^2_\phi\le 3||\bar\partial^* f||^2_{\phi-2\psi}+2||\bar\partial f||^2_\phi+\epsilon^{-1}c||f||^2_\phi+4|||\partial\psi|f||^2_\phi.$$

We then choose $\epsilon=\frac{\lambda}{2\nu}$ and replace ϕ by $\chi(\phi)+6|z|^2$ where

$$\dot\chi(t)\ge\sup_{\{z:\phi(z)\le t\}}\frac{2}{\lambda}\left(\frac{2\nu c}{\lambda}+4|\partial\psi|+3e^\psi-3\right).$$

This gives the same estimate as (20) (but with no uniform control for ϕ). With this estimate in hands we get existence in L^2 and then gain of one derivative for solutions of the system $(\bar\partial,\bar\partial^*)$, in the same way as in Theorem 3 (*ii*). This entails existence in C^∞. □

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Pervenuta il 23 ottobre 1998,
in forma definitiva il 13 maggio 1999.

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