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Some existence results for the scalar curvature problem via Morse theory

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Analisi matematica. — *Some existence results for the scalar curvature problem via Morse theory.* Nota (*) di Andrea Malchiodi, presentata dal Corrisp. A. Ambrosetti.

Abstract. — We prove existence of positive solutions for the equation $-\Delta_{g_0} u + u = (1 + \varepsilon K(x)) u^{2^*-1}$ on S^n, arising in the prescribed scalar curvature problem. Δ_{g_0} is the Laplace-Beltrami operator on S^n, 2^* is the critical Sobolev exponent, and ε is a small parameter. The problem can be reduced to a finite dimensional study which is performed with Morse theory.

Key words: Elliptic equations; Critical exponent; Scalar curvature; Perturbation method; Morse theory.

Riassunto. — *Alcuni risultati di esistenza per il problema della curvatura scalare tramite la teoria di Morse.* Si dimostra l'esistenza di soluzioni positive per l'equazione $-\Delta_{g_0} u + u = (1 + \varepsilon K(x)) u^{2^*-1}$ su S^n, che nasce del problema della curvatura scalare prescritta. Δ_{g_0} è l'operatore di Laplace-Beltrami su S^n, 2^* è l'esponente critico di Sobolev, ed ε un parametro piccolo. Il problema si riduce a uno studio finito-dimensionale che è affrontato con la teoria di Morse.

1. Introduction and statement of the results

In this *Note* we state some existence results for the problem on S^n

$$-4\frac{(n-1)}{(n-2)}\Delta_{g_0} u + Ru = Su^{\frac{n+2}{n-2}}, \qquad u > 0, \tag{1.1}$$

where Δ_{g_0} is the Laplace-Beltrami operator on S^n, and $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Such a problem, which has been widely investigated, arises in Differential Geometry, when the metric g of a Riemannian manifold M of dimension greater or equal than 3, with scalar curvature R, is conformally deformed to a metric with prescribed scalar curvature S.

Some difficulties arise in studying this problem by means of variational methods, because of the lack of compactness, and some topological obstructions may occur, see [11].

We consider the case when S is close to a constant, *i.e.* when S is of the form $1 + \varepsilon K$ wiht $|\varepsilon|$ small. Using the stereographic projection, the problem reduces to find solutions of

$$\begin{cases} -4\dfrac{(n-1)}{(n-2)}\Delta u = (1 + \varepsilon K(x)) u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n, \\ u > 0,\ u \in D^1(\mathbb{R}^n). \end{cases} \tag{1.2}$$

We will be concerned with functions K which have nondegeneracy properties between some levels. So we introduce the condition

$$x \in Crit(K) \cap K_a^b \qquad \Rightarrow \qquad \Delta K(x) \neq 0, \tag{L_a^b}$$

(*) Pervenuta in forma definitiva all'Accademia il 14 luglio 1999.

where $Crit(K) = \{K' = 0\}$, $a, b \in \mathbb{R}$, and $K_a^b = \{a \leq K \leq b\}$. Our main results are the following Theorems 1.1 and 1.2.

THEOREM 1.1. *Suppose* $K \in C^2(\mathbb{R}^n)$ *is a Morse function which satisfies* (L_a^b) *with* $a = \inf K$ *and* $b = \sup K$. *For* $j = 0, \dots, n-1$, *let* D_j *denote the number of critical points of* K *with Morse index* $n - j$ *and with* $\Delta K < 0$. *Suppose that* K *satisfies*

$$\sum_{j=0}^{q} (-1)^{q-j} D_j - (-1)^q \leq -1\,, \qquad \text{for some} \quad q = 1, \dots, n-1. \tag{1.3}$$

Then for $|\varepsilon|$ *sufficiently small, Problem* (1.2) *has solution.*

When $n = 2$ (hence $q = 1$), (1.3) becomes $D_0 > D_1 + 1$. In [6], for $n = 2$, it has been introduced the condition $D_0 \neq D_1 + 1$. Thus our result can be wiewed as a partial extension of it (see also [15] for $n = 3$). It is worth pointing out that condition (1.3) is different from the well known assumption in [3], which also extends [6], see (2.2) below.

If x is a critical point of K, we define $m(x, K)$ to be the Morse index of K at x.

THEOREM 1.2. *Suppose that* K *has a local minimum* x_0, *and that there exists* x_1 *with* $K(x_1) \leq K(x_0)$. *Suppose also that there exists a curve* $x(t) : [0, 1] \to \mathbb{R}^n$ *with* $x(0) = x_0$, $x(1) = x_1$, *such that, letting* $a = K(x_0)$, $b = \max_t K(x(t))$, *the following condition holds*

$$z \in Crit(K) \cap K_a^b\,,\ m(z, K) = 1 \quad \Rightarrow \quad \Delta K(z) < 0. \tag{1.4}$$

Suppose also that K *is a Morse function in* K_a^b, *and that condition* (L_a^b) *is satisfied. Then for* $|\varepsilon|$ *small, Problem* (1.2) *admits a solution.*

In [4] there is a non perturbative existence result similar to Theorem 1.2, but condition (1.4) is required for all the saddle points in K_a^b, and not only for the critical points with Morse index 1. For $n = 2$, analogous results have been previously given in [5] under the assumption that $\Delta K < 0$ at all the saddle points of K, and in [10] under the hypothesis that there is no critical point of K in $\{a < K < b\}$.

The proofs rely on an abstract perturbation result developed in [1], see also [2] for an application to (1.2), which leads to study a reduced, finite dimensional functional Γ. We show that Morse theory under general boundary conditions (see [8]) applies to Γ, and allows us to obtain the preceding results. An infinite dimensional Morse theoretical approach has been used to face the scalar curvature problem in [9] for $n = 2$, and in [15] for $n = 3$. The new feature here is that we can deal with all dimension, and that, differently from [15], we can also restrict our attention to some prescribed levels of K, and work with relative homology.

2. OUTLINE OF THE PROOFS AND GENERALIZATIONS

Solutions are found as critical points of some functional $f_\varepsilon(u) = f_0(u) - \varepsilon G(u)$, where the f_0 possesses a manifold Z of critical points, $Z \simeq \{(\mu, \xi), \mu > 0, \xi \in \mathbb{R}^n\}$. For $|\varepsilon|$

small, it is shown that Z perturbs to a manifold Z_ε which is a natural constrain for f_ε, and $f_\varepsilon|_{Z_\varepsilon} = b - \varepsilon\Gamma(z) + o(\varepsilon)$, where b is a constant. Solutions are obtained, roughly, by finding «stable» critical points of Γ.

The behaviour of the function Γ has been studied in [2]: we are particularly interested in the following proposition.

PROPOSITION 2.1. *The function* Γ *can be extended to the hyperplane* $\{\mu = 0\}$ *by setting* $\Gamma(0, \xi) = c_0 K(\xi)$, $c_0 > 0$. *Moreover, for some* $c_1 > 0$ *there holds*

$$\Gamma_\mu(0, \xi) = 0, \quad \Gamma_{\mu\xi_i}(0, \xi) = 0, \quad \Gamma_{\mu\mu}(0, \xi) = c_1 \Delta K(\xi); \quad \forall \xi \in \mathbb{R}^n. \tag{2.1}$$

Using Proposition 2.1, one can study the gradient flow of Γ on the boundary of a great ball B which is close to ∂Z. The flow is inward B when $\Delta K > 0$, and is outward B when $\Delta K < 0$. This enables us to prove the following proposition.

PROPOSITION 2.2. *Suppose* $K \in C^2(\mathbb{R}^n)$ *is a Morse function in* K_a^b, *and such that* (L_a^b) *holds. For* $s > 0$, *let* $\widetilde{B}_s$ *the the* $n+1$*-dimensional ball centred in* $((s^2+1)/2s, 0)$ *and with radius* $(s^2-1)/2s$. *Then, for* s *sufficiently large,* Γ *satisfies the general boundary conditions on* $B \equiv \widetilde{B}_s$ *between the levels* a *and* b.

Theorems 1.1 and 1.2 are proved using Morse inequalities for manifolds with boundary; with the same method, we can also prove existence if K is a Morse function which satisfies

$$\sum_{x \in Crit(K), \Delta K(x) < 0} (-1)^{m(x,K)} \neq (-1)^n. \tag{2.2}$$

This condition has been used in [3] for $n = 3$, and in [6] for $n = 2$; in the case $n > 3$ there are analogous results under some flatness assumptions, see [12, 13, 2]. Other perturbation results have been given in [7].

Theorem 1.2 can be easily generalized to the following situation, where existence of critical points of Morse index 1 and with positive Laplacian is admitted.

THEOREM 2.3. *Suppose* K *possesses a local minimum* x_0 *and* l *connected components* $A_1, \dots, A_l$ *of* $(K^{K(x_0)} \setminus x_0)$. *For* $i = 1, \dots, l$, *let* $c_i : [0, 1] \to S^n$ *be a curve with* $c_i(0) = x_0$, $c_i(1) \in A_i$; *set* $a = K(x_0)$, $b = \sup_i \sup_t K(c_i(t))$. *Suppose that* K *is a Morse function in* K_a^b, *that satisfies* (L_a^b), *and that possesses at most* $l - 1$ *saddle points of Morse index* 1 *in* K_a^b. *Then for* $|\varepsilon|$ *small, Problem* (1.2) *admits a solution.*

Theorems 1.2 and 2.3 can be modified by substituting one dimensional curves with m-spheres, $m < n$. Moreover, in all the above results, we can suppose the critical points of K to be degenerate of an order $\beta \in (1, n)$. For complete proofs we refer to the forthcoming paper [14].

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