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Some existence results for the scalar curvature problem via Morse theory


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**Analisi matematica. — Some existence results for the scalar curvature problem via Morse theory.** Nota (*) di Andrea Malchiodi, presentata dal Corrisp. A. Ambrosetti.

**Abstract.** — We prove existence of positive solutions for the equation $-\Delta_{g_0} u + u = (1 + \varepsilon K(x))u^{2^* - 1}$ on $S^n$, arising in the prescribed scalar curvature problem. $\Delta_{g_0}$ is the Laplace-Beltrami operator on $S^n$, $2^*$ is the critical Sobolev exponent, and $\varepsilon$ is a small parameter. The problem can be reduced to a finite dimensional study which is performed with Morse theory.

**Key words:** Elliptic equations; Critical exponent; Scalar curvature; Perturbation method; Morse theory.

**Riassunto.** — Alcuni risultati di esistenza per il problema della curvatura scalare tramite la teoria di Morse. Si dimostra l'esistenza di soluzioni positive per l'equazione $-\Delta_{g_0} u + u = (1 + \varepsilon K(x))u^{2^* - 1}$ su $S^n$, che nasce del problema della curvatura scalare prescritta. $\Delta_{g_0}$ `e l'operatore di Laplace-Beltrami su $S^n$, $2^*$ `e l'esponente critico di Sobolev, ed $\varepsilon$ un parametro piccolo. Il problema si riduce a uno studio finito-dimensionale che `e affrontato con la teoria di Morse.

1. **Introduction and statement of the results**

In this Note we state some existence results for the problem on $S^n$

\[
(1.1) \quad -4 \frac{(n-1)}{(n-2)} \Delta_{g_0} u + Ru = Su^{\frac{n+2}{n-2}}, \quad u > 0,
\]

where $\Delta_{g_0}$ is the Laplace-Beltrami operator on $S^n$, and $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Such a problem, which has been widely investigated, arises in Differential Geometry, when the metric $g$ of a Riemannian manifold $M$ of dimension greater or equal than 3, with scalar curvature $R$, is conformally deformed to a metric with prescribed scalar curvature $S$.

Some difficulties arise in studying this problem by means of variational methods, because of the lack of compactness, and some topological obstructions may occur, see [11].

We consider the case when $S$ is close to a constant, i.e. when $S$ is of the form $1 + \varepsilon K$ with $|\varepsilon|$ small. Using the stereographic projection, the problem reduces to find solutions of

\[
(1.2) \quad \begin{cases} 
-4 \frac{(n-1)}{(n-2)} \Delta u = (1 + \varepsilon K(x))u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n, \\
u > 0, \ u \in D^1(\mathbb{R}^n). 
\end{cases}
\]

We will be concerned with functions $K$ which have nondegeneracy properties between some levels. So we introduce the condition

\[
(L^b_a) \quad x \in \text{Crit}(K) \cap K_a^b \quad \Rightarrow \quad \Delta K(x) \neq 0,
\]

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where $\text{Crit}(K) = \{K' = 0\}$, $a, b \in \mathbb{R}$, and $K^b_a = \{a \leq K \leq b\}$. Our main results are the following Theorems 1.1 and 1.2.

**Theorem 1.1.** Suppose $K \in C^2(\mathbb{R}^n)$ is a Morse function which satisfies $(L_a^K)$ with $a = \inf K$ and $b = \sup K$. For $j = 0, \ldots, n - 1$, let $D_j$ denote the number of critical points of $K$ with Morse index $n - j$ and with $\Delta K < 0$. Suppose that $K$ satisfies

\begin{equation}
(1.3) \quad \sum_{j=0}^{q} (-1)^{q-j}D_j - (-1)^q \leq -1, \quad \text{for some} \quad q = 1, \ldots, n - 1.
\end{equation}

Then for $|\varepsilon|$ sufficiently small, Problem (1.2) has solution.

When $n = 2$ (hence $q = 1$), (1.3) becomes $D_0 > D_1 + 1$. In [6], for $n = 2$, it has been introduced the condition $D_0 \neq D_1 + 1$. Thus our result can be viewed as a partial extension of it (see also [15] for $n = 3$). It is worth pointing out that condition (1.3) is different from the well known assumption in [3], which also extends [6], see (2.2) below.

If $x$ is a critical point of $K$, we define $m(x, K)$ to be the Morse index of $K$ at $x$.

**Theorem 1.2.** Suppose that $K$ has a local minimum $x_0$, and that there exists $x_i$ with $K(x_i) \leq K(x_0)$. Suppose also that there exists a curve $x(t) : [0, 1] \to \mathbb{R}^n$ with $x(0) = x_0$, $x(1) = x_i$, such that, letting $a = K(x_0)$, $b = \max K(x(t))$, the following condition holds

\begin{equation}
(1.4) \quad z \in \text{Crit}(K) \cap K^b_a, \quad m(z, K) = 1 \implies \Delta K(z) < 0.
\end{equation}

Suppose also that $K$ is a Morse function in $K^b_a$, and that condition $(L_a^K)$ is satisfied. Then for $|\varepsilon|$ small, Problem (1.2) admits a solution.

In [4] there is a non perturbative existence result similar to Theorem 1.2, but condition (1.4) is required for all the saddle points in $K^b_a$, and not only for the critical points with Morse index 1. For $n = 2$, analogous results have been previously given in [5] under the assumption that $\Delta K < 0$ at all the saddle points of $K$, and in [10] under the hypothesis that there is no critical point of $K$ in $\{a < K < b\}$.

The proofs rely on an abstract perturbation result developed in [1], see also [2] for an application to (1.2), which leads to study a reduced, finite dimensional functional $\Gamma$. We show that Morse theory under general boundary conditions (see [8]) applies to $\Gamma$, and allows us to obtain the preceding results. An infinite dimensional Morse theoretical approach has been used to face the scalar curvature problem in [9] for $n = 2$, and in [15] for $n = 3$. The new feature here is that we can deal with all dimension, and that, differently from [15], we can also restrict our attention to some prescribed levels of $K$, and work with relative homology.

2. Outline of the proofs and generalizations

Solutions are found as critical points of some functional $f_\varepsilon(u) = f_0(u) - \varepsilon G(u)$, where $f_0$ possesses a manifold $Z$ of critical points, $Z \simeq \{(\mu, \xi), \mu > 0, \xi \in \mathbb{R}^n\}$. For $|\varepsilon|$
small, it is shown that $Z$ perturbs to a manifold $Z_\varepsilon$ which is a natural constrain for $f_\varepsilon$, and $f_\varepsilon|_{Z_\varepsilon} = b - \varepsilon \Gamma(z) + o(\varepsilon)$, where $b$ is a constant. Solutions are obtained, roughly, by finding «stable» critical points of $\Gamma$.

The behaviour of the function $\Gamma$ has been studied in [2]: we are particularly interested in the following proposition.

**Proposition 2.1.** The function $\Gamma$ can be extended to the hyperplane $\{\mu = 0\}$ by setting $\Gamma(0, \xi) = c_0 K(\xi)$, $c_0 > 0$. Moreover, for some $c_1 > 0$ there holds

$$
(2.1) \quad \Gamma_\mu(0, \xi) = 0, \quad \Gamma_{\mu\xi_i}(0, \xi) = 0, \quad \Gamma_{\mu\mu}(0, \xi) = c_1 \Delta K(\xi); \quad \forall \xi \in \mathbb{R}^n.
$$

Using Proposition 2.1, one can study the gradient flow of $\Gamma$ on the boundary of a great ball $B$ which is close to $\partial Z$. The flow is inward $B$ when $\Delta K > 0$, and is outward $B$ when $\Delta K < 0$. This enables us to prove the following proposition.

**Proposition 2.2.** Suppose $K \in C^2(\mathbb{R}^n)$ is a Morse function in $K^b_a$, and such that $(L^b_a)$ holds. For $s > 0$, let $\tilde{B}$ be the $n + 1$-dimensional ball centred in $(s^2 + 1)/2s, 0)$ and with radius $(s^2 - 1)/2s$. Then, for $s$ sufficiently large, $\Gamma$ satisfies the general boundary conditions on $B \equiv \tilde{B}$, between the levels $a$ and $b$.

Theorems 1.1 and 1.2 are proved using Morse inequalities for manifolds with boundary; with the same method, we can also prove existence if $K$ is a Morse function which satisfies

$$
(2.2) \quad \sum_{x \in \text{Crit}(K), \Delta K(x) < 0} (-1)^{m(x, K)} \neq (-1)^n.
$$

This condition has been used in [3] for $n = 3$, and in [6] for $n = 2$; in the case $n > 3$ there are analogous results under some flatness assumptions, see [12, 13, 2]. Other perturbation results have been given in [7].

Theorem 1.2 can be easily generalized to the following situation, where existence of critical points of Morse index 1 and with positive Laplacian is admitted.

**Theorem 2.3.** Suppose $K$ possesses a local minimum $x_0$ and $l$ connected components $A_1, \ldots, A_l$ of $(K^{K(x_0)} \setminus x_0)$. For $i = 1, \ldots, l$, let $c_i : [0, 1] \to S^n$ be a curve with $c_i(0) = x_0$, $c_i(1) \in A_i$; set $a = K(x_0)$, $b = \sup_i \sup_t K(c_i(t))$. Suppose that $K$ is a Morse function in $K^b_a$, that satisfies $(L^b_a)$, and that possesses at most $l - 1$ saddle points of Morse index 1 in $K^b_a$. Then for $|\varepsilon|$ small, Problem (1.2) admits a solution.

Theorems 1.2 and 2.3 can be modified by substituting one dimensional curves with $m$-spheres, $m < n$. Moreover, in all the above results, we can suppose the critical points of $K$ to be degenerate of an order $\beta \in (1, n)$. For complete proofs we refer to the forthcoming paper [14].

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References


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