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Alberto Farina

Simmetria delle soluzioni di equazioni ellittiche semilineari in $\mathbb{R}^N$

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Analisi matematica. — *Symmetry for solutions of semilinear elliptic equations in $\mathbb{R}^N$ and related conjectures.* Nota di Alberto Farina, presentata (*) dal Socio E. Magenes.

Abstract. — In the first part of this *Note* we prove one-dimensional and radial symmetry results for solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^N$. These results are connected with two conjectures (De Giorgi, 1978 and Gibbons, 1994) about the classification of solutions of the equation $\Delta u + u(1-u^2) = 0$ in $\mathbb{R}^N$. In particular we prove a stronger version of Gibbons' conjecture in any dimension $N > 1$, namely: if the set of zeros of u is bounded with respect to one direction, say ν, then u is one-dimensional, *i.e.*, $u(x) = u_0(\nu \cdot x)$. In the second part we consider the reaction-convection-diffusion equations of type $a^{ij}(x)\partial_{ij}u + b^i(x)\partial_i u + f(x, u) = 0$ in $\mathbb{R}^N$ and prove monotonicity and symmetry results which, when combined, lead to another stronger version of Gibbons's conjecture in any dimension.

Key words: Symmetry and monotonicity properties; Semilinear elliptic PDE; Moving planes method; Maximum principles.

Riassunto. — *Simmetria delle soluzioni di equazioni ellittiche semilineari in $\mathbb{R}^N$.* Nella prima parte di questa *Nota* si dimostrano dei risultati di simmetria unidimensionale e radiale per le soluzioni di $\Delta u + f(u) = 0$ in $\mathbb{R}^N$. Questi risultati sono legati a due congetture (De Giorgi, 1978 e Gibbons, 1994) riguardanti la classificazione delle soluzioni dell'equazione $\Delta u + u(1-u^2) = 0$ in $\mathbb{R}^N$. Si dimostra, in particolare, la seguente generalizzazione della congettura di Gibbons: se $N > 1$ e se l'insieme degli zeri di u è limitato nella direzione ν, allora $u(x) = u_0(\nu \cdot x)$, ovvero, u è unidimensionale. Nella seconda parte si considerano le equazioni di reazione-convezione-diffusione del tipo $a^{ij}(x)\partial_{ij}u + b^i(x)\partial_i u + f(x, u) = 0$ in $\mathbb{R}^N$ e si dimostrano dei risultati di monotonia e simmetria che, una volta combinati, conducono ad un'altra generalizzazione della congettura di Gibbons.

1. Introduction

This *Note* deals with symmetry properties of the solutions of semilinear elliptic equations in $\mathbb{R}^N$ and is motivated by two questions concerning the scalar Ginzburg-Landau equation:

$$\Delta u + u(1-u^2) = 0 \quad \text{on} \quad \mathbb{R}^N. \tag{1.1}$$

In 1978 E. De Giorgi formulated the following:

De Giorgi's conjecture [8]. *Assume $N > 1$ and consider a solution $u \in C^2(\mathbb{R}^N)$ of* (1.1), *such that, for every $x = (x_1, \dots, x_N) \in \mathbb{R}^N$,*

$$|u(x)| \leq 1, \qquad \frac{\partial u}{\partial x_N}(x) > 0, \tag{1.2}$$

$$\lim_{x_N \to \pm\infty} u(x_1, \dots, x_N) = \pm 1. \tag{1.3}$$

(*) Nella seduta del 16 giugno 1999.

Then, is it true that the level sets are parallel hyperplanes ? Or, equivalently, do there exist $g \in C^2(\mathbb{R})$ *and* $a = (a_1, ... a_{N-1}) \in \mathbb{R}^{N-1}$, *such that :*

$$u(x) = g(a_1 x_1 + ... + a_{N-1} x_{N-1} - x_N) \tag{1.4}$$

whenever $x \in \mathbb{R}^N$?

Later, (see [7]), G. W. Gibbons, proposed a weaker version of the above conjecture, namely:

GIBBONS' CONJECTURE [7]. *Assume* $N > 1$ *and consider a bounded solution* u *of* (1.1) *in* $C^2(\mathbb{R}^N)$, *such that, for every* $x' := (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$,

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \tag{1.5}$$

the limits being uniform with respect to x'.

Then, is it true that

$$u(x) = \tanh\left(\frac{x_N - \alpha}{\sqrt{2}}\right) \tag{1.6}$$

for every $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ *and some* $\alpha \in \mathbb{R}$?

De Giorgi's conjecture was proved by L. Modica and Mortola [17] in the case $N = 2$ and if the level sets of u are the graphs of an equi-lipschitzian family of functions and in any dimension N if there exists a point $x \in \mathbb{R}^N$ such that $|\nabla u(x)|^2 = \frac{1}{2}(1 - u^2(x))^2$ (see Caffarelli, Garofalo and Segala [6] and also L. Modica [16]).

In 1997, the author [11] gave a positive answer to De Giorgi's conjecture in any dimension provided there exists an infinite open connected cylinder $\Sigma := \omega \times \mathbb{R}$, such that u minimizes the energy $\int_\Sigma \left[\frac{|\nabla u|^2}{2} + \frac{(|u|^2-1)^2}{4}\right] dx$ in the class of functions satisfying

$$\lim_{\substack{x_N \to \pm\infty, \\ x' \in \omega}} u(x', x_N) = \pm 1. \tag{1.7}$$

Furthermore, this result also holds when the Laplacian operator is replaced by the p-Laplacian operator.

Recently Ghoussoub and Gui [13] proved De Giorgi's conjecture in dimension 2 without any additional assumption. They used some recent results about the spectrum of linear Scroedinger operators in $\mathbb{R}^2$. Similar results are also obtained by Berestycki, Caffarelli and Nirenberg in [4].

For $N \geq 3$ De Giorgi's conjecture is still open. Concerning this case, it has to be remarked that the method used in [13] and [4] does not work for $N \geq 3$ (see [1, 13]).

For $N = 2, 3$ Ghoussoub and Gui proved Gibbons' conjecture in [13]. For $N \geq 4$ this conjecture is still open.

In this *Note*, we propose an approach to the proof of these conjectures by investigating the more general problem of the symmetry properties of the solutions of semilinear elliptic equations in $\mathbb{R}^N$. The point of view adopted here, in studying problem (1.1), is the following: in $\mathbb{R}^N$ symmetry properties are a consequence of the «shape» of the set of zeros of the solutions. Following this idea we are able to obtain various stronger versions of Gibbons' conjecture and several other symmetry and monotonicity results for a wide class of semilinear problems. Our proofs are based on the moving planes method, on various versions of the maximum principle and the translation invariance of the structure of the considered equations.

The *Note* is organized as follows. The first part of Section 2 is devoted to the model problem (1.1). In particular we prove a stronger version of Gibbons' conjecture in any dimension $N > 1$, namely: if the set of zeros of u is bounded with respect to one direction, say ν, then u is one-dimensional, *i.e.*, $u(x) = u_0(\nu \cdot x)$. In the second part of Section 2 we consider a class of semilinear equations relevant in many different physical contexts. More precisely we study the qualitative properties of the solutions of the problem $\Delta u + f(u) = 0$, where f is a locally Lipschitz continuous function of «bistable» type. We prove a stronger version of Gibbons' conjecture, *i.e.*, if the level set of u corresponding to the value of the nonstable equilibrium point is bounded with respect to one direction, then «u depends only on that direction». Section 3 deals with radial symmetry of the solutions of semilinear problems. We prove that a solution whose level set corresponding to the value of the nonstable equilibrium point is bounded, must be radial. The fourth section is concerned with reaction-convection-diffusion equations of the type $a^{ij}(x)\partial_{ij}u + b^i(x)\partial_i u + f(x, u) = 0$. We prove monotonicity and symmetry results which, when combined, lead to another generalized Gibbons's conjecture. The proofs of these results are detailed in [12].

After we announced these results we were informed by R. Monneau (private communication) that he had obtained some results similar to those proved in Section 4 of this *Note*. These results are included in a forthcoming paper [5] jointly with H. Berestycki and F. Hamel. We also learned that similar questions were investigated by M.T. Barlow, R. Bass and C. Gui [2] by probabilistic methods.

2. One-dimensional symmetry

We start this section with the study of the classical Ginzburg-Landau equation (1.1). To state our results we need to define the sets

$$\Gamma := \{x \in \mathbb{R}^N \ : \ u(x) = 0\}\,,$$

$$\Omega^+ := \{x \in \mathbb{R}^N \ : \ u(x) > 0\}\,, \qquad \Omega^- := \{x \in \mathbb{R}^N \ : \ u(x) < 0\}.$$

Our first result is the following

Theorem 2.1. *Assume $N > 1$ and let u be a solution of* (1.1) *in* $C^2(\mathbb{R}^N)$ (*without any assumption about boundedness or monotonicity*)*. Suppose that the set Γ of zeros of u is bounded with respect to some direction $\nu \in \mathbf{S}^{\mathbf{N-1}}$* (*the unit sphere in $\mathbb{R}^N$*)*, and both Ω^+ and Ω^- are unbounded with respect to ν then,*

$$u(x) = \pm \tanh\left(\frac{\nu \cdot x - \alpha}{\sqrt{2}}\right) \tag{2.1}$$

for every $x \in \mathbb{R}^N$ and some $\alpha \in \mathbb{R}$.

Remark 2.2. The assumptions of Theorem 2.1 are exactly equivalent to the existence of an infinite cylinder $\Sigma\,(=\mathbb{R}^{N-1}\times I$, up to rotation, where I is an open and bounded interval) containing Γ and such that its complement intersects both Ω^+ and Ω^-.

It is immediate to see that Gibbons' conjecture follows from Theorem 2.1. Actually Theorem 2.1 implies a somewhat stronger version of that conjecture since we do not assume anything about boundedness of the solution u. More precisely we have

Corollary 2.3. *Assume* $N>1$, $\nu\in \mathbf{S}^{\mathbf{N-1}}$ *and let* u *be a solution of* (1.1) *in* $C^2(\mathbb{R}^N)$. *The following are equivalent*

i) *u satisfies*: $\lim_{(\nu\cdot x)\to\pm\infty} u(x)=\pm 1$, *the limits being uniform in the* ν*-direction.*

ii) *The set* Γ *of zeros of* u *is bounded with respect to direction* $\nu\in\mathbf{S}^{\mathbf{N-1}}$ *and both* Ω^+ *and* Ω^- *are unbounded with respect to direction* ν.

Proof of Theorem 2.1. We divide the proof into several steps.

Step 1. Boundedness of solutions. Any L^3_{loc} solution u, in the sense of distribution, of (1.1) is smooth and satisfies $u^2\le 1$ on $\mathbb{R}^N$ (see [10]). Moreover, by the strong maximum principle, $|u(x)|<1$ for all $x\in\mathbb{R}^N$, if and only if, u is not identically equals to 1 (or -1).

Step 2. Asymptotic behaviour. It is clear that, up to a translation and a rotation, we may suppose $u(0)=0$ and $\nu=e_N:=(0,\dots,1)$, *i.e.*, the set Γ of zeros of u is bounded with respect to direction x_N. Up to a reflection with respect to the (N-1)-dimensional hyperplane $H:=\{\,x=(x',x_N)\in\mathbb{R}^N\ :\ x_N=0\,\}$, we see that there exists a positive number M^+ such that $\Gamma\subset\{x\in\mathbb{R}^N\ :\ |x_n|<M^+\}$ and

$$\forall\ x=(x',x_N)\ :\ x_N>M^+,\qquad u(x)>0,$$

$$\forall\ x=(x',x_N)\ :\ x_N<-M^+,\qquad u(x)<0.$$

Now, we apply the following Lemma 2.4 to obtain

$$\lim_{x_N\to\pm\infty} u(x',x_N)=\pm 1 \tag{2.2}$$

the limits being uniform with respect to x'.

Lemma 2.4. *Assume* $\mu>0$ *and let* f *be a positive Lipschitz continuous function on* $(0,\mu)$, *vanishing on* μ *and satisfying* $f(t)\ge\delta_0 t$ *on* $(0,t_0]$ *for some* $\delta_0>0$ *and* $t_0>0$.

Let u *be a* C^2 *function on an affine half-space* $\Sigma_M:=\{x\in\mathbb{R}^N\ :\ x_N>M\}$ *satisfying*

$$\begin{cases}\Delta u+f(u)\le 0 & \text{on}\quad \Sigma_M,\\[2mm] 0<u\le\mu & \text{on}\quad \Sigma_M.\end{cases}$$

Then

$$\lim_{x_N\to+\infty} u(x',x_N)=\mu.$$

the limit being uniform with respect to x'. □

The above Lemma is a consequence of Lemma 3.2 and 3.3 of [3] (notice that the result does not explicitely appear in this form in [3]).

Step 3. u is strictly increasing with respect to x_N*, more precisely,* $\frac{\partial u}{\partial x_N} > 0$ *in* $\mathbb{R}^N$.

From the previous step we know that $u(x', x_N)$ converges to ± 1 uniformly as x_N tends to $\pm\infty$, hence we may apply Lemma 3.2 of [13] to prove the above claim.

From now on, the statement *u is strictly increasing with respect to the direction* ν, will be used to mean $\frac{\partial u}{\partial \nu} > 0$ in $\mathbb{R}^N$.

Step 4. For every $\gamma > 0$ *there exists* $\epsilon = \epsilon(\gamma) > 0$ *such* $\frac{\partial u}{\partial x_N}(x) \geq \epsilon$ *for every* $x \in S_\gamma := \mathbb{R}^{N-1} \times (-\gamma, \gamma)$.

Suppose the claim does not hold, then there is a sequence of points $x_n \in S_\gamma$ such that $\lim_{n\to+\infty} \frac{\partial u}{\partial x_N}(x_n) = 0$. Set $u_n(x) = u(x + x_n)$. By standard regularity theory for elliptic equations, up to extraction of a subsequence, the functions u_n converge to u_∞ in $C^2_{\text{loc}}(\mathbb{R}^N)$. Then the function $u_{\infty,N} := \frac{\partial u_\infty}{\partial x_N}$ satisfies

$$-\Delta(u_{\infty,N}) + (3u^2 - 1)(u_{\infty,N}) = 0 \quad \text{on} \quad \mathbb{R}^N, \tag{2.3}$$

$$\frac{\partial u_\infty}{\partial x_N}(0) = 0, \quad \frac{\partial u_\infty}{\partial x_N}(x) \geq 0, \quad \forall\, x \in \mathbb{R}^N, \tag{2.4}$$

furthermore, from (2.2), we have

$$\lim_{x_N\to\pm\infty} u_\infty(x', x_N) = \pm 1 \tag{2.5}$$

the limits being uniform with respect to x'.

From (2.3)-(2.4) and the strong maximum principle, applied to $-u_{\infty,N}$, we know that

$$\forall\, x \in \mathbb{R}^N, \qquad \frac{\partial u_\infty}{\partial x_N}(x) = 0, \tag{2.6}$$

but the latter contradicts (2.5). Hence the claim holds true.

Step 5. The map $\nu \to \frac{\partial u}{\partial \nu}$ *belongs to* $C^{0,1}\left(\mathbf{S^{N-1}}, C^0(\mathbb{R}^N)\right)$.

Since u is bounded, by standard elliptic estimates, we know that also $\nabla u \in$ $\in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Thus,

$$\left|\frac{\partial u}{\partial \nu}(x) - \frac{\partial u}{\partial \tau}(x)\right| = |\nabla u(x) \cdot (\nu - \tau)| \leq \|\nabla u\|_\infty |\nu - \tau|. \tag{2.7}$$

The claim follows immediately from (2.7).

Step 6. u is strictly increasing with respect to all unit vectors of an a open subset of $\mathbf{S^{N-1}}$.

Since (2.2) holds true, there is a positive constant $M \geq M^+$ such that $u(x) > \frac{1}{\sqrt{3}}$ for all $x \in \mathbb{R}^N$ satisfying $x_N > M$ and $u(x) < -\frac{1}{\sqrt{3}}$ for all $x \in \mathbb{R}^N$ satisfying $x_N < -M$.

Fix a $\gamma > M$. By making use of Steps 4 and 5 we obtain the existence of an open neighbourhood $\mathcal{O}_{e_N}$ of e_N in $\mathbf{S^{N-1}}$ such that

$$\forall \nu \in \mathcal{O}_{e_N}, \quad \forall x \in S_\gamma, \qquad \frac{\partial u}{\partial \nu}(x) > 0. \tag{2.8}$$

Now, for $\nu \in \mathcal{O}_{e_N}$, functions $u_\nu := \frac{\partial u}{\partial \nu}$ are smooth and bounded and satisfy:

$$-\Delta u_\nu + (3u^2 - 1)u_\nu = 0 \quad \text{on} \quad \mathbb{R}^N \setminus \overline{S_M}\,,$$

with the boundary conditions $u_\nu > 0$. Notice also that $3u^2 - 1 \geq 0$ in $\mathbb{R}^N \setminus \overline{S_M}$.

By invoking a standard version of the maximum principle for unbounded domains (see for example [3]), applied separately to the affine half-spaces $H_M^+ := \{x \in \mathbb{R}^N : : x_N > M\}$ and $H_M^- := \{x \in \mathbb{R}^N \ : \ x_N < -M\}$, and the strong maximum principle we obtain that

$$\forall \nu \in \mathcal{O}_{e_N}\,, \quad \forall x \in \mathbb{R}^N \setminus \overline{S_M}\,, \qquad \frac{\partial u}{\partial \nu}(x) > 0.$$

The latter combined with (2.8) completes the proof of Step 6.

Step 7. u is strictly increasing with respect to all directions of the upper hemisphere $(\mathbf{S}^{\mathbf{N-1}})^+$. We consider the upper hemisphere

$$(\mathbf{S}^{\mathbf{N-1}})^+ \ := \{\nu \in \mathbf{S}^{\mathbf{N-1}} \ : \nu_n > 0\}\,,$$

and we denote by ω the set of $\nu \in (\mathbf{S}^{\mathbf{N-1}})^+$ such that there exists an open neighbourhood $\mathcal{O}_\nu \subset (\mathbf{S}^{\mathbf{N-1}})^+$ of ν, satisfying $\forall\, n \in \mathcal{O}_\nu\,,\ \forall\, x \in \mathbb{R}^N\,,\ \frac{\partial u}{\partial n}(x) > 0$.

The set ω is open by definition and contains e_N.

We claim that ω is also closed in $(\mathbf{S}^{\mathbf{N-1}})^+$ so that, it is equal to $(\mathbf{S}^{\mathbf{N-1}})^+$.

Consider a cluster point $\overline{\nu}$ of ω ; then there exists a sequence $\{\nu_n\}_{n\in\mathbf{N}}$ in ω and converging to $\overline{\nu}$ in ω, satisfying $\frac{\partial u}{\partial \nu_n}(x) > 0\,,\ \forall n \in \mathbf{N}$ and $\forall x \in \mathbb{R}^N$. Hence

$$\forall x \in \mathbb{R}^N\,, \qquad \frac{\partial u}{\partial \overline{\nu}}(x) \geq 0.$$

From the strong maximum principle we have that either $\frac{\partial u}{\partial \overline{\nu}}(x) > 0$ on $\mathbb{R}^N$ or u is constant in the $\overline{\nu}$-direction. The second case is clearly impossible, since $\overline{\nu}$ is not orthogonal to e_N and Γ is bounded in the x_N-direction. Hence

$$\forall x \in \mathbb{R}^N\,, \qquad \frac{\partial u}{\partial \overline{\nu}}(x) > 0.$$

To complete the proof of this step we have to show the existence of an open neighbourhood $\mathcal{O}_{\overline{\nu}}$ of $\overline{\nu}$ on the upper hemisphere $(\mathbf{S}^{\mathbf{N-1}})^+$ such that $\forall\, n \in \mathcal{O}_{\overline{\nu}}\,,\ \forall\, x \in \mathbb{R}^N\,,\ \frac{\partial u}{\partial n}(x) > 0$.

To do so, we apply the same proofs as in steps 4, 5 and 6 with e_N replaced by $\overline{\nu}$, to obtain an open neighbourhood $\mathcal{O}_{\overline{\nu}}$ of $\overline{\nu}$ such that

$$\forall\, n \in \mathcal{O}_{\overline{\nu}}\,, \quad \forall\, x \in \mathbb{R}^N\,, \quad \frac{\partial u}{\partial n}(x) > 0.$$

The latter implies that the direction $\overline{\nu} \in \omega$, so that ω is also closed in $(\mathbf{S}^{\mathbf{N-1}})^+$. This concludes Step 7.

Step 8. End of proof. Since $\omega = (\mathbf{S}^{\mathbf{N-1}})^+$ we have that $\frac{\partial u}{\partial \tau} \geq 0$ for any direction τ orthogonal to e_N, but also $-\tau$ is orthogonal to e_N thus, $\frac{\partial u}{\partial \tau} = 0$, *i.e.*, u is one-dimensional. It is well-known that, in the one-dimensional case, all the solutions of

(1.1) satisfying $\lim_{t\to\pm\infty} u(t) = \pm 1$ are given by $u(t) = \tanh\left(\frac{t-\alpha}{\sqrt{2}}\right)$, for some $\alpha \in \mathbb{R}$. This ends the proof of Theorem 2.1. □

Theorem 2.1 and Corollary 2.3 extend to bounded solutions of the following general semilinear elliptic equation

$$\Delta u + f(u) = 0 \quad \text{on} \quad \mathbb{R}^N , \tag{2.9}$$

where f is a locally Lipschitz continuous function of bistable type on $\mathbb{R}$, *i.e.*, fulfilling the following properties. There exist numbers $\mu^- < \mu_0 < \mu^+$ such that

$$\forall\ t \in (-\infty\,, \mu^-) \qquad f(t) \geq 0\,, \tag{2.10}$$

$$\forall\ t \in (\mu^-\,, \mu_0) \qquad f(t) < 0\,, \tag{2.11}$$

$$\forall\ t \in (\mu_0\,, \mu^+) \qquad f(t) > 0\,, \tag{2.12}$$

$$\forall\ t \in (\mu^+\,, +\infty) \qquad f(t) \leq 0; \tag{2.13}$$

there exist numbers $t_1^-\,, t_0^-\,, t_0^+\,, t_1^+\,, \delta_0^-\,, \delta_0^+$ satisfying $\mu^- < t_1^- < t_0^- < \mu_0 < t_0^+ < t_1^+ < \mu^+$ and $\delta_0^- > 0\,, \delta_0^+ > 0$ such that

$$\forall\, t \in [t_0^-\,, \mu_0] \qquad f(t) \leq \delta_0^-(t-\mu_0)\,, \tag{2.14}$$

$$\forall\, t \in [\mu_0\,, t_0^+] \qquad f(t) \geq \delta_0^+(t-\mu_0)\,, \tag{2.15}$$

$$f(t) \text{ is nonincreasing on } (\mu^-\,, t_1^-) \text{ and on } (t_1^+\,, \mu^+). \tag{2.16}$$

We define, for $\lambda \in \mathbb{R}$, the sets

$$\Gamma_\lambda := \{x \in \mathbb{R}^N \ :\ u(x) = \lambda\}\,,$$

$$\Omega_\lambda^+ := \{x \in \mathbb{R}^N \ :\ u(x) > \lambda\}\,, \qquad \Omega_\lambda^- := \{x \in \mathbb{R}^N \ :\ u(x) < \lambda\}.$$

Under the above conditions we have

THEOREM 2.5. *Assume $N > 1$ and let u be a bounded solution of* (2.9) *in* $C^2(\mathbb{R}^N)$. *Suppose that the level set Γ_{μ_0} of u is bounded with respect to some direction $\nu \in \mathbf{S}^{\mathbf{N-1}}$, and both $\Omega_{\mu_0}^+$ and $\Omega_{\mu_0}^-$ are unbounded with respect to ν, then u is one-dimensional,* i.e., $u(x) = g(x\cdot\nu)$ *for all $x \in \mathbb{R}^N$, where*

$$g''(t) + f(g(t)) = 0 \quad \forall\ t \in \mathbb{R}\,,$$

and

$$\begin{cases} \textit{either} & \lim\limits_{t\to\pm\infty} g(t) = \mu^\pm \quad \textit{and} \quad g'(t) > 0 \quad \forall\ t \in \mathbb{R}\,, \\ \textit{or} & \lim\limits_{t\to\pm\infty} g(t) = \mu^\mp \quad \textit{and} \quad g'(t) < 0 \quad \forall\ t \in \mathbb{R}. \end{cases}$$

PROOF OF THEOREM 2.5. The proof follows the same lines of that of Theorem 2.1 (only some modifications are needed to handle the weakened assumptions) and we refer to [12] for all details.

3. Radial symmetry

The following result, together with theorems 2.1 and 2.5, corroborates our point of view in studying semilinear elliptic problems of the considered type.

Theorem 3.1. *Assume* $N > 1$ *and let* u *be a bounded non-constant* $C^2(\mathbb{R}^N)$ *solution of* (2.9)*, where* f *satisfies properties* (2.10)-(2.16)*. Suppose that the level set* Γ_{μ_0} *of* u *is bounded. Then* u *must be radially symmetric about some point* $x_0 \in \mathbb{R}^N$*, moreover* u *is strictly monotone with respect to the radial variable* $r = |x - x_0| > 0$.

Proof. As above we have

$$\mu^- < u(x) < \mu^+.$$

The boundedness of Γ_{μ_0} yields that u is $> \mu_0$ (or $< \mu_0$) near infinity, say $u > \mu_0$. Hence by Lemma 2.4 we have

$$\lim_{|x|\to+\infty} u(x) = \mu^+.$$

Now, the function $v = \mu^+ - u > 0$ solves

$$\begin{cases} \Delta v + g(v) = 0 & \text{on} \quad \mathbb{R}^N \\ \\ \lim\limits_{|x|\to+\infty} v(x) = 0\,, \end{cases}$$

where $g(s) = -f(\mu^+ - s)$. Since g is non-increasing in a right-neighbourhood of 0, we may apply Theorem 1 in [15] to find that v is radially symmetric about some point $x_0 \in \mathbb{R}^N$ and $v_r < 0$ for $r = |x - x_0| > 0$. (Theorem 1 of [15] actually is stated for g which are differentiable in a right neighbourhood of 0 and satisfying $g' \leq 0$ in this neighbourhood. It is easy to see that this is true also for locally Lipschitz continuous function g which are only non-increasing in a right neighbourhood of 0). Thus, the claim follows immediately. □

4. A stronger version of Gibbons' conjecture

This section is concerned with a further generalization of Gibbons' conjecture. We are interested in monotonicity and symmetry properties of solutions of reaction-convection-diffusion equations naturally arising in many differents physical contexts such as biology (population dynamics, epidemiology, gene developements) and combustion theory (flame propagation). We consider the linear operator $L := a^{ij}(x)\partial_{ij} + b^i(x)\partial_i$, where the summation convention is used, and the following semilinear problem

$$L(u) + f(x, u) = a^{ij}(x)\partial_{ij}u + b^i(x)\partial_i u + f(x, u) = 0 \quad \text{on} \quad \mathbb{R}^N\,, \tag{4.1}$$

where functions f and u satisfy some suitable properties coming directly from physical assumptions. The typical (and simplest) examples are given by the well-known N-dimensional Fisher equation or N-dimensional Kolmogorov-Petrovskii-Piskunov type equation (KPP equation): $\Delta u + b\frac{\partial u}{\partial x_N} + f(u) = 0$ in $\mathbb{R}^N$, where b is a constant and f is a Lipschitz continuous function.

The assumptions about functions a^{ij}, b^i and f are as follows, a^{ij} and b^i are bounded and Holder continuous functions (with Holder exponent $0 < \alpha \leq 1$), in addition the matrix a^{ij} is always supposed to be symmetric and uniformly positive definite over $\mathbb{R}^N$. The function $f \in C^0(\mathbb{R}^{N+1})$ satisfies the following assumptions

$$\forall\, \mathcal{K} \subset\subset \mathbb{R}\,,\ \exists\, \mathcal{L}_{\mathcal{K}} > 0 \quad : \quad \forall\, (x, y) \in\ \mathbb{R}^N \times \mathbb{R}^N\,, \quad \forall\, (s, t) \in \mathcal{K}^2\,,$$

$$|f(x, s) - f(y, t)| \leq \mathcal{L}_{\mathcal{K}}[|x - y|^\alpha + |s - t|]\,, \tag{4.2}$$

$$\exists \quad \theta\,,\ \mu^+ \quad : \quad \theta \leq f(x, \mu^+) \leq 0\,, \quad \forall\, x \in\ \mathbb{R}^N. \tag{4.3}$$

There exist constants $M^+ > 0$, $\delta^+ > 0$ and $C^+ > 0$ such that $\forall\, x' \in\ \mathbb{R}^{N-1}\,,\ \forall\, x_N >$ $> M^+$, $\forall\, (s, s') \in (\mu^+ - \delta^+\,, \mu^+]^2$, with $s \neq s'$, we have

$$\frac{f(x'\,, x_N\,, s) - f(x'\,, x_N\,, s')}{s - s'} \leq -C^+ \quad , \tag{4.4}$$

and $\forall\, x' \in\ \mathbb{R}^{N-1}\,,\ \forall\, s \in \mathbb{R}\,,$

$$\text{functions} \quad x_N \longrightarrow f(x'\,, x_N\,, s) \quad \text{are non-decreasing.} \tag{4.5}$$

REMARK 4.1. In case f is independent of x, assumption (4.5) is automatically satisfied while (4.4) is always satisfied when f is differentiable at the point μ^+ and $f'(\mu^+) < 0$. These assumptions are exactly those required by the classical Fisher or KPP equations (see for example [14]).

Under these assumptions we are able to prove monotonicity and symmetry results for the problem (4.1). The proofs are based on the moving planes method, on various versions of the maximum principle, on the translation invariance of the structure of the considered equations and are detailed in [12].

We start with the following monotonicity result.

THEOREM 4.2. *Assume* $N > 1$ *and let* u *be a bounded* $C^2(\mathbb{R}^N)$ *solution of* (4.1), *where* f *satisfies assumptions* (4.2)-(4.5). *Suppose that*

i) *functions* a^{ij} *and* b^i *are independent on* x_N, i.e., $a^{ij}(x) = a^{ij}(x')$ *and* $b^i(x) = b^i(x')$ *for every* $x \in \mathbb{R}^N$,

ii) $u(x) \leq \mu^+$ *for every* $x \in \mathbb{R}^N$,

iii) $\lim_{x_N \to +\infty} u(x'\,, x_N) = \mu^+$,

the limit being uniform with respect to x'.

iv) *There are constants* $\mu < \mu^+$ *and* $M_1 > 0$ *such that*

$$u(x) \leq \mu \quad \forall\, x \in \mathbb{R}^N \quad \textit{with} \quad x_N < -M_1.$$

Then, $\frac{\partial u}{\partial x_N}(x) > 0$ *for every* $x \in \mathbb{R}^N$.

Making use of Theorem 4.2, we prove Gibbons' conjecture in the wider context of the reaction-convection-diffusion equations. In particular our result applies to various types of Fisher or KPP equations.

THEOREM 4.3. *Assume* $N > 1$. *Suppose* f *satisfies assumptions* (4.2)-(4.5) *and* u *is a* $C^2(\mathbb{R}^N)$ *solution of*

$$\begin{cases} a^{ij}\partial_{ij}u + b^i\partial_i u + f(x_N\,, u) = 0 \quad on \quad \mathbb{R}^N\,, \\ \mu^- \le u \le \mu^+\,, \\ \lim\limits_{x_N \to \pm\infty} u(x'\,, x_N) = \mu^\pm\,, \end{cases} \tag{4.6}$$

where a^{ij}, b^i *are constants and* $\mu^- \in (-\infty\,, \mu^+)$. *Suppose furthermore that there exist constants* $M^- > 0$, $\delta^- > 0$, $C^- > 0$ *such that* $\forall\, x_N < -M^-$, $\forall (s, s') \in [\mu^-\,, \mu^- + \delta^-)^2$, *with* $s \ne s'$, *we have*

$$\frac{f(x_N\,, s) - f(x_N\,, s')}{s - s'} \le -C^-\,. \tag{4.7}$$

Then, u *is one-dimensional,* i.e., $u(x) = g(x_N)$, *where* g *is a function satisfying*

$$\begin{cases} a^{NN} g''(t) + b^N g'(t) + f(t, g(t)) = 0 \quad on \quad \mathbb{R}\,, \\ \lim_{t\to\pm\infty} g(t) = \mu^\pm \quad and \quad g'(t) > 0 \quad \forall\ t \in \mathbb{R}. \end{cases} \tag{4.8}$$

REMARK 4.4. In view of Theorem 4.2 and assumption (4.7), it is easy to see that the above symmetry result also holds, if we replace the assumption $\lim_{x_N\to-\infty} u(x'\,, x_N) = \mu^-$ by the weaker one: there are constants $\mu \in [\mu^-\,, \min\{\mu^- + \delta^-\,, \mu^+\})$ and $M_3 > 0$ such that $u(x) \le \mu$ for all $x \in \mathbb{R}^N$ with $x_N < -M_3$. In this case the limit, as $t \to -\infty$, appearing in (4.8) has to be replaced by $\lim_{t\to-\infty} g(t) = l^- \le \mu$.

In the two-dimensional case we can prove the above Theorem 4.3 even letting b^1 depend on x_1-variable, more precisely we have

THEOREM 4.5. *Assume* $N = 2$. *Suppose that* f *is differentiable and satisfies* (4.2)-(4.5) *and* (4.7). *Let* u *be a* $C^2(\mathbb{R}^2)$ *solution of* (4.6), *where* a^{ij}, b^2 *and* μ^- *fulfill the assumptions of Theorem* 4.3, *while* b^1 *is a differentiable function satisfying* $b^1(x) = b^1(x_1)$ *and* $\frac{\partial b^1}{\partial x_1} \le 0$ *everywhere. Then, the conclusion of Theorem* 4.3 *also holds.*

REMARK 4.6. In case f is independent on x_2, the above Theorem 4.5 holds true even without the assumption about the differentiability of f.

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LAMFA, CNRS UPRES-A 6119
Université de Picardie Jules Verne
Faculté de Mathématiques et d'Informatique
33, rue Saint-Leu - 80039 Amiens (Francia)