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Parametric representations of semi-complete vector fields on the unit balls in $\mathbb{C}^n$ and in Hilbert space

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Analisi matematica. — *Parametric representations of semi-complete vector fields on the unit balls in $\mathbb{C}^n$ and in Hilbert space.* Nota di Dov Aharonov, Mark Elin, Simeon Reich e David Shoikhet, presentata (*) dal Socio E. Vesentini.

Abstract. — We present several characterizations and representations of semi-complete vector fields on the open unit balls in complex Euclidean and Hilbert spaces.

Key words: Flow invariance; Hilbert ball; Holomorphic mapping; Hyperbolic metric; Monotone mapping.

Riassunto. — *Rappresentazioni parametriche di campi vettoriali semi-completi sulle palle unitarie in $\mathbb{C}^n$ ed in uno spazio di Hilbert.* Vengono presentate alcune caratterizzazioni e rappresentazioni di campi vettoriali semi-completi sulle palle unitarie aperte degli spazi complessi euclidei e di Hilbert.

1. Introduction and preliminaries

Let H be a complex Hilbert space with the inner product $\langle \cdot , \cdot \rangle$. If H is finite dimensional, we will identify H with $\mathbb{C}^n$ and then $\langle x , y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i$ for all $x = (x_1 , \dots , x_n)$ and $y = (y_1 , \dots , y_n)$ in $\mathbb{C}^n$.

If D is a domain (open connected subset) in H, we will denote by $\mathrm{Hol}(D, H)$ the family of all holomorphic mappings from D into H. By $\mathrm{Hol}(D)$ we will denote the semigroup (with respect to composition) of all holomorphic self-mappings of D and by $\mathrm{Aut}(D)$ the group of all automorphisms of D.

Definition 1.1. *A mapping $f \in \mathrm{Hol}(D, H)$ is said to be a semi-complete vector field on D if the Cauchy problem*

$$\begin{cases} \frac{\partial u(t,z)}{\partial t} + f(u(t, z)) = 0 \\ u(0 , z) = z , \end{cases} \tag{1.1}$$

has a unique global solution $\{u(t, z) : t \geq 0\} \subset D$ for each $z \in D$.

If this solution can be extended from $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$ to all of $\mathbb{R} = (-\infty , \infty)$, then f is said to be a complete (or an integrable) vector field on D (see, for example, [10, 20, 4, 17]).

It is well known (see, for instance, [1, 17]) that if $f \in \mathrm{Hol}(D, H)$ is semi-complete, then the family $S_f = \{F_t\}_{t \geq 0}$, defined by $F_t(z) := u(t, z)$, is a one-parameter semigroup (flow) of biholomorphic self-mappings of D, *i.e.*,

$$\begin{cases} F_{t+s} = F_t \circ F_s , \quad t, s \geq 0 , \\ F_0 = I, \end{cases} \tag{1.2}$$

(*) Nella seduta del 14 maggio 1999.

where I is the restriction of the identity operator on H to D.

In addition, $S_f = \{F_t\}$ defined by (1.1) is a locally uniformly continuous semigroup, *i.e.*,

$$\lim_{t\to 0^+} F_t(z) = z\,, \quad z \in D\,, \tag{1.3}$$

uniformly on each ball which is strictly inside D (see [17]).

Furthermore, F_{t_0} is an automorphism of D for some $t_0 > 0$ if and only if all the mappings $F_t\,,\ t \geq 0$, are automorphisms of D. In this case the solution of the Cauchy problem (1.1) can be extended to all of $\mathbb{R}$ by $u(-t, z) := F_t^{-1}(z)$. Hence f is a complete vector field.

If D is a bounded convex domain in H and f is a bounded holomorphic mapping in a neighborhood U of $\overline{D}$, the closure of D, such that $\overline{D}$ is strictly inside U, then the following boundary flow invariance condition is necessary and sufficient for f to be semi-complete:

$$\inf_{z^*} \mathrm{Re}\langle f(z)\,, z^*\rangle \geq 0\,, \quad z \in \partial D\,, \tag{1.4}$$

where z^* is a support functional of $\overline{D}$ at $z \in \partial D$, *i.e.*, $\mathrm{Re}\,\langle z\,, z^*\rangle \geq \mathrm{Re}\,\langle w\,, z^*\rangle$ for all $w \in \overline{D}$ (see [12-15, 17]).

Consequently, $f \in \mathrm{Hol}\,(\overline{D}\,, H)$ is a complete vector field on D if and only if

$$\mathrm{Re}\,\langle f(z)\,, z^*\rangle = 0\,, \quad z \in \partial D\,, \tag{1.5}$$

for each support functional z^* of $\overline{D}$ at z.

In particular, when $D = \mathbb{B}$ is the open unit ball of H, it is well known that each automorphism of $\mathbb{B}$ can be holomorphically extended to a larger ball containing $\overline{\mathbb{B}}$ (see, for example, [9, 5]). Therefore (1.5) can be useful for describing the family of all complete vector fields on $\mathbb{B}$. In fact, it is known that this family (usually denoted by $\mathrm{aut}\,(\mathbb{B})$) is a real Banach Lie algebra, and each element of $\mathrm{aut}\,(\mathbb{B})$ has the following parametric representation:

$$f(z) = a + Az - \langle z\,, a\rangle z\,, \quad z \in \mathbb{B}\,, \tag{1.6}$$

where a is an arbitrary element of H and A is a conservative linear operator, *i.e.*, $\mathrm{Re}\langle Ax\,, x\rangle = 0\,,\ x \in H$ (see, for example, [20, 4]).

At the same time, the family of semi-complete vector fields on $\mathbb{B}$ (which we will denote by $\mathrm{hol}\,(\mathbb{B})$) contains many members which have no holomorphic extension to $\overline{\mathbb{B}}$. Therefore condition (1.4) is not an appropriate tool for characterizing this class. This is one of the facts leading to the problem of finding an interior flow invariance condition which will characterize $\mathrm{hol}\,(\mathbb{B})$. Another geometric reason leading to the same question is the hyperbolic structure of the Hilbert ball and the good behavior of holomorphic flows with respect to the hyperbolic metric on $\mathbb{B}$. Indeed, we recall that if ρ is the Poincaré hyperbolic metric on $\mathbb{B}$ (see the definition below in Section 3), then $(\mathbb{B}\,, \rho)$ is a complete metric space and each holomorphic self-mapping of $\mathbb{B}$ is nonexpansive with respect to this metric. From this standpoint, condition (1.4) may be useful only for a domain which is strictly inside $\mathbb{B}$, but not for all of $\mathbb{B}$, since the boundary $\partial\mathbb{B}$ of $\mathbb{B}$

plays the role of «infinity» for $(\mathbb{B}, \rho)$. Here we encounter other important questions from the point of view of dynamical systems in metric spaces: *If f is a semi-complete vector field on $\mathbb{B}$, is it also a semi-complete vector field on certain domains which are strictly inside $\mathbb{B}$?* In other words: *Are there flow invariant subsets of $\mathbb{B}$? If so, what is their geometric structure? Are they attractors?*

To answer some of these questions the following assertion was established in [17, 18].

Let D be a bounded convex domain in H. Then a bounded holomorphic f is a semi-complete vector field on D if and only if it satisfies the range condition

$$(I + \lambda f)(D) \supset D \tag{RC}$$

for each $\lambda \geq 0$, and the mapping $R(\lambda, f) = (I + \lambda f)^{-1}$ is a well-defined holomorphic self-mapping of D. In other words, if $f \in \mathrm{Hol}\,(D, H)$, then the Cauchy problem (1.1) has a unique global solution $\{u(t, x) : t \geq 0\} \subset D$ for each $x \in D$ if and only if the operator equation

$$x + \lambda f(x) = y \tag{1.7}$$

has a unique solution $x = x(\lambda, y)$ for each $y \in D$ and $\lambda \geq 0$.

In addition, $u(t, x)$ can be found by the exponential formula

$$u(t, x) = \lim_{n\to\infty} R(\frac{t}{n}, f)^n x, \quad x \in D, \quad t \geq 0, \tag{1.8}$$

where the limit in (1.8) is taken with respect to the norm of H, and is actually uniform on each ball strictly inside D.

The mapping $R(\lambda, f) = (I + \lambda f)^{-1}$ is called *the (nonlinear) resolvent of f*.

If again $D = \mathbb{B}$ is the open unit ball in H, then it was shown in [18] that f satisfies the range condition (RC) if and only if the following inequality

$$\mathrm{Re}\langle f(x), x\rangle \geq m(1 - \|x\|^2), \quad x \in \mathbb{B}, \tag{1.9}$$

holds for some real m. (In fact, m must be non-positive).

Thus (1.9) can be considered an interior flow-invariance condition for a bounded $f \in \mathrm{Hol}\,(\mathbb{B}, H)$ to be a semi-complete vector field on $\mathbb{B}$.

Such a condition is usually called a «one-sided estimate». When $H = \mathbb{C}^n$ and $\mathbb{B}$ is its unit (Euclidean) ball, another characterization of the class $\mathrm{hol}\,(\mathbb{B})$ can be found in [1].

Earlier, for the one-dimensional case, E. Berkson and H. Porta [2] also established a parametric representation of the class $\mathrm{hol}\,(\Delta)$ consisting of the semi-complete vector fields on the open unit disk of the complex plane $\mathbb{C}$. Their representation has been effectively applied to the eigenvalue problem for composition operators on a Hardy space on Δ (see also [3, and the references there]).

More precisely, they proved that $f \in \mathrm{Hol}\,(\Delta, \mathbb{C})$ is a semi-complete vector field if and only if it has the form

$$f(z) = (z - \tau)(1 - \overline{\tau} z)g(z), \quad z \in \Delta, \tag{1.10}$$

where g is a function of the class of Carathéodory, *i.e.*, $\operatorname{Re} g(z) \geq 0$ everywhere, and $\tau \in \overline{\Delta}$.

If $\tau \in \Delta$, then it must be the unique null point of f in Δ; hence it is also the common fixed point of the flow generated by f. If $\tau \in \partial\Delta$, then it can be shown that f has no null point in Δ, and τ is the unique attractive point for the semigroup $\{F_t\}_{t\geq 0}$ generated by f, *i.e.*, $\{F_t(z)\}_{t\geq 0}$ converges to τ as t tends to infinity uniformly on each compact subset of Δ (see details below).

Usually such a point τ is referred to as the Denjoy-Wolff point for a flow in Δ.

One of the goals of this paper is to point out different approaches concerning the description of the class $\operatorname{hol}(\mathbb{B})$, in order to show interactions between the theory of dynamical systems, complex analysis and monotone operator theory, and to solve additional problems in these areas.

In the next section of this paper we consider the one-dimensional case. By using the methods of complex analysis we will establish directly the equivalence of two parametric representations of the class $\operatorname{hol}(\Delta)$. This will also be useful for the higher dimensional case, because in certain steps one can employ a reduction to one dimension.

Another look at the problem, using hyperbolic geometry on the Hilbert ball, will be presented in Section 3. There we will establish a generalized condition which contains both one-dimensional representations. Also, we will describe the subcone of $\operatorname{hol}(\mathbb{B})$ consisting of all those semi-complete vector fields which vanish at a given point of $\mathbb{B}$.

Combining this approach with the Hefer formula on generating ideals, we give in the last section two forms of parametric representations of semi-complete vector fields in $\mathbb{C}^n$, both of which coincide with the Berkson-Porta formula in the one-dimensional case.

2. The one-dimensional case

Let Δ be the open unit disk in the complex plane $\mathbb{C}$, and let $\operatorname{Hol}(\Delta, \mathbb{C})$ be the family of all holomorphic mappings from Δ into $\mathbb{C}$.

Theorem 2.1. *Let* $f \in \operatorname{Hol}(\Delta, \mathbb{C})$, $f \not\equiv 0$, *have the representation*

$$f(z) = a - \overline{a}z^2 + zh(z), \tag{2.1}$$

where $a \in \mathbb{C}$ *and* $h \in \operatorname{Hol}(\Delta, \mathbb{C})$ *with* $\operatorname{Re} h(z) \geq 0$ *for all* $z \in \Delta$. *Then* f *has a unique representation in the form*

$$f(z) = (z - \tau)(1 - \overline{\tau}z)g(z) \tag{2.2}$$

for some $\tau \in \overline{\Delta}$ *and* $g \in \operatorname{Hol}(\Delta, \mathbb{C})$ *with* $\operatorname{Re} g(z) \geq 0$ *for all* $z \in \Delta$.

Moreover, if τ *lies in* Δ, *then it is the unique and simple zero of* f *in* Δ; *if* $\tau \in \partial\Delta$, *then* f *has no zero in* Δ.

Conversely, if $f \in \operatorname{Hol}(\Delta, \mathbb{C})$ *has the representation* (2.2), *then it can also be represented by* (2.1) *with* $a = f(0)$ *and* $h \in \operatorname{Hol}(\Delta, \mathbb{C})$ *with* $\operatorname{Re} h \geq 0$ *everywhere.*

For the proof of this theorem we will need the following lemma.

LEMMA 2.1. *Let $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ have a continuous extension to $\overline{\Delta}$ and let f have the form*

$$f(z) = a - \overline{a}z^2 + zh(z)\,, \tag{2.1'}$$

where $a \in \mathbb{C}$, $h \in \mathrm{Hol}(\Delta, \mathbb{C})$, $h(0) = 1$ and $\mathrm{Re}\, h(z) > 0$ for all $z \in \overline{\Delta}$. Then f has a simple zero in Δ at some point $\tau \in \Delta$, and has no other zero in Δ.

PROOF. If $a = 0$, the assertion is obvious. So, we may assume $a \neq 0$. Since $\mathrm{Re}\, h > 0$ everywhere, we can rewrite f in the form

$$f(z) = a - \overline{a}z^2 + z\left(\frac{1 + Q(z)}{1 - Q(z)}\right)$$

where $Q \in \mathrm{Hol}(\Delta, \mathbb{C}) \cap C(\overline{\Delta})$, and $|Q(z)| < 1$ for $z \in \overline{\Delta}$. This implies that for $z = e^{i\theta}$ we get

$$\begin{aligned}(1 - Q(e^{i\theta}))f(e^{i\theta}) = (a - \overline{a}e^{2i\theta})(1 - Q(e^{i\theta})) + e^{i\theta}(1 + Q(e^{i\theta})) =\\ = a - e^{2i\theta}\overline{a} + e^{i\theta} + Q(e^{i\theta})(e^{i\theta} - a + e^{2i\theta}\overline{a}).\end{aligned}$$

We now claim:

$$|a - e^{2i\theta}\overline{a} + e^{i\theta}| > |Q(e^{i\theta})|\; |e^{i\theta} - a + e^{2i\theta}\overline{a}|. \tag{2.3}$$

If claim (2.3) is correct, then by Rouché's theorem, we have

$$N_0((1 - Q)f) = N_0(f) = N_0(a - \overline{a}z^2 + z)\,,$$

where N_0 denotes the number of zeros in Δ, because $|Q(z)| < 1$ for $z \in \Delta$.

So, it remains to prove claim (2.3) and to analyze the behavior of the function $a - \overline{a}z^2 + z$.

We have

$$\frac{a - e^{2i\theta}\overline{a} + e^{i\theta}}{e^{i\theta} - a + e^{2i\theta}\overline{a}} = \frac{1 + (u - \overline{u})}{1 - (u - \overline{u})}$$

where $u = ae^{-i\theta}$. Since $u - \overline{u}$ is purely imaginary, the latter expression is of absolute value 1. Putting this information back into (2.3), and recalling that $|Q(z)| < 1$ for all $z \in \overline{\Delta}$, we indeed obtain the validity of (2.3).

Finally, it is easy to see that the quadratic equation $\overline{a}z^2 - z - a = 0$ has exactly one root with absolute value smaller than 1, and another with absolute value bigger than 1. This ends the proof of the lemma.

PROOF OF THEOREM 2.1. We begin with the proof of the equivalence of the representations (2.1) and (2.2). We first assume that g in (2.2) is holomorphic in $\overline{\Delta}$, $\mathrm{Re}\, g(z) > 0$ for $z \in \overline{\Delta}$ and $|\tau| < 1$. As for (2.1), we also assume a stronger condition at first, namely, $h \in \mathrm{Hol}(\overline{\Delta}, \mathbb{C})$ and $\mathrm{Re}\, h(z) > 0$ for $z \in \overline{\Delta}$. In this case, by the above lemma the mapping f defined by the formula (2.1′), $f(z) = a - \overline{a}z^2 + zh(z)$, has a unique, simple zero $\tau \in \Delta$. Hence one can write

$$f(z) = (z - \tau)(1 - \overline{\tau}z)g(z)\,, \tag{2.4}$$

where $g \in \mathrm{Hol}(\overline{\Delta}, \mathbb{C})$.

Equating (2.1′) and (2.4) we see that

$$f(0) = a = -\tau g(0). \tag{2.5}$$

Hence,

$$f(z) = -\tau g(0) + z^2\overline{\tau g(0)} + zh(z) = (z-\tau)(1-\overline{\tau}z)g(z)$$

or

$$h(z) = (1-\overline{\tau}z + |\tau|^2)g(z) - \frac{g(z)-g(0)}{z}\tau - z\overline{\tau g(0)}.$$

Once again, substituting $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, we calculate:

$$\operatorname{Re} h(e^{i\theta}) = \operatorname{Re}\{(1+|\tau|^2 - \overline{\tau}e^{i\theta})g(e^{i\theta}) - \tau e^{-i\theta}g(e^{i\theta})\} = (1+|\tau|^2 - 2\operatorname{Re}\overline{\tau}e^{i\theta})\operatorname{Re} g(e^{i\theta}) = \\ = |1-\overline{\tau}e^{i\theta}|^2 \operatorname{Re} g(e^{i\theta}).$$

Since, by our assumptions, $\tau \in \Delta$, and the functions $\operatorname{Re} h$ and $\operatorname{Re} g$ are harmonic, we see that $\operatorname{Re} h(z) > 0$ if and only if $\operatorname{Re} g(z) > 0$. This ends the proof when we assume our stronger conditions on h and g.

For the general case we use an approximation argument. Given h and g in $\mathrm{Hol}(\Delta, \mathbb{C})$, with $\operatorname{Re} h(z)$ and $\operatorname{Re} g(z)$ nonnegative in Δ, set $h_n(z) = r_n^{-1}h(r_n z)$ and $g_n(z) = r_n^{-1}g(r_n z)$ for $r_n \in (0, 1)$, $r_n \to 1^-$. Except for the trivial case $h \equiv 0$ (or $g \equiv 0$) it follows that $\operatorname{Re} h_n(z)$ and $\operatorname{Re} g_n(z)$ are positive everywhere.

Note also that the families $\{h_n(z)\}$ and $\{g_n(z)\}$ are normal in Δ.

Assume now that (2.1) is given. Then it follows by our approximation process and the discussion above that there is a sequence $\{\tau_n\}$, $|\tau_n| < 1$, such that

$$f_n(z) = (z-\tau_n)\,(1-\overline{\tau}_n z)g_n(z) = a - \overline{a}z^2 + zh_n(z),$$

where $\operatorname{Re} g_n(z) > 0$ in $\overline{\Delta}$. Hence, taking if necessary subsequences of $\{\tau_n\}$ and $\{g_n\}$ and passing to their limits, we get (2.2), with $\tau = \lim_{k\to\infty} \tau_{n_k} \in \overline{\Delta}$ and $g \in \mathrm{Hol}(\Delta)$ with $\operatorname{Re} g \ge 0$.

Similarly, we can derive the representation (2.1) from (2.2).

Now we will prove the uniqueness of the representation (2.2). Assume to the contrary that we are given two representations of the form (2.2), namely:

$$f(z) = (z-\tau_j)(1-\overline{\tau}_j z)g_j(z), \tag{2.6}$$

where $|\tau_j| \le 1$ and $g_j(z) \in \mathrm{Hol}(\Delta, \mathbb{C})$, with $\operatorname{Re} g_j(z) \ge 0$, $j = 1, 2$.

We now intend to show that $\tau_1 = \tau_2$.

Without loss of generality we can assume that $f(z) \not\equiv 0$. We consider two cases:

(*i*) $|\tau_1| < 1$, $|\tau_2| \le 1$,

and

(*ii*) $|\tau_1| = |\tau_2| = 1$.

In the first case we have $f(z) = (z-\tau_1)(1-\overline{\tau}_1 z)g_1(z) = (z-\tau_2)\,(1-\overline{\tau}_2 z)\,g_2(z)$ leading to $0 = f(\tau_1) = (\tau_1-\tau_2)(1-\overline{\tau}_2\tau_1)g_2(\tau_1)$.

Hence, if $\tau_2 \neq \tau_1$, we must conclude that $g_2(\tau_1) = 0$, which is impossible as $\operatorname{Re} g_2 \geq 0$ and $f \not\equiv 0$.

Next, assume that (*ii*) holds. Except in trivial cases, since $\operatorname{Re} g_j \geq 0$ in Δ, one can suppose that actually $\operatorname{Re} g_j > 0$ in Δ, $j = 1, 2$. In this case, it follows by (2.6) that

$$\frac{g_1(z)}{g_2(z)} = \frac{(z-\tau_2)(1-\overline{\tau}_2 z)}{(z-\tau_1)(1-\overline{\tau}_1 z)} = Q(z),$$

where $z \in \Delta$ and $|\tau_1| = |\tau_2| = 1$.

Now if $\tau_1 \neq \tau_2$, then we claim that there is a point $z_0 \in \Delta$ such that $Q(z_0) = -1$. If this indeed holds, then it will be a desirable contradiction because in this case $g_1(z_0) = -g_2(z_0)$, while both $\operatorname{Re} g_1(z_0)$ and $\operatorname{Re} g_2(z_0)$ are positive.

To this end we consider the quadratic equation $(z-\tau_1)(1-\overline{\tau}_1 z) = -(z-\tau_2)(1-\overline{\tau}_2 z)$ which is equivalent to the equation $Q(z) = -1$. We have, after simple calculations,

$$(\overline{\tau}_1 + \overline{\tau}_2)z^2 - 4z + (\tau_1 + \tau_2) = 0. \tag{2.7}$$

If $\tau_1 + \tau_2 = 0$, then $z_0 = 0$ and we are done. If $\tau_1 + \tau_2 \neq 0$, then it follows by Viète's theorem that (2.7) has two solutions z_1 and z_2 such that $|z_1 z_2| = 1$ and $|z_1 + z_2| > 2$. Hence, one of them, say z_1, lies in Δ, and the second root z_2 lies in $\mathbb{C}\backslash\overline{\Delta}$. Setting $z_0 = z_1$, we obtain our claim. The theorem is proved.

Corollary 2.1. *Let* $f \in \operatorname{Hol}(\Delta, \mathbb{C})$. *Then the following are equivalent*:

(*a*) $\operatorname{Re} f(z)\overline{z} \geq (1-|z|^2)\operatorname{Re} f(0)\overline{z}$, $z \in \Delta$;
(*b*) $\operatorname{Re} f(z)\overline{z} \geq (1-|z|^2)\operatorname{Re} \frac{f(z)\overline{\tau}}{1-z\overline{\tau}}$ *for some* $\tau \in \overline{\Delta}$ *and for all* $z \in \Delta$;
(*c*) f *is a semi-complete vector field.*

In addition, if (*b*) *holds with* $\tau \in \Delta$, *then* τ *must be the zero of* f *in* Δ. *If* $\tau \in \partial\Delta$, *then* f *has no zero in* Δ.

Proof. Condition (*a*) is another form of condition (2.1), while condition (*b*) is a reformulation of condition (2.2).

Now let $f \in \operatorname{Hol}(\Delta, \mathbb{C})$ have the form (2.2) with $\tau \in \Delta$. Then the function h in (2.1) is uniquely determined by τ, and satisfies the condition $\operatorname{Re} h(z) \geq 0$. It follows by condition (*a*) of Corollary 2.1 that the mapping $f_1(z) = zh(z)$ is also a semi-complete vector field and $f_1(0) = 0$. In addition, the constant a in (2.1) is also uniquely determined by τ, because of (2.5). Thus we have the following fact.

Corollary 2.2. *Let* $\tau \in \Delta$ *and let* K_τ *be the set of all semi-complete vector fields vanishing at* τ. *Then there is a one-to-one correspondence between* K_τ *and* K_0, *the set of all semi-complete vector fields on* Δ *preserving zero.*

Actually, we will show in the sequel that K_τ and K_0 are linearly isomorphic. More precisely, there is an isomorphism between these sets which is a restriction of a linear involution.

3. The property of monotonicity with respect to the hyperbolic metric on the Hilbert ball

Since in this section our arguments do not depend on the dimension of the space, we are able to assume that $\mathbb{B}$ is the open unit ball in a complex Hilbert space H with the inner product $\langle \cdot\,, \cdot \rangle$.

First we recall that a mapping $f : C \to H$, where C is a subset of H, is said to be monotone on C (with respect to the norm of H) if for each x and y in C,

$$\operatorname{Re}\langle x - y, f(x) - f(y)\rangle \geq 0.$$

This inequality is equivalent to the following condition:

For each x, y in C, and for all $r \geq 0$,

$$\|x - y\| \leq \|x + rf(x) - (y + rf(y))\|. \tag{3.1}$$

If, in addition, f satisfies the range condition

$$(I + rf)(C) \supset C, \quad r > 0\,, \tag{RC}$$

then (3.1) implies that for each $r \geq 0$ the mapping $J_r = (I + rf)^{-1}$ is a single-valued and nonexpansive (with respect to the norm of H) self-mapping of C. This mapping is usually called the nonlinear resolvent of f. A similar situation (which is sometimes even better) can be described with respect to the hyperbolic metric of $\mathbb{B}$.

We recall that the hyperbolic (Poincaré) metric ρ on $\mathbb{B}$ can be defined by the formula

$$\rho(x, y) = \operatorname{arctanh}(1 - \sigma(x, y))^{\frac{1}{2}}\,, \tag{3.2}$$

where

$$\sigma(x, y) = \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{|1 - \langle x, y\rangle|^2}. \tag{3.3}$$

Note that $\rho(x, y) \leq \rho(u, v)$ if and only if $\sigma(x, y) \geq \sigma(u, v)$ and that $x = y$ if and only if $\sigma(x, y) = 1$.

Definition 3.1. *A mapping $F : \mathbb{B} \to \mathbb{B}$ is said to be ρ-nonexpansive if for each pair $(x, y) \in \mathbb{B} \times \mathbb{B}$,*

$$\rho(F(x)\,, F(y)) \leq \rho(x, y). \tag{3.4}$$

It is well known that each holomorphic self-mapping of $\mathbb{B}$ is ρ-nonexpansive.

Definition 3.2. *A mapping $f : \mathbb{B} \to H$ is said to be ρ-monotone on $\mathbb{B}$ if for each pair $(x, y) \in \mathbb{B} \times \mathbb{B}$,*

$$\rho(x, y) \leq \rho(x + rf(x)\,, y + rf(y)) \tag{3.5}$$

for all $r \geq 0$ such that $x + rf(x)$ and $y + rf(y)$ belong to $\mathbb{B}$.

The following characterization of ρ-monotone mappings was given in [18].

PROPOSITION 3.1. *A mapping* $f : \mathbb{B} \to H$ *is* ρ*-monotone if and only if for each* $x, y \in \mathbb{B}$ *the following inequality holds:*

$$\frac{\operatorname{Re}\langle x, f(x)\rangle}{1-\|x\|^2} + \frac{\operatorname{Re}\langle y, f(y)\rangle}{1-\|y\|^2} \geq \operatorname{Re}\frac{\langle f(x), y\rangle + \langle x, f(y)\rangle}{1-\langle x, y\rangle}. \tag{3.6}$$

If, in addition, f satisfies the range condition (RC)*, then for each* $r \geq 0$ *the resolvent* $J_r = (I + rf)^{-1}$ *is a single-valued* ρ*-nonexpansive self-mapping of* $\mathbb{B}$.

The latter fact, in turn, obviously implies the ρ-monotonicity of f by definition.

We remark in passing that in the proof of Lemma 2.2 in [18] (which leads to Proposition 3.1), the function

$$\psi(r) = \sigma(tx + (1-t)(x+\delta u), ty + (1-t)(y+\delta v)), \qquad \text{where } \ t = 1 - r/\delta,$$

while

$$\varphi(t) := \rho\big(tx + (1-t)(x+\delta u), ty + (1-t)(y+\delta v)\big)^2, \ \ 0 \leq t \leq 1.$$

Furthermore, it was shown in [17, 18] that a bounded holomorphic mapping $f : D \to H$, where D is a bounded convex domain in H, satisfies the range condition (RC) with a resolvent $(I + rf)^{-1}$, which is a well-defined holomorphic self-mapping of D, if and only if it is a semi-complete vector field.

So, combining this result with Proposition 3.1, we obtain the following assertion [18].

PROPOSITION 3.2. *A bounded holomorphic mapping* $f : \mathbb{B} \to H$ *is a semi-complete vector field if and only if it is a* ρ*-monotone mapping on* $\mathbb{B}$, i.e., *if and only if condition* (3.6) *holds.*

In addition, in this case the semigroup generated by f can be obtained by the exponential formula:

$$F_t(z) = \lim_{n\to\infty}(I + \frac{t}{n}f)^{-n}(z) \tag{3.7}$$

(see [17, 18]).

Condition (3.6) is the focus of our further investigations.

If we substitute in this condition $y = 0$ we get

$$\operatorname{Re}\langle f(x), x\rangle \geq (1-\|x\|^2)\operatorname{Re}\langle f(0), x\rangle. \tag{3.8}$$

In the one-dimensional case, (3.8) immediately implies the representation (2.1) (see Theorem 2.1) and actually, it is also sufficient for f to be a semi-complete vector field (see [11]).

At the same time, if f has a null point τ in $\mathbb{B}$, then setting $y = \tau$, we get the following necessary condition:

$$\frac{\operatorname{Re}\langle x, f(x)\rangle}{1-\|x\|^2} \geq \operatorname{Re}\frac{\langle f(x), \tau\rangle}{1-\langle x, \tau\rangle}, \tag{3.9}$$

or

$$\operatorname{Re}\langle x, f(x)\rangle \geq \frac{1-\|x\|^2}{|1-\langle x,\tau\rangle|^2}\operatorname{Re}\langle f(x)\,,\tau\rangle(1-\langle\tau\,,x\rangle).$$

Again, in the one-dimensional case, (3.9) becomes condition (2.2), which is also sufficient for f to be a semi-complete vector field on the unit disk. The first question is: *Is condition* (3.9) *also sufficient for a vector field f defined on the Hilbert ball to be semi-complete* ?

The second question (which also arises from the analogy with the one-dimensional case) is: *Can this condition characterize a semi-complete vector field when the point τ in* (3.9) *lies on the boundary of $\mathbb{B}$* ? In addition, we may also ask: *What is the geometric meaning of such a point τ* ?

To answer these and other questions, we need some additional notions.

For a point $a \in \overline{\mathbb{B}}$, the closure of $\mathbb{B}$, define the function

$$\varphi_a(x) = |1-\langle x\,,a\rangle|^2/(1-\|x\|^2)\,, \tag{3.10}$$

and consider the sets

$$E(a\,,k) = \{x \in \mathbb{B} : \varphi_a(x) < k\}\,,$$

where $k > 1-\|a\|^2$. Geometrically, these sets are ellipsoids in $\mathbb{B}$ [6].

If $a \in \mathbb{B}$, then $E(a\,,k)$ is actually the open ρ-ball

$$E(a\,,k) = \{x \in \mathbb{B} : \rho(a\,,x) < r\}\,,\ \text{ where } r = \operatorname{arctanh}\sqrt{1-\frac{1-\|a\|^2}{k}}.$$

If $a \in \partial\mathbb{B}$, then the sets $E(a\,,k)$ are ρ-unbounded and therefore are no longer ρ-balls. But the norm closure of each such ellipsoid is a subset of $\overline{\mathbb{B}}$ and for each $k > 0$, $\overline{E(a\,,k)} \cap \partial B = \{a\}$. In other words, the norm closures of these ellipsoids intersect the boundary of $\mathbb{B}$ at the point a.

DEFINITION 3.3. *Let $f \in \operatorname{Hol}(\mathbb{B}\,,H)$. A point $\tau \in \overline{\mathbb{B}}$ is said to be a flow-invariance point for f if f is a semi-complete vector field on each ellipsoid $E(\tau\,,k)$, $k > 1-|\tau|^2$. In other words, τ is a flow-invariance point for f if for each $k > 1-|\tau|^2$, and for each $x \in E(\tau\,,k)$ the Cauchy problem*

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + f(u(t\,,x)) = 0 \\ u(0\,,x) = x \end{cases} \tag{3.11}$$

has a unique global solution $\{u(t\,,x) : t \geq 0\} \subset E(\tau\,,k)$.

It is clear that in this case $f \in \operatorname{hol}(\mathbb{B})$, *i.e.*, f is semi-complete on all of $\mathbb{B}$. We will see that the converse is also true, *i.e.*, if $f \in \operatorname{hol}(\mathbb{B})$, then it has a flow-invariance point. If $\tau \in \mathbb{B}$ is a flow-invariance point for $f \in \operatorname{Hol}(\mathbb{B}\,,H)$, then it is actually a null point for f. Indeed, in this case we have $\rho(u(t\,,\tau)\,,\tau) < r$ for each $r > 0$ and $t \geq 0$. Hence $u(t\,,\tau) = \tau$ for all $t \geq 0$ and $f(\tau) = 0$.

Conversely, if $f \in \mathrm{hol}\,(\mathbb{B})$ with a null point τ inside $\mathbb{B}$, then τ is a common fixed point for the semigroup $\{F_t\}_{t\geq 0}$ ($F_t(x) := u(t, x)$) generated by f, and we get

$$\rho(F_t(x)\,,\tau) = \rho(F_t(x)\,,F_t(\tau)) \leq \rho(x\,,\tau)\,,$$

which is equivalent to

$$\varphi_\tau(F_t(x)) \leq \varphi_\tau(x)\,,$$

because $F_t \in \mathrm{Hol}\,(\mathbb{B})$ is a ρ-nonexpansive mapping on $\mathbb{B}$.

Now, if $f \in \mathrm{hol}\,(\mathbb{B})$ is null point free, then for each $r > 0$, the mapping $J_r = (I + rf)^{-1} : \mathbb{B} \to \mathbb{B}$ is holomorphic in $\mathbb{B}$ and is fixed point free. It was shown in [18] that in this case for each $x \in \mathbb{B}$ the strong limit $\lim_{r\to\infty} J_r(x) = \tau$ exists and does not depend on $x \in \mathbb{B}$, with $\|\tau\| = 1$. In addition, $\varphi_\tau(J_r(x)) \leq \varphi_\tau(x)\,,\ r \geq 0$. Hence, it follows by the exponential formula that

$$\varphi_\tau(F_t(x)) \leq \varphi_\tau(x) \text{ for all } t \geq 0.$$

Thus each ellipsoid $E(\tau\,,k)\,,\ k > 0$, is F_t-invariant and τ is a flow-invariance point for f. Note also that in this case τ is the unique flow-invariance point for f.

Now we can formulate the main result of this section.

Theorem 3.1. *A bounded holomorphic mapping f on a neighborhood strictly containing $\mathbb{B}$ is semi-complete on $\mathbb{B}$ if and only if for some $\tau \in \overline{\mathbb{B}}$ the following inequality holds for all $z \in \mathbb{B}$:*

$$\frac{\mathrm{Re}\,\langle f(z)\,,z\rangle}{1-\|z\|^2} \geq \mathrm{Re}\,\frac{\langle f(z)\,,\tau\rangle}{1-\langle z\,,\tau\rangle}. \tag{3.12}$$

Moreover, this point τ must be a flow-invariance point for f.

Proof. *Necessity.* In this direction we only assume that $f \in \mathrm{hol}(\mathbb{B})$. As we mentioned above, if $\tau \in \mathbb{B}$ is a null point of a semi-complete vector field f on $\mathbb{B}$, then condition (3.12) is a direct consequence of Propositions 3.1 and 3.2.

Now let $f \in \mathrm{hol}\,(\mathbb{B})$ be null point free. For $\epsilon > 0$ consider the mapping $f_\epsilon = f + \epsilon I$. Since $\mathrm{hol}(\mathbb{B})$ is a real cone, f_ϵ also belongs to $\mathrm{hol}\,(\mathbb{B})$. We claim that f_ϵ has a unique null point in $\mathbb{B}$. In fact, the equation $f_\epsilon(x) = 0$ is equivalent to the equation $(I + (1/\epsilon)f\,)(x) = 0$. But we already know that $f \in \mathrm{hol}\,(\mathbb{B})$ satisfies the range condition (RC), *i.e.*,

$$(I + \frac{1}{\epsilon}f)(B) \supset B,$$

and this implies that f_ϵ has a unique null point $\tau_\epsilon \in B$, $\tau_\epsilon = J_{1/\epsilon}(0) = (I + (1/\epsilon)f\,)^{-1}(0)$. In addition, since f is null point free, for each x in $\mathbb{B}$, $J_{1/\epsilon}(x)$ converges strongly as $\epsilon \to 0^+$ to a unique point $\tau \in \partial B$ which is a flow-invariance point of f. In particular, $\tau_\epsilon \to \tau$ as $\epsilon \to 0^+$. Furthermore, for $f_\epsilon \in \mathrm{hol}\,(\mathbb{B})$ we have the inequality

$$\frac{\mathrm{Re}\,\langle f_\epsilon(x)\,,x\rangle}{1-\|x\|^2} \geq \frac{\langle f_\epsilon(x)\,,\tau_\epsilon\rangle}{1-\langle x\,,\tau_\epsilon\rangle}\,,\quad x \in \mathbb{B}\,,$$

which immediately implies (3.12).

Sufficiency. Suppose that (3.12) holds for some $\tau \in \mathbb{B}$. In this case we merely assume that $f \in \mathrm{Hol}(\mathbb{B}, H)$. We clearly have

$$\operatorname{Re}\langle f(x), x\rangle \geq m(1-\|x\|^2) \tag{3.13}$$

for some $m \in \mathbb{R}$ and for all $x \in \mathbb{B}$. Hence $f \in \mathrm{hol}(\mathbb{B})$ [18]. If the point τ in (3.12) belongs to the boundary of $\mathbb{B}$, then we first consider the function

$$g(\lambda) = \langle f(\lambda\tau), \tau\rangle,$$

where $\lambda \in \Delta = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. This function is holomorphic in the unit disk Δ and by (3.12) we get

$$\operatorname{Re} g(\lambda)\overline{\lambda} \geq (1-|\lambda|^2)\operatorname{Re}\frac{g(\lambda)}{1-\lambda}.$$

By Theorem 2.1 and Corollary 2.1, $g(\lambda)$ has the form

$$g(\lambda) = g(0) - \overline{g(0)}\lambda^2 + \lambda h(\lambda),$$

with $\operatorname{Re} h(\lambda) \geq 0$ everywhere. This, in turn, implies that

$$\operatorname{Re} g(\lambda)\overline{\lambda} \geq \operatorname{Re} g(0)\overline{\lambda}(1-|\lambda|^2).$$

Thus we have

$$\operatorname{Re}\langle f(\lambda\tau), \lambda\tau\rangle \geq \operatorname{Re}\langle f(0), \lambda\tau\rangle(1-|\lambda|^2)$$

for all $\lambda \in (0, 1)$.

To complete the proof of the sufficiency of (3.12) for f to be a semi-complete vector field on $\mathbb{B}$, and of Theorem 3.1 itself, we will show that this condition ensures, in fact, that τ is a flow-invariance point for f, *i.e.*, that f is a semi-complete vector field on each ellipsoid $E(\tau, k)$, $k > 1-|\tau|^2$, where τ satisfies (3.12). To this end, we rewrite (3.12) in the form

$$\operatorname{Re}\left\langle f(z), \frac{z}{1-\|z\|^2} - \frac{\tau}{1-\langle \tau, z\rangle}\right\rangle \geq 0. \tag{3.14}$$

Fixing any $z \in \partial E(\tau, k)$, we now want to show that the non-zero vector

$$\frac{z}{1-\|z\|^2} - \frac{\tau}{1-\langle \tau, z\rangle} = z^* \tag{3.15}$$

is a support functional of the smooth convex set $E(\tau, k)$ at z. (In the case when $\tau \in \partial B$, we assume $z \neq \tau$). That is, we have to prove that for each $y \in E(\tau, k)$,

$$\operatorname{Re}\langle y, z^*\rangle \leq \operatorname{Re}\langle z, z^*\rangle. \tag{3.16}$$

By the definition of $E(\tau, k)$ we can find $\varphi \in [0, 2\pi]$ and $\psi \in [0, 2\pi]$ such that

$$\langle y, \tau\rangle = 1 - re^{i\varphi}, \quad 0 \leq r \leq \sqrt{k(1-\|y\|^2)}, \tag{3.17}$$

and

$$\langle z, \tau\rangle = 1 - \sqrt{k(1-\|z\|^2)}e^{i\psi}. \tag{3.18}$$

Then, by using (3.17) and (3.18), we calculate

$$
\begin{aligned}
\operatorname{Re}\langle y, z^*\rangle &= \operatorname{Re}\left(\frac{\langle y, z\rangle}{1-\|z\|^2} - \frac{\langle y, \tau\rangle}{1-\langle z, \tau\rangle}\right) = \operatorname{Re}\left(\frac{\langle y, z\rangle}{1-\|z\|^2} - \frac{1-re^{i\varphi}}{1-\langle z, \tau\rangle}\right) = \\
&= \operatorname{Re}\left(\frac{\langle y, z\rangle}{1-\|z\|^2} - \frac{1}{1-\langle z, \tau\rangle} + \frac{re^{i\varphi}}{\sqrt{k(1-\|z\|^2)e^{i\psi}}}\right) \le \\
&\le \operatorname{Re}\left(\frac{\langle y, z\rangle}{1-\|z\|^2} - \frac{1}{1-\langle z, \tau\rangle} + \frac{\sqrt{1-\|y\|^2}}{\sqrt{1-\|z\|^2}}\right) = \\
&= \operatorname{Re}\left(\frac{\langle y, z\rangle + \sqrt{(1-\|z\|^2)(1-\|y\|^2)}}{1-\|z\|^2} - \frac{1}{1-\langle z, \tau\rangle}\right).
\end{aligned} \tag{3.19}
$$

Since the inequality

$$(1-\|y\|^2)(1-\|z\|^2) \le (1-\operatorname{Re}\langle y, z\rangle)^2$$

is true for all $z, y \in \mathbb{B}$, we get

$$\operatorname{Re}\langle y, z\rangle + \sqrt{(1-\|z\|^2)(1-\|y\|^2)} \le 1\,,$$

and continuing (3.19) we have

$$\operatorname{Re}\langle y, z^*\rangle \le \operatorname{Re}\left(\frac{1}{1-\|z\|^2} - \frac{1}{1-\langle z, \tau\rangle}\right) = \operatorname{Re}\langle z, z^*\rangle.$$

Now, if $\tau \in \mathbb{B}$, then by (3.14) for each $z \in \partial E(\tau\,, k)$ we have

$$\operatorname{Re}\langle f(z)\,, z^*\rangle \ge 0\,, \tag{3.20}$$

where z^* is the support functional of $E(\tau\,, k)$ at z. This is equivalent to the so-called «flow-invariance condition» for f on $\overline{E(\tau\,, k)}$:

$$\lim_{h\to 0+} \frac{\operatorname{dist}(x - hf(x)\,, \overline{E(\tau\,, k)})}{h} = 0$$

(see [14, 15]).

Since $E(\tau\,, k)$ is strictly inside $\mathbb{B}$, the mapping f is Lipschitzian on this set and our assertion is a direct consequence of Proposition 2 in [14].

To complete our proof, we return to the case when $\tau \in \partial\mathbb{B}$. In this case (3.20) holds not only for all $z \in \partial E(\tau\,, k)$, $z \neq \tau$, but also for $z = z^* = \tau$ by the discussion following (3.13). By our assumptions, the mapping f is, in fact, Lipschitzian on $\overline{E(\tau\,, k)}$ in this case too. Therefore we see, once again, that $\overline{E(\tau\,, k)}$ is flow-invariant for f by (3.20). It follows that f is semi-complete on $\mathbb{B}$. However, once we know this, we can show that τ is a flow-invariance point for f even if we only assume that f is merely uniformly continuous on $\overline{\mathbb{B}}$. Indeed, if we show that f satisfies the range condition (RC) on each open ellipsoid $E(\tau\,, k)$, we will be done. To this end, fix $k > 0$ and

$y \in E(\tau, k)$ and consider the mapping $h \in \mathrm{Hol}(\mathbb{B}, H)$ defined by the formula

$$h(x) = x + rf(x) - y, \tag{3.21}$$

where $r \geq 0$. The argument leading to (3.20) also implies that if $\operatorname{dist}(y, \partial E(\tau, k)) = \delta > 0$ and $\langle y, \tau \rangle = 1 - \mu\sqrt{k(1 - \|y\|^2)}$ with $0 < \mu < 1$, then for each $z \in \partial E(\tau, k)$,

$$\mathrm{Re}\,\langle h(z), z^* \rangle = r\mathrm{Re}\,\langle f(z), z^* \rangle + \mathrm{Re}\,\langle z - y, z^* \rangle \geq \frac{(1-\mu)\delta^2}{2(1 - \|z\|^2)}.$$

(The vector z^* is again the support functional of $E(\tau, k)$ at z defined by (3.15)).

Since a computation shows that

$$\|z^*\|^2 = \frac{1}{(1 - \|z\|^2)^2}\left(1 - \frac{\|z\|^2 - |\langle z, \tau \rangle|^2}{k}\right),$$

we deduce that

$$\mathrm{Re}\langle h(z), \hat{z} \rangle \geq (1-\mu)\delta^2/2 = \epsilon > 0$$

for all $z \in \partial E(\tau, k)$, where $\hat{z}$ is the support functional of unit norm of $E(\tau, k)$ at z. We also note that since $\mathrm{hol}(\mathbb{B})$ is a real cone, h is also a semi-complete vector field on $\mathbb{B}$, and its resolvent $J := (I + h)^{-1}$ is a holomorphic self-mapping of $\mathbb{B}$. Moreover, h is uniformly continuous on $\overline{\mathbb{B}}$.

Now one can take a homothety of the boundary $\partial E(\tau, k)$ to $\partial \widetilde{E}$, such that the convex set $\widetilde{E}$ lies strictly inside $E(\tau, k)$, $y \in \widetilde{E}$, and for each $\omega \in \partial \widetilde{E}$,

$$\mathrm{Re}\,\langle h(\omega), \hat{\omega} \rangle \geq \epsilon/2.$$

(Here $\hat{\omega}$ is the support functional of unit norm of $\widetilde{E}$ at ω). Again this means that h satisfies the flow-invariance condition on the closure of $\widetilde{E}$. Hence its resolvent $J = (I + h)^{-1}$ is a holomorphic self-mapping of $\widetilde{E}$. Since $\widetilde{E}$ is a ρ-bounded convex subset of $(\mathbb{B}, \rho)$, it follows by Theorem 23.1 in [6] and Theorem 5 in [16] that $J = (I + h)^{-1}$ has a fixed point u in the closure of $\widetilde{E}$. This point u belongs to $E(\tau, k)$ and it is obvious that $g(u) = 0$. In other words, we have shown that for each $y \in E(\tau, k)$ and for each $r \geq 0$, the equation

$$x + rf(x) = y$$

has a solution in $E(\tau, k)$. Actually, this solution is unique because $u = (I + rf)^{-1}(y)$. This concludes the proof of our theorem.

COROLLARY 3.1. *Suppose that for some bounded $f \in \mathrm{Hol}(\overline{\mathbb{B}}, H)$ the inequality* (3.12) *can be solved by some τ in $\overline{\mathbb{B}}$. Then f is a ρ-monotone mapping on $\mathbb{B}$ and this solution τ is a null point of f in $\overline{\mathbb{B}}$.*

From now we will concentrate on the case when $\tau \in \mathbb{B}$.

The first question which follows from the one-dimensional case is: *How can all semi-complete vector fields vanishing at τ be represented*? (In the one-dimensional case this set can be represented by the class of Carathéodory functions).

The second question is specific to higher dimensions. We already know that if f has a null point in $\mathbb{B}$, then the set of null points of f in $\mathbb{B}$ is an affine submanifold of $\mathbb{B}$. Let us suppose that an affine submanifold L of $\mathbb{B}$ is given. So the question is: *How can all semi-complete vector fields vanishing on L be described*?

For the finite dimensional case we are able to give complete answers to these questions. This will be done in the next section. Here we establish a preliminary general result which is interesting in itself.

For $\tau \in \mathbb{B}$, consider the subset K_τ of $\mathrm{hol}(\mathbb{B})$ defined as follows:

$$K_\tau = \{f \in \mathrm{hol}(\mathbb{B}) : f(\tau) = 0\}.$$

It is clear that this set is a real cone. We will show that for each pair τ_1 and τ_2 in $\mathbb{B}$ the cones K_{τ_1} and K_{τ_2} are linearly isomorphic. Moreover, for each $\tau \in \mathbb{B}$ there is an isomorphism T of K_τ onto K_0 which is the restriction of a linear involution.

To this end, let us denote by M_τ a Möbius transformation of the unit ball $\mathbb{B}$ such that $M_\tau(0) = \tau$ and $M_\tau(\tau) = 0$, where τ is a point in $\mathbb{B}$. In other words, we choose M_τ so that it will be an involution of $\mathbb{B}$, *i.e.*, $M_\tau = M_\tau^{-1}$.

Then, if $\{F_t\}$, $t \geq 0$, is a semigroup on $\mathbb{B}$ such that $F_t(\tau) = \tau$, the family $\{G_t\}_{t\geq 0}$ defined by

$$G_t(x) = (M_\tau \circ F_t \circ M_\tau)(x) \tag{3.22}$$

is also a one-parameter semigroup on $\mathbb{B}$ preserving 0 as a common fixed point. Furthermore, if $S = \{F_t\}_{t\geq 0}$ is differentiable at $t = 0^+$, *i.e.*, $S = S_f$ is a semigroup generated by $f \in \mathrm{hol}(\mathbb{B})$, then $S_g = \{G_t\}_{t\geq 0}$, where G_t is defined by (3.22), is a semigroup generated by the following mapping g:

$$g(x) = A(x)[(f \circ M_\tau)(x)]\,, \tag{3.23}$$

where $A(x)$, for a fixed $x \in \mathbb{B}$, is a linear operator on H defined as follows:

$$A(x) = (M_\tau)'(M_\tau(x)). \tag{3.24}$$

That is, $A(x)$ is the Fréchet derivative of the mapping M_τ at the point $M_\tau(x)$. By $A(\cdot)[b]$ we denote the value of this operator at an element $b \in H$. In fact, to see that g is the generator of the semigroup $\{G_t\}_{t\geq 0}$ defined by (3.22) it is enough to check that if $u = u(t)$ is a solution of the equation $u' + f(u) = 0$, then $v = M_\tau(u)$ is a solution of the equation $v' + g(v) = 0$. Indeed, since M_τ is an involution, we have $u = M_\tau(v)$. Hence

$$v'(t) = (M_\tau)'(u(t))[u'(t)] = (M_\tau)'(M_\tau(v))[u'(t)] = (M_\tau)'(M_\tau(v(t)))[-f(M_\tau(v(t)))] = -g(v(t)),$$

and we are done.

Finally, we note that for each fixed $x \in \mathbb{B}$, the operator $A(x)$ defined by (3.24) is invertible and, moreover,

$$(A(x))^{-1} = A(M_\tau(x)). \tag{3.25}$$

Indeed, since $A(M_\tau(x)) = (M_\tau)'(x)$ and $M_\tau(M_\tau(x)) = x$ for all $x \in \mathbb{B}$, by differentiating the latter equality we get

$$[M_\tau(M_\tau(x))]' = A(x)[A(M_\tau(x))] = I,$$

where I is the identity operator on H, and this implies (3.25). Now, using (3.23) and (3.25) we also have

$$f(x) = A(x)[(g \circ M_\tau)(x)]. \tag{3.26}$$

That is, the linear operator T on $\mathrm{Hol}(\mathbb{B}, H)$ defined by the formula

$$T(f) = A(\cdot)[(f \circ M_\tau)] \tag{3.27}$$

is invertible and $T^2 = I$.

So, we have proved the following assertion.

THEOREM 3.2. *Let τ be a point in $\mathbb{B}$, and let $K_\tau = \{f \in \mathrm{hol}(\mathbb{B}) : f(\tau) = 0\}$ be the cone of all semi-complete vector fields vanishing at τ. Then the linear operator T on $\mathrm{Hol}(\mathbb{B}, H)$ given by the formula (3.27) is an involution and takes K_τ to the cone $K_0 = \{g \in \mathrm{hol}(\mathbb{B}) : g(0) = 0\}$ of all semi-complete vector fields preserving 0. That is, for $f \in \mathrm{hol}(\mathbb{B})$ we have*

$$g(x) = A(x)[f(M_\tau(x))], \quad g \in \mathrm{hol}(\mathbb{B}),$$
$$f(x) = A(x)[g(M_\tau(x))],$$

and $f(\tau) = 0$ if and only if $g(0) = 0$.

REMARK 3.1. If we define a Möbius transformation $M_\tau : \mathbb{B} \to \mathbb{B}$ by the formula

$$M_\tau(x) = \frac{1}{1 - \langle x, \tau \rangle} \left[\tau - sx + \frac{(s-1)\langle x, \tau \rangle}{\|\tau\|^2} \tau \right], \tag{3.28}$$

where $s = \sqrt{1 - \|\tau\|^2}$, then one can calculate the operator $A(x)$ in (3.24) in the explicit form

$$A(x)[b] = \frac{1 - \langle x, \tau \rangle}{1 - \|\tau\|^2} \left[\langle b, \tau \rangle x - sb + \frac{(s-1)\langle b, \tau \rangle}{\|\tau\|^2} \tau \right]. \tag{3.29}$$

We will use such a representation in the sequel.

4. Semi-complete vector fields on the unit ball in $\mathbb{C}^n$ with null points

In this Section we consider the case when $H = \mathbb{C}^n$ is a complex Euclidean space with the inner product $\langle z, w \rangle = \sum_{i=1}^n z_i \overline{w}_i$ and the induced norm $\|\cdot\|$ defined by $\|z\| = \sqrt{\langle z, z \rangle}$. In addition, we introduce the following sesquilinear matrix valued form in $\mathbb{C}^n$:

$$[z; w] = \begin{pmatrix} z_1\overline{w}_1 & z_1\overline{w}_2 & \cdots & z_1\overline{w}_n \\ \vdots & \vdots & & \vdots \\ z_n\overline{w}_1 & z_n\overline{w}_2 & \cdots & z_n\overline{w}_n \end{pmatrix}.$$

It is clear that for $n = 1$, $[z; w] = \langle z, w \rangle = z\overline{w}$. Now in $\mathbb{C}^n$ the operator $A(\cdot)$ defined by (3.24) can be considered an $n \times n$ square matrix, the entries of which are holomorphic functions, $a_j^i(\cdot)$, $i, j = 1, \ldots, n$. Thus, by (3.29), we have for a given τ, $\|\tau\| < 1$, that the operator A has the form $A = (a_j^i)$, with

$$a_j^i(z) = \frac{1 - \langle z, \tau\rangle}{1 - \|\tau\|^2}\left(z_i\overline{\tau_j} + \frac{\tau_i\overline{\tau_j}}{\|\tau\|^2}(s-1) - s\delta_{ij}\right), \tag{4.1}$$

where $z = (z_1, \ldots, z_n)$, $\tau = (\tau_1, \ldots, \tau_n)$ and δ_{ij} is Kronecker's symbol. Now let $\mathbb{B} = \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the open unit ball. We first describe the cone K_τ of all semi-complete vector fields vanishing at a given point $\tau \in \mathbb{B}$, *i.e.*,

$$K_\tau = \{f \in \mathrm{hol}(\mathbb{B}) : f(\tau) = 0\}.$$

THEOREM 4.1. *A mapping $f = (f_1, \ldots, f_n) \in \mathrm{Hol}(\mathbb{B}, \mathbb{C}^n)$ belongs to K_τ, $\tau \in \mathbb{B}$, if and only if there is a square $n \times n$ matrix $Q(\cdot) = (q_j^i(\cdot))$, $i, j = 1, \ldots, n$, the entries of which are holomorphic functions q_j^i, $i, j = 1, \ldots, n$, on $\mathbb{B}$, such that*

$$\mathrm{Re}\,\langle Q(z)(z-\tau), z-\tau\rangle \geq 0, \quad z \in \mathbb{B}, \tag{4.2}$$

and

$$f(z) = (I - [z;\tau])\,Q(z)(z-\tau), \tag{4.3}$$

where I is the identity matrix on $\mathbb{C}^n$.

REMARK 4.1. For $n = 1$ condition (4.2) means that $Q(z) = q_1^1(z)$ is a function of the class of Carathéodory. Hence, in this case formula (4.3) coincides with the Berkson-Porta formula (1.10).

REMARK 4.2. Conditions (4.2) and (4.3) can be rewritten in an equivalent coordinate form:

$$\mathrm{Re}\sum_{\ell, m=1}^{n} q_m^\ell(z)(z_\ell - \tau_\ell)(\overline{z_m - \tau_m}) \geq 0, \quad z \in \mathbb{B}, \tag{4.2$'$}$$

and

$$f_i(z) = \sum_{\ell, m=1}^{n} (\delta_{im} - z_i\overline{\tau}_m)q_m^\ell(z)(z_\ell - \tau_\ell), \quad 1 \leq i \leq n. \tag{4.3$'$}$$

Therefore, we have to prove the representation (4.3$'$) with condition (4.2$'$) for the mappings f belonging to the cone K_τ.

The key to our arguments is the following well-known lemma due to H. Hefer (see [7, 19, 8]).

LEMMA 4.1. *Let $f \in \mathrm{Hol}(\mathbb{B}, \mathbb{C})$ be a holomorphic function from $\mathbb{B}$ into $\mathbb{C}$ such that for some $1 \leq k \leq n$ the set $\{z \in \mathbb{B} : z_1 = z_2 = \cdots = z_k = 0\}$ is contained in $\mathrm{Null}_{\mathbb{B}}\, f$. Then there exist k holomorphic functions $h_1(z), \ldots, h_k(z)$ on $\mathbb{B}$ such that*

$$f(z) = \sum_{i=1}^{k} z_i h_i(z).$$

PROOF OF THEOREM 4.1. Let $f \in K_\tau$, $\tau \in \mathbb{B}$. By Theorem 3.2, there is a linear operator T defined on $\mathrm{Hol}(\mathbb{B}, \mathbb{C}^n)$ such that $T^2 = I$ and $g = Tf \in K_0$, *i.e.*, $g(0) = 0$. By Lemma 4.1,

$$g_j(z) = \sum_{k=1}^{n} h_j^k(z) z_k , \tag{4.4}$$

where g_j is the j-th coordinate of g. Since g is semi-complete, we also have

$$\mathrm{Re}\,\langle g(z), z\rangle \geq 0, \quad \forall z \in \mathbb{B}. \tag{4.5}$$

For the functions $\{h_j^k\}_{j,k=1,\dots,n}$ in (4.4), condition (4.5) implies that

$$\mathrm{Re} \sum_{j,k=1}^{n} h_j^k(z) z_k \bar{z}_j \geq 0, \quad z \in \mathbb{B}. \tag{4.5$'$}$$

Since $T = T^{-1}$, we have $f = Tg$, that is,

$$f_i(z) = \sum_{j=1}^{n} a_j^i(z) g_j(M_\tau(z)), \tag{4.6}$$

where M_τ is the Möbius transformation on $\mathbb{B}$ defined by (3.28). Substituting into (4.6) the expressions for the elements $a_j^i(z)$ in (4.1) and using (4.4) we get

$$f_i(z) = \sum_{j=1}^{n} \frac{1 - \langle z, \tau\rangle}{1 - \|\tau\|^2} \left(z_i \overline{\tau}_j + \frac{(s-1)\tau_i \overline{\tau}_j}{\|\tau\|^2} - s\delta_{ij} \right) \left(\sum_{k=1}^{k} h_j^k(M_\tau(z))(M_\tau(z))_k \right).$$

In addition, from (3.28) we see that

$$(M_\tau(z))_k = \frac{1}{(1 - \langle z, \tau\rangle)} \left[\tau_k - s z_k + \frac{(s-1)\langle z, \tau\rangle}{\|\tau\|^2} \tau_k \right]. \tag{4.7}$$

Hence,

$$f_i(z) = \sum_{j,k=1}^{n} \frac{1}{1 - \|\tau\|^2} h_j^k(M_\tau(z)) \left(z_i \overline{\tau}_j + \frac{(s-1)\tau_i \overline{\tau}_j}{\|\tau\|^2} - s\delta_{ij} \right) \left(\tau_k - s z_k + \frac{(s-1)\langle z, \tau\rangle}{\|\tau\|^2} \tau_k \right) =$$

$$= \sum_{j,k=1}^{n} \frac{h_j^k(M_\tau(z))}{1 - \|\tau\|^2} \sum_{m=1}^{n} (\delta_{im} - z_i \overline{\tau}_m) \left((s-1)\frac{\tau_m \overline{\tau}_j}{\|\tau\|^2} - s\delta_{jm} \right) \sum_{\ell=1}^{n} (z_\ell - \tau_\ell) \left((s-1)\frac{\tau_k \overline{\tau}_\ell}{\|\tau\|^2} - s\delta_{k\ell} \right) =$$

$$= \sum_{\ell,m=1}^{n} (\delta_{im} - z_i \overline{\tau}_m)(z_\ell - \tau_\ell) \sum_{j,k=1}^{n} \frac{h_j^k(M_\tau(z))}{1 - \|\tau\|^2} \left((s-1)\frac{\tau_m \overline{\tau}_j}{\|\tau\|^2} - s\delta_{jm} \right) \left((s-1)\frac{\tau_k \overline{\tau}_\ell}{\|\tau\|^2} - s\delta_{k\ell} \right).$$

Now if we define for $m, \ell = 1, \dots, n$,

$$q_m^\ell(z) = \sum_{j,k=1}^n \frac{h_j^k(M_\tau(z))}{1-\|\tau\|^2}\left((s-1)\frac{\tau_m\overline{\tau}_j}{\|\tau\|^2} - s\delta_{jm}\right)\left((s-1)\frac{\tau_k\overline{\tau}_\ell}{\|\tau\|^2} - s\delta_{k\ell}\right), \tag{4.8}$$

then we immediately obtain (4.3′). It remains to check that the functions $q_m^\ell(z)$ satisfy (4.2′). To this end, we substitute in the right hand side of (4.2′) the expressions (4.8) for $q_m^\ell(z)$ and change the order of summation:

$$\begin{aligned}\sum_{\ell,m=1}^n q_m^\ell(z)(z_\ell-\tau_\ell)(\overline{z_m-\tau_m}) &= \sum_{j,k=1}^n \frac{h_j^k(M_\tau(z))}{1-\|\tau\|^2}\sum_{\ell,m=1}^n(\overline{z}_m-\overline{\tau}_m)\times\\ &\times\left((s-1)\frac{\tau_m\overline{\tau}_j}{\|\tau\|^2} - s\delta_{jm}\right)(z_\ell-\tau_\ell)\left((s-1)\frac{\tau_k\overline{\tau}_\ell}{\|\tau\|^2} - s\delta_{k\ell}\right) =\\ &= \sum_{j,k=1}^n \frac{h_j^k(M_\tau(z))}{1-\|\tau\|^2}\left(\overline{\tau}_j + \frac{\langle\tau, z\rangle}{\|\tau\|^2}\overline{\tau}_j(s-1) - s\overline{z}_j\right)\times\\ &\times\left(\tau_k + \frac{\langle z, \tau\rangle}{\|\tau\|^2}\tau_k(s-1) - sz_k\right).\end{aligned}$$

Comparing with (4.7), we obtain

$$\sum_{\ell,m=1}^n q_m^\ell(z)(z_\ell-\tau_\ell)(\overline{z_m-\tau_m}) = \frac{|1-\langle z,\tau\rangle|^2}{1-\|\tau\|^2}\sum_{j,k=1}^n h_j^k(M_\tau(z))(M_\tau(z))_k\overline{(M_\tau(z))_j}.$$

The latter equality together with (4.5′) implies (4.2′). Thus the necessity is proved. Conversely, let the representation (4.3′) hold for some matrix $Q(z) = (q_j^i(z))$ which satisfies (4.2′). We have to show that $f \in K_\tau$. It is clear from (4.3′) that $f(\tau) = 0$. So, we only have to prove that $f \in \operatorname{hol}(\mathbb{B})$. Again, consider the operator T defined by (3.27) and let $g = Tf$. It is enough to show that $g \in \operatorname{hol}(\mathbb{B})$, *i.e.*, that $\operatorname{Re}\langle g(z), z\rangle \geq 0$ for all $z \in \mathbb{B}$. Indeed, we have

$$\langle g(z), z\rangle = \sum_{j=1}^n g_j(z)\overline{z}_j = \sum_{j=1}^n\left(\sum_{i=1}^n d_i^j(z)f_i(M_\tau(z))\right)\overline{z}_j =$$

$$\begin{aligned}&= \sum_{i,j=1}^n \frac{1-\langle z,\tau\rangle}{1-\|\tau\|^2}\left(z_j\overline{\tau}_i + \frac{\tau_j\overline{\tau}_i}{\|\tau\|^2}(s-1) - s\delta_{ij}\right)\times\\ &\times \overline{z}_j\sum_{\ell,m=1}^n(\delta_{im} - (M_\tau(z))_i\overline{\tau}_m)q_m^\ell(M_\tau(z))((M_\tau(z))_\ell - \tau_\ell).\end{aligned} \tag{4.9}$$

Note that

$$\sum_{i,j=1}^{n}\left(z_j\overline{\tau}_i+\frac{\tau_j\overline{\tau}_i}{\|\tau\|^2}(s-1)-s\delta_{ij}\right)\overline{z}_j\big(\delta_{im}-(M_\tau(z))_i\overline{\tau}_m\big)=$$

$$=\sum_{j=1}^{n}\overline{z}_j\left(\big(1-\langle M_\tau(z)\,,\tau\rangle\big)\left(z_j\overline{\tau}_m+\frac{\tau_j\overline{\tau}_m}{\|\tau\|^2}(s-1)\right)+s(M_\tau(z))_j\overline{\tau}_m-s\delta_{jm}\right)=$$

$$=\frac{1-\|\tau\|^2}{1-\langle z\,,\tau\rangle}\left(\|z\|^2\overline{\tau}_m+\frac{\overline{\tau}_m\langle\tau,z\rangle}{\|\tau\|^2}(s-1)\right)+\frac{s\overline{\tau}_m}{1-\langle z,\tau\rangle}\left(\langle\tau,\,z\rangle+\frac{|\langle\tau,z\rangle|^2(s-1)}{\|\tau\|^2}-s\|z\|^2\right)-s\overline{z}_m=$$

$$=\overline{\tau}_m\langle\tau\,,z\rangle\left(1+\frac{s-1}{\|\tau\|^2}\right)-s\overline{z}_m=(1-\langle\tau\,,z\rangle)\left(\overline{(M_\tau(z))_m-\tau_m}\right).$$

Substituting this into (4.9) we get

$$\langle g(z)\,,z\rangle=\sum_{\ell,m=1}^{n}\frac{|1-\langle z\,,\tau\rangle|^2}{1-\|\tau\|^2}q_m^\ell(M_\tau(z))\overline{((M_\tau(z))_m-\tau_m)}((M_\tau(z))_\ell-\tau_\ell).$$

Recalling (4.2′) we now see that $\operatorname{Re}\langle g(z)\,,z\rangle\geq 0$. This completes the proof of Theorem 4.1.

We now turn to the second question raised towards the end of Section 3, namely, let Γ be an $(n-k)$-dimensional affine submanifold of $\mathbb{C}^n$ such that $\Gamma\cap\mathbb{B}\neq\emptyset$.

We wish to describe the real cone K_Γ in $\operatorname{hol}(\mathbb{B})$ defined by

$$K_\Gamma=\left\{f\in\operatorname{hol}(\mathbb{B}):\quad f|_{\Gamma\cap\mathbb{B}}=0\right\}. \tag{4.10}$$

First we consider the following linear subspace Γ_0 of $\mathbb{C}^n$:

$$\Gamma_0=\{z\in\mathbb{C}^n:\langle z,\,b^j\rangle=0\,,j=1\,,\ldots\,,k\}\,, \tag{4.11}$$

where $\{b^i:\ 1\leq i\leq n\}\subset\mathbb{C}^n$ is a basis and $1\leq k\leq n$.

The following assertion is a direct consequence of Lemma 4.1 and condition (4.5).

Lemma 4.2. *Let* $g\in\operatorname{Hol}(\mathbb{B}\,,\mathbb{C}^n)$. *Then* $g\in K_{\Gamma_0}$ *if and only if there exist* $n\times k$ *holomorphic functions* $h_i^j(z)\,,\ i=1\,,\ldots\,,n\,,\ j=1\,,\ldots\,,k$, *such that*

$$\operatorname{Re}\sum_{i=1}^{n}\sum_{j=1}^{k}h_i^j(z)\langle z,\,b^j\rangle\langle b^i\,,z\rangle\geq 0\,,\ z\in\mathbb{B}\,, \tag{4.12}$$

and

$$g(z)=\sum_{i=1}^{n}\sum_{j=1}^{k}h_i^j(z)\langle z,\,b^j\rangle b^i. \tag{4.13}$$

Remark 4.3. Note that the sum in (4.13) is the representation of the vector g using the basis $\{b^1\,,\ldots\,,b^n\}$.

In the sequel, without loss of generality, we will assume that $(0\,,\ldots\,,0)\notin\Gamma$. To obtain a description of K_Γ we need to represent Γ in some special form. More precisely, choose $\tau \in \Gamma$ such that for at least one $z \in \Gamma$,

$$\langle \tau - z , \tau \rangle \neq 0.$$

This implies that

$$\langle \tau + \frac{(s-1)\langle z, \tau \rangle}{\|\tau\|^2}\tau - sz , \tau \rangle \neq 0 ,$$

where $s = \sqrt{1 - \|\tau\|^2}$, and consequently,

$$\langle M_\tau(z) , \tau \rangle \neq 0 ,$$

where M_τ is the Möbius transformation on $\mathbb{B}$ defined by (3.28). Since $0 \notin \Gamma$, $\tau \notin$ $\notin M_\tau(\Gamma)$. We also know that $\tau \notin M_\tau(\Gamma)^\perp$, where $L^\perp$ is the subspace orthogonal to the set $L \subset \mathbb{C}^n$. Hence, one can choose an orthonormal basis $\{b^1 , \dots , b^n\}$ in $\mathbb{C}^n$ such that $M_\tau(\Gamma)$ will be represented by Γ_0, as in (4.11). In addition, $\langle \tau , b^j \rangle \neq 0$ for all $j = 1 , \dots , n$. Now consider the pre-image of the hyperplane $\{z \in \mathbb{C}^n : \langle z, b^j \rangle = 0\}$, where $1 \leq j \leq n$, under the mapping M_τ. We have the following series of calculations:

$$\langle \tau , b^j \rangle + (s-1)\frac{\langle z, \tau \rangle}{\|\tau\|^2}\langle \tau , b^j \rangle - s\langle z, b^j \rangle = 0;$$

$$s\langle z, b^j \rangle - (s-1)\frac{\langle z, \tau \rangle}{\|\tau\|^2}\langle \tau , b^j \rangle = \langle \tau , b^j \rangle;$$

and

$$\langle z, \frac{sb^j}{\langle \tau , b^j \rangle} - (s-1)\frac{\tau}{\|\tau\|^2} \rangle = 1.$$

If we denote $\zeta^j = \frac{sb^j}{\langle \tau , b^j \rangle} - (s-1)\frac{\tau}{\|\tau\|^2}$, $j = 1 , \dots , n$, we obtain

$$\Gamma = \{z : \langle z, \zeta^j \rangle = 1 , \ j = 1 , \dots , k\} \tag{4.14}$$

$$\langle \tau , \zeta^j \rangle = 1 , \ j = 1 , \dots , n \text{ and } \langle \zeta^i , \zeta^j \rangle = 1 , \ i \neq j.$$

Next, we express b^j in terms of ζ^j , $j = 1 , \dots , n$. Since M_τ is an involution, the affine hyperplane $\{z \in \mathbb{C}^n : \langle z, \zeta^j \rangle = 1\}$ is transformed into $\{z \in \mathbb{C}^n : \langle M_\tau(z) , \zeta^j \rangle = 1\}$. Hence,

$$\langle \tau + (s-1)\frac{\langle z, \tau \rangle}{\|\tau\|^2}\tau - sz, \zeta^j \rangle = 1 - \langle z, \tau \rangle$$

and

$$\langle z, \tau + \frac{(s-1)}{\|\tau\|^2}\tau - s\zeta^j \rangle = 0$$

because $\langle \tau , \zeta^j \rangle = 1$. It is clear that we can take $\{b^j\}_{j=1}^n$ as $b^j = (\tau + \frac{(s-1)}{\|\tau\|^2}\tau - s\zeta)\lambda_j$, where $\lambda_j \in \mathbb{C}$, $j = 1 , \dots , n$, are chosen such that $\{b^j\}_{j=1}^n$ is an orthonormal basis. As a matter of fact, simple calculations show that

$$\langle b^i , b^j \rangle = \lambda_i \overline{\lambda}_j (1 - \|\tau\|^2)(\langle \zeta^i , \zeta^j \rangle - 1).$$

Therefore, we can set

$$b^j = \frac{1}{s\sqrt{\|\zeta^j\|^2 - 1}} \left(\tau + \frac{(s-1)}{\|\tau\|^2}\tau - s\zeta^j \right). \tag{4.15}$$

It is also not difficult to see that

$$\langle M_\tau(z), b^j \rangle = \left(\frac{s}{\sqrt{\|\zeta^j\|^2 - 1}} \right) \left(\frac{\langle z, \zeta^j \rangle - 1}{1 - \langle z, \tau \rangle} \right). \tag{4.16}$$

In particular, setting $z = 0$ in (4.16) we get,

$$\langle \tau, b^j \rangle = \langle b^j, \tau \rangle = -\frac{s}{\sqrt{\|\zeta^j\|^2 - 1}}. \tag{4.16$'$}$$

We can now describe the cone K_Γ of all those semi-complete vector fields on $\mathbb{B}$ vanishing on Γ.

Theorem 4.2. *Let Γ be given by* (4.14) *and let $K_\Gamma = \{f \in \mathrm{hol}(\mathbb{B}) : f|_{\Gamma\cap\mathbb{B}} = 0\}$. Then $f \in K_\Gamma$ if and only if there are $n \times k$ holomorphic functions $q_i^j(z)$, $i = 1, \dots, n$, $j = 1, \dots, k \leq n$, such that*

$$\mathrm{Re} \sum_{i=1}^{n} \sum_{j=1}^{k} q_i^j(z)(1 - \langle z, \zeta^j \rangle)(1 - \langle \zeta^i, z \rangle) \geq 0, \quad z \in \mathbb{B}, \tag{4.17}$$

and

$$f(z) = \sum_{i=1}^{n} \sum_{j=1}^{k} q_i^j(z)(1 - \langle z, \zeta^j \rangle)(z - \zeta^i). \tag{4.18}$$

Proof. Let $f \in K_\Gamma$. Again, by Theorem 3.2 and Lemma 4.2, the mapping $g = Tf \in K_{\Gamma_0}$ and

$$g_\ell(z) = \sum_{i=1}^{n} \sum_{j=1}^{k} h_i^j(z) \langle z, b^j \rangle (b^i)_\ell.$$

Furthermore, by (4.16) we have

$$g_\ell(M_\tau(z)) = \sum_{i=1}^{n} \sum_{j=1}^{k} h_i^j(M_\tau(z)) \left(\frac{s}{\sqrt{\|\zeta^j\|^2 - 1}} \right) \left(\frac{\langle z, \zeta^j \rangle - 1}{1 - \langle z, \tau \rangle} \right) (b^i)_\ell.$$

We now compute $f = Tg$:

$$f_p(z) = \sum_{\ell=1}^{n} a_\ell^p(z) g_\ell(M_\tau(z)) =$$

$$= \sum_{\ell=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{k}(b^i)_\ell h_i^j(M_\tau(z))\frac{s}{\sqrt{\|\zeta^j\|^2-1}}\frac{\langle z,\zeta^j\rangle-1}{1-\langle z,\tau\rangle}\left(\frac{1-\langle z,\tau\rangle}{1-\|\tau\|^2}\right)\times$$

$$\times\left(z_p\overline{\tau}_\ell+\frac{\tau_p\overline{\tau}_\ell}{\|\tau\|^2}(s-1)-s\delta_{\ell p}\right)=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{k} h_i^j(M_\tau(z))\frac{\langle z,\tau^j\rangle-1}{s\sqrt{\|\zeta^j\|^2-1}}\left(z_p\langle b^i,\tau\rangle+\tau_p\frac{\langle b^i,\tau\rangle}{\|\tau\|^2}(s-1)-s(b^i)_p\right).$$

The expression for b^i in (4.15) and formula (4.16′) imply that

$$f_p(z)) = \sum_{i=1}^{n}\sum_{j=1}^{k} h_i^j(M_\tau(z))\frac{(1-\langle z,\zeta^j\rangle)(z_p-(\zeta^i)_p)}{\sqrt{(\|\zeta^i\|^2-1)(\|\zeta^j\|^2-1)}}.$$

Now setting

$$q_i^j(z) = h_i^j(M_\tau(z))\frac{1}{\sqrt{(\|\zeta^i\|^2-1)(\|\zeta^j\|^2-1)}},$$

we get (4.18).

To prove (4.17) we can use (4.16):

$$\sum_{i=1}^{n}\sum_{j=1}^{k} q_i^j(z)(1-\langle z,\zeta^j\rangle)(1-\langle\zeta^i,z\rangle) = \sum_{i=1}^{n}\sum_{j=1}^{k} h_i^j(M_\tau(z))\frac{(\langle z,\zeta^j\rangle-1)(\langle\zeta^i,z\rangle-1)}{\sqrt{(\|\zeta^i\|^2-1)(\|\zeta^j\|^2-1)}} =$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{k} h_i^j(M_\tau(z))\frac{\langle M_\tau(z),b^j\rangle(1-\langle z,\tau\rangle)}{s}\frac{\langle b^i,M_\tau(z)\rangle(1-\langle\tau,z\rangle)}{s}=$$

$$=\frac{|1-\langle z,\tau\rangle|^2}{1-\|\tau\|^2}\sum_{i=1}^{n}\sum_{j=1}^{k} h_i^j(M_\tau(z))\langle M_\tau(z),b^j\rangle\langle b^i,M_\tau(z)\rangle.$$

It remains to apply inequality (4.12) to complete the proof of the necessity. Conversely, let $f\in \mathrm{Hol}(\mathbb{B},\mathbb{C}^n)$ have the form (4.18) with the functions q_i^j satisfying (4.17). To show that $f\in K_\Gamma$, we first note that by (4.14), the functions $1-\langle z,\zeta^j\rangle$, $1\le j\le k$, vanish on Γ. Hence, for all $z\in\Gamma$, $f(z)=0$. So, it is enough to show that $g=Tf\in$

$\in \mathrm{hol}(\mathbb{B})$. In other words, we have to show that condition (4.5) holds. We have

$$\langle g(z)\,,\,z\rangle = \sum_{m=1}^{n} g_m(z)\overline{z}_m = \sum_{m=1}^{n}\sum_{\ell=1}^{n} a_\ell^m(z) f_\ell(M_\tau(z))\overline{z}_m =$$

$$= \sum_{\ell=1}^{n} \frac{1-\langle z\,,\,\tau\rangle}{1-\|\tau\|^2}\left[\|z\|^2\overline{\tau}_\ell + \frac{\langle \tau\,,\,z\rangle\overline{\tau}_\ell(s-1)}{\|\tau\|^2} - s\overline{z}_\ell\right] f_\ell(M_\tau(z)) =$$

$$= \sum_{\ell=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{k} \frac{1-\langle z\,,\,\tau\rangle}{1-\|\tau\|^2}\left(\|z\|^2\overline{\tau}_\ell + \frac{\langle \tau\,,\,z\rangle}{\|\tau\|^2}\overline{\tau}_\ell(s-1) - s\overline{z}_\ell\right)\times$$

$$\times\ q_i^j(M_\tau(z))(1-\langle M_\tau(z)\,,\,\zeta^j\rangle)((M_\tau(z))_\ell - (\zeta^i)_\ell).$$

Recalling that $\langle \zeta^i\,,\,\tau\rangle = 1$, and summing up from $\ell = 1$ to $\ell = n$ we get,

$$\langle g(z)\,,\,z\rangle = \sum_{i=1}^{n}\sum_{j=1}^{k} \frac{1-\langle z\,,\,\tau\rangle}{1-\|\tau\|^2} q_i^j(M_\tau(z))(1-\langle M_\tau(z)\,,\,\zeta^j\rangle)(1-\langle \tau, z\rangle)(1-\langle \zeta^i, M_\tau(z)\rangle) =$$

$$= \frac{|1-\langle z\,,\,\tau\rangle|^2}{1-\|\tau\|^2}\sum_{i=1}^{n}\sum_{j=1}^{k} q_i^j(M_\tau(z))(1-\langle M_\tau(z)\,,\,\zeta^j\rangle)(1-\langle \zeta^i, M_\tau(z)\rangle).$$

Finally, using (4.17) we obtain

$$\mathrm{Re}\,\langle g(z)\,,\,z\rangle \geq 0$$

and we are done. Theorem 4.2 is proved.

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References

[1] M. Abate, *The infinitesimal generators of semigroups of holomorphic maps*. Ann. Mat. Pura Appl., 161, 1992, 161-180.
[2] E. Berkson - H. Porta, *Semigroups of analytic functions and composition operators*. Michigan Math. J., 25, 1978, 101-115.
[3] C. C. Cowen - B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*. CRC Press, Boca Raton, FL 1995.
[4] S. Dineen, *The Schwarz Lemma*. Clarendon Press, Oxford 1989.
[5] T. Franzoni - E. Vesentini, *Holomorphic Maps and Invariant Distances*. North Holland, Amsterdam 1980.
[6] K. Goebel - S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*. Dekker, New York-Basel 1984.
[7] H. Hefer, *Zur Funktionentheorie mehrerer Veränderlichen. Über eine Zerlegung analytischer Funktionen und die Weilsche Integraldarstellung*. Math. Ann., 122, 1950, 276-278.
[8] G. Henkin - J. Leiterer, *Theory of Functions on Complex Manifolds*. Birkhäuser, Basel 1984.

[9] T. L. HAYDEN - T. J. SUFFRIDGE, *Biholomorphic maps in Hilbert space have a fixed point*. Pacific J. Math., 38, 1971, 419-422.
[10] J. M. ISIDRO - L. L. STACHO, *Holomorphic Automorphism Groups in Banach Spaces: An Elementary Introduction*. North-Holland, Amsterdam 1984.
[11] V. KHATSKEVICH - S. REICH - D. SHOIKHET, *Complex dynamical systems on bounded symmetric domains*. Electronic J. Differential Equations, 19, 1997, 1-9.
[12] R. H. MARTIN JR., *Differential equations on closed subsets of a Banach space*. Trans. Amer. Math. Soc., 179, 1973, 399-414.
[13] R. H. MARTIN JR., *Nonlinear Operators and Differential Equations in Banach Spaces*. Wiley, New York 1976.
[14] S. REICH, *Minimal displacement of points under weakly inward pseudo-Lipschitzian mappings*. Rend. Mat. Acc. Lincei, s. 8, v. 59, 1975, 40-44.
[15] S. REICH, *On fixed point theorems obtained from existence theorems for differential equations*. J. Math. Anal. Appl., 54, 1976, 26-36.
[16] S. REICH, *Averaged mappings in the Hilbert ball*. J. Math. Anal. Appl., 109, 1985, 199-206.
[17] S. REICH - D. SHOIKHET, *Generation theory for semigroups of holomorphic mappings*. Abstract and Applied Analysis, 1, 1996, 1-44.
[18] S. REICH - D. SHOIKHET, *Semigroups and generators on convex domains with the hyperbolic metric*. Rend. Mat. Acc. Lincei, s. 9, v. 8, 1997, 231-250.
[19] B.V. SHABAT, *Introduction to Complex Analysis Part II. Functions of Several Variables*. Trans. Math. Monographs, 110, American Math. Soc., Providence RI 1992.
[20] H. UPMEIER, *Jordan Algebras in Analysis, Operator Theory and Quantum Mechanics*. CBMS-NSF Regional Conference Series in Math., American Math. Soc., Providence RI 1987.

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