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## A note on Jeu de Taquin

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**Algebra.** — *A note on Jeu de Taquin.* Nota (*) di Rocco Chirivì, presentata dal Corrisp. C. De Concini.

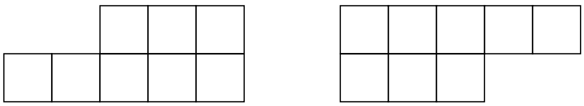
Abstract. — A direct formula for jeu de taquin applied to the swap of two rows of standard tableaux is given. A generalization of this formula to non standard tableaux is used to describe combinatorially a path basis isomorphism for the algebra of type $A_\ell$.

Key words: Jeu de taquin; Skew tableaux; LS paths; Root operators.

Riassunto. — *Una nota sul Jeu de Taquin.* Otteniamo una formula diretta per il *jeu de taquin* applicato allo scambio di due righe di un tableau standard. Una generalizzazione di questa formula ai tableaux non standard è usata per descrivere, dal punto di vista combinatorio, un isomorfismo di basi di cammini per l'algebra di tipo $A_\ell$.

## 1. Introduction

In this paper we mainly deal with tableaux consisting of two rows. We are interested in skew tableaux with shape $(m, m) \setminus (m-n)$ or shape $(m, n)$, where $m, n$ are positive integers and $m \geq n$. These two shapes can be obtained by swapping the two rows of their diagrams. In the following figure we see an example for $m = 5$, $n = 3$.



In what follows we will describe a lifting of this «swapping» map from diagrams to (also non standard) tableaux. If $T$ is a skew tableau with two rows of length $m, n$, then we write $T = R_1, R_2$, and $R_1 = r_{1,1} r_{1,2} \cdots r_{1,m}$, $R_2 = r_{2,1} r_{2,2} \cdots r_{2,n}$. We always fix the shape of $T$ in the following sense: if $m \geq n$ then $T$ has shape $(m, n)$ else $T$ has shape $(n, n) \setminus (n - m)$. If $R_1$ and $R_2$ are two rows then by $R_1 \leq R_2$ we mean that in the tableau $T = R_1, R_2$ the numbers do not decrease in coloumns from top to bottom.

Let $T = R_1, R_2$ be a *standard* skew tableau, *i.e.* $R_1 \leq R_2$ and in the two rows $R_1$ and $R_2$ the numbers increase from left to right, with entries out of $\{1, \ldots, \ell, \ell + 1\}$ of shape $(m, m) \setminus (m - n)$ with $n \leq m$. Schützenberger's «jeu de taquin» [2, 5] can be performed step by step to reduce $T$ to a standard tableau $T'$ of shape $(m, n)$. We give here a direct map that avoids this step by step procedure. For $T$ as above, define $H(T)$ as the set of (standard) rows $R$ of length $n$ such that $R \subset R_2$ (*i.e.* $R$

(*) Pervenuta in forma definitiva all'Accademia il 6 luglio 1999.

is a subrow of $R_2$) and $R_1 \leq R$. The set $H(T)$ has a minimum element $\overline{R_2}$ (see Proposition 2.1 below). Let $\overline{R_1} = [(R_1 \cup R_2) \setminus \overline{R_2}]$ be the (standard) row obtained by reordering the set $(R_1 \cup R_2) \setminus \overline{R_2}$ (the elements are counted with multiplicity). We have $\overline{R_1} \leq \overline{R_2}$, *i.e.* the tableau $\overline{R_1}\,, \overline{R_2}$ of shape $(m\,, n)$ is standard. The main result of Section 2 is the that the tableau $\overline{R_1}\,, \overline{R_2}$ equals the tableau $T'$ obtained using jeu de taquin (see Theorem 2.1).

In Section 3 we change a bit our approach. We briefly introduce LS paths and root operators (see [3, 4]), we give an «interpretation» of rows as integral weights and an interpretation of tableaux as LS paths for the algebra of type $A_\ell$. Next we describe root operators for tableaux corresponding to root operators for LS paths under the interpretation. Then we define a generalization of the swapping map to non standard tableaux. This map combinatorially describes a path isomorphism in terms of tableaux with two rows (see the Problem 3.1).

In this generalization we introduce the notion of *index* of two rows. Roughly speaking the index is a «measure of nonstandardness» for tableaux. Such notion turns out to be invariant under root operators and under the swapping map. Finally we consider tableaux with $p$ rows and we define an action of the symmetric group $\mathfrak{S}_p$ on these tableaux. This action can be used to define standard tableaux of any shape.

## 2. Standard tableaux and jeu de taquin

Let $T = R_1\,, R_2$ be a standard tableau of shape $(m\,, m) \setminus (m - n)$ with $m \geq n$, $R_1 = r_{1\,,1} r_{1\,,2} \cdots r_{1\,,n}$, $R_2 = r_{2\,,1} r_{2\,,2} \cdots r_{2\,,m}$. We attach to the tableau $T$ the set $H(T)$ of standard rows $R$ of length $n$ such that $R \subset R_2$ and $R_1 \leq R$. In the next proposition we see that $H(T)$ has a minimum element.

Proposition 2.1. *Set* $i_1 = \min\{i \mid r_{1\,,1} \leq r_{2\,,i}\}$ *and for* $k = 1\,, \ldots\,, n-1$ *set* $i_{k+1} = \min\{i \mid r_{1\,,k+1} \leq r_{2\,,i}\,,\ i > i_k\}$. *Then* $\overline{R_2} = r_{2\,,i_1} r_{2\,,i_2} \cdots r_{2\,,i_n}$ *is the minimum element of* $H(T)$. *Further if we set* $\overline{R_1} = [(R_1 \cup R_2) \setminus \overline{R_2}]$, *then* $\overline{R_1} \leq \overline{R_2}$, i.e. *the tableau* $\overline{T} = \overline{R_1}\,, \overline{R_2}$ *of shape* $(m\,, n)$ *is standard.*

Proof. Notice that $m - n + 1 \in \{i \mid r_{1\,,1} \leq r_{2\,,i}\}$ since $T$ is standard, so $i_1$ is well defined and $i_1 \leq m - n + 1$. Hence using induction, $i_1\,, \ldots\,, i_n$ are well defined with $i_k \leq m - n + k$. Now it is clear that $\overline{R_2} \in H(T)$ (which is therefore non void). Let $R = r_{2\,,j_1} r_{2\,,j_2} \cdots r_{2\,,j_n}$ be a row in $H(T)$ and let $h$ be such that $j_1 = i_1\,, j_2 = i_2\,, \ldots\,, j_h = i_h$ and $j_{h+1} \neq i_{h+1}$ (or $h = -1$ if $j_1 \neq i_1$). Then $r_{2\,,j_{h+1}} \geq r_{1\,,h+1}$ forces $j_{h+1} > i_{h+1}$. Hence $j_{h+2} > j_{h+1}$, $r_{2\,,j_{h+2}} \geq r_{1\,,h+2}$ imply in turn $j_{h+2} \geq i_{h+2}$ and so on. This proves the first statement.

We claim that $\overline{R_1} = r_{2\,,1} \cdots r_{2\,,i_1-1} r_{1\,,1} r_{2\,,i_1+1} \cdots r_{2\,,i_2-1} r_{1\,,2} r_{2\,,i_2+1} \cdots r_{2\,,i_n-1} r_{1\,,n} r_{2\,,i_n+1} \cdots \cdots r_{2\,,n}$. This is clear once we show that the right hand is a standard row, and this follows from the definition of $i_1\,, i_2\,, \ldots i_n$.

Finally notice that the standardness of $\overline{T}$ is clear since $i_k \geq k$. □

In the figure below we see an example where the boxes of position $i_k$ are highlighted.

| | | | | |
|---|---|---|---|---|
| | | 2 | 5 | 7 |
| 1 | **3** | 4 | **6** | **7** |

Using this proposition we define the *swapping map* as $\sigma : T \mapsto \overline{T}$. We define also $j : T \mapsto j(T)$, where $j(T)$ is the tableau of shape $(m, n)$ obtained from $T$ by applying the jeu de taquin. The aim of this first section is to prove that $j(T) = \sigma(T)$ for any standard tableau $T$. We will use induction on the length of the rows of $T$ and the following lemma will be useful.

Lemma 2.1. *Let $T = R_1, R_2$ be a standard tableau of shape $(m, m) \setminus (m-n)$. Let $\sigma(T) = \overline{R_1}, \overline{R_2}$, $j(T) = R'_1, R'_2$ and*

$$S_1, S_2 = r_{1,2}\cdots r_{1,n}, r_{2,2}\cdots r_{2,m}$$
$$\overline{S_1}, \overline{S_2} = \overline{r}_{1,2}\cdots \overline{r}_{1,n}, \overline{r}_{2,2}\cdots \overline{r}_{2,m}$$
$$S'_1, S'_2 = r'_{1,2}\cdots r'_{1,n}, r'_{2,2}\cdots r'_{2,m}.$$

*If $r_{2,1} \ge r_{1,1}$ then $\sigma(S_1, S_2) = \overline{S_1}, \overline{S_2}$, $j(S_1, S_2) = S'_1, S'_2$ and $\overline{r}_{1,1} = r'_{1,1} = r_{1,1}$.*

Proof. We have $r_{2,i} > r_{2,1} \ge r_{1,1}$ for $i = 2, \ldots, m$.

($\sigma$) $R_1 \subset \overline{R_1}$ implies $\overline{r}_{1,1} = r_{1,1}$. Now $\sigma(S_1, S_2) = \overline{S_1}, \overline{S_2}$ follows from the definition of $\sigma$.

($j$) Each step of jeu de taquin preserves $r_{1,1}$ as the first entry of the upper row. Then $r'_{1,1} = r_{1,1}$ and $j(S_1, S_2) = S'_1, S'_2$ follows from the definition. □

In the next lemma a sort of «associativity» for $\sigma$ is proved.

Lemma 2.2. *Let $T = R_1, R_2$ be a standard skew tableau of shape $(m, m) \setminus (m-n)$. Let $\overline{R_1}, \overline{R_2} = \sigma(T)$, $R'_1, R'_2 = \sigma(R_1, r_{2,2}r_{2,3}\cdots r_{2,n})$, $R''_1, R''_2 = \sigma(r'_{1,1}\cdots r'_{1,n}, r_{2,1}r'_{2,1}\cdots r'_{2,n})$. If we suppose $r_{1,1} > r_{2,1}$ then $\overline{R_2} = R''_2$.*

Proof. If $r''_{2,1} = r_{2,1}$ then $r_{2,1} \ge r'_{1,1} \ge r_{1,1}$, hence $r''_{2,1} \ne r_{2,1}$. So $R''_2 = r'_{2,1}\cdots r'_{2,n} = R'_2$ and using again $r_{1,1} > r_{2,1}$ we see $R'_2 = R_2$. □

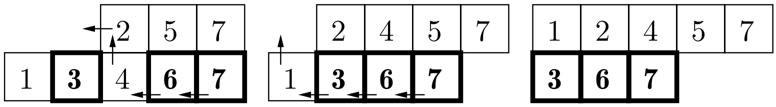
Finally using the lemmas above we can prove

Theorem 2.1. *If $T = R_1, R_2$ is a standard skew tableau of shape $(m, m) \setminus (m-n)$, then $j(T) = \sigma(T)$.*

Proof. If $n = 1$ or $m = n + 1$ it is obvious that $\sigma(T) = j(T)$, we need just to use the definition of $\sigma$ and $j$. If $r_{2,1} \ge r_{1,1}$ we can use the Lemma 2.1 and the induction on $n$. So we can suppose $r_{2,1} < r_{1,1}$.

Now we use induction on $m - n$. If $m - n > 1$ then the Lemma 2.2 and the case $m - n = 1$ prove the inductive step. □

In the following figure a simple example is treated with jeu de taquin, the highlighted boxes represent $\overline{R_2}$.




## 3. The path basis isomorphism

We briefly recall the principal definition of LS paths language in the case $A_\ell$ (see [3] for a general introduction to LS paths).

Let $X \subset \mathbb{R}^\ell$ be the weight lattice of the Lie algebra $\mathfrak{g}$ of type $A_\ell$. We denote by $X^+ \subset X$ the set of dominant weights, by $\Pi$ the set of piecewise linear paths $\pi : [0\,, 1] \longrightarrow X \otimes \mathbb{Q}$ such that $\pi(0) = 0$ and $\pi(1) \in X$, with $\pi$ and $\pi'$ identified if $\pi = \pi'$ up to reparametrization. Let $\Pi^+ \subset \Pi$ be the subset of all paths whose image is contained in the dominant Weyl chamber corresponding to the usual choise (see [1]) of the simple roots $\alpha_1\,, \ldots\,, \alpha_\ell$ of $\mathfrak{g}$. Let $\omega_1\,, \ldots\,, \omega_\ell$ be the fundamental weights correspondig to these simple roots.

Let $\mathbb{Z}\Pi$ be the free $\mathbb{Z}$-module with basis $\Pi$. We denote by $\pi_1 * \pi_2$ the concatenation of the two paths $\pi_1$ and $\pi_2$. Taking $\alpha$ to be a simple root, in [3] root operators $e_\alpha$ and $f_\alpha$ on $\Pi$ are introduced. Let $\mathcal{A} \subset \operatorname{End}_{\mathbb{Z}} \mathbb{Z}\Pi$ be the subalgebra generated by $e_\alpha$, $f_\alpha$. Denote by $\mathbb{B}\pi$ the basis of the $\mathcal{A}$-module $\mathcal{A}\pi$ for $\pi \in \Pi^+$.

This LS paths machinery has allowed a straight generalization of the Littelwood-Richardson rule. Indeed the same language can be introduced in the more general setting of symmetrizable Kac-Moody algebras and the following results hold.

Theorem 3.1 [3]. *If $\pi \in \Pi^+$ and $\pi(1) = \lambda$ then $\sum_{\eta \in \mathbb{B}\pi} e^{\eta(1)} = \operatorname{ch} V_\lambda$, where $V_\lambda$ is the irreducible $\mathfrak{g}$-module of highest weight $\lambda$.*

Theorem 3.2 [3]. *If $\pi_1\,, \pi_2 \in \Pi^+$ and $\pi(1) = \lambda = \pi(2)$ then $\mathcal{A}\pi_1$ and $\mathcal{A}\pi_2$ are $\mathcal{A}$-modules isomorphic via a map extending $\pi_1 \mapsto \pi_2$.*

Consider now two paths $\pi_1$, $\pi_2$ in $\Pi^+$, define $\mathbb{B}\pi_1 * \mathbb{B}\pi_2$ as the set of all concatenations $\eta_1 * \eta_2$ with $\eta_1 \in \mathbb{B}\pi_1$ and $\eta_2 \in \mathbb{B}\pi_2$ and let $M\pi_1 * M\pi_2$ be the $\mathbb{Z}$-module spanned by $\mathbb{B}\pi_1 * \mathbb{B}\pi_2$. This is an $\mathcal{A}$-module and it decomposes in the following way: $M\pi_1 * M\pi_2 = \oplus \mathcal{A}(\pi_1 * \eta)$ where the sum is over all $\eta \in \mathbb{B}\pi_2$ such that $\pi_1 * \eta \in \Pi^+$ (see [3]). Then, using the character formula above (Theorem 3.1) we have:

Theorem 3.3 [3]. *Let $\lambda_1\,, \lambda_2 \in X^+$ and let $\pi_1\,, \pi_2 \in \Pi^+$ be such that $\pi(1) = \lambda_1$, $\pi_2(1) = \lambda_2$. Then $V_{\lambda_1} \otimes V_{\lambda_2} = \oplus V_{\epsilon(1)}$, where the sum is over all LS paths $\epsilon = \pi_1 * \eta \in \Pi^+$ with $\eta \in \mathbb{B}\pi_2$.*

Our first aim is to define an «interpretation» of tableaux in terms of paths for $\mathfrak{g}$ of type $A_\ell$ and to define operators $e_j$, $f_j$ on tableaux behaving as $e_\alpha$, $f_\alpha$ for $\alpha = \alpha_j$. Then we will consider the following problem (see [4]):

PROBLEM 3.1. *Using the theorems above is clear that there exists an $\mathcal{A}$-bijection* $\mathbb{B}\pi_{\omega_m} * * \mathbb{B}\pi_{\omega_n} \longrightarrow \mathbb{B}\pi_{\omega_n} * \mathbb{B}\pi_{\omega_m}$ *such that* $\pi_{\omega_m} * \eta \in \Pi^+$ *correspond to* $\pi_{\omega_n} * \eta' \in \Pi^+$ *where* $\omega_m + \eta(1) = \omega_n + \eta'(1)$. *How can we combinatorially describe this map in terms of tableaux?*

We will give an answer to this problem generalizing the jeu de taquin seen in Section 2. Now we see the definitions of various maps used in the sequel.

Let $R = r_1 \dots r_k$ be a row with entries out of $\{1, \dots, \ell, \ell+1\}$ and fix $1 \le j \le \ell$. Define $\nu_j(R) =+1$ if $j \in R, j+1 \notin R$, define $\nu_j(R) = 0$ if $j, j+1 \in R$ or $j, j+1 \notin R$ and define $\nu_j(R) = -1$ if $j \notin R, j+1 \in R$. By $R(\kappa_1, \dots, \kappa_h \longleftarrow \hat{\kappa}_1, \dots, \hat{\kappa}_h)$ we mean the row obtained by 1) replacing each occurrence of $\kappa_1, \dots, \kappa_h$ in $R$ with $\hat{\kappa}_1, \dots, \hat{\kappa}_h$ and 2) rearranging in non decreasing order the new entries. Now define $s_j(R) = R(j, j+1 \longleftarrow j+1, j)$ and notice that $s_j(R)$ is a standard row if $R$ is a standard row. Notice also that $\nu_j s_j(R) = -\nu_j(R)$, indeed $\nu_j$ can be seen as the «scalar product» of $R$ and $\alpha_j$ and $s_j$ is a sort of «symmetry» with respect to $\alpha_j$. This has a precise meaning once we introduce the map $\lambda : R \mapsto \sum_{i=1}^{k}(\omega_{r_i} - \omega_{r_i - 1})$. We have $(\lambda(R), \alpha_j) = \nu_j(R)$ and $s_{\alpha_j}\lambda(R) = \lambda(s_j R)$.

Now let $T = R_1, \dots, R_s$ be a (skew) tableau with $s$ rows and entries out of $\{1, \dots, \ell, \ell+1\}$. We attach to such a tableau a map $h_T : \{0, \dots, s\} \longrightarrow \mathbb{Z}$ defined as follows

$$t \;\mapsto\; h_T(t) = \sum_{i=1}^{t} \nu_j(R_{s-i+1}).$$

Notice that the index $s - i + 1$ just «reads» the tableau from the bottom to the top. Now we finally come to the definition of the operator $f_j$ on tableaux. Let $t_0$ be the maximum such that $h_T(t_0) = \min h_T$. If $t_0 = s$ define $f_j(T) = 0$, otherwise define

$$f_j(T) = R_1, \dots, R_{s-t_0-1}, s_j(R_{s-t_0}), R_{s-t_0+1}, \dots, R_s.$$

In the same way, let $t_1$ be the minimum such that $h_T(t_1) = \min h_T$. If $t_1 = 0$ define $e_j(T) = 0$, otherwise define

$$e_j(T) = R_1, \dots, R_{s-t_1}, s_j(R_{s-t_1+1}), R_{s-t_1+2}, \dots, R_s.$$

Given any tableau $T = R_1, \dots, R_s$ we define its interpretation as path in the following way $\pi(T) = \pi_{\lambda(R_s)} * \pi_{\lambda(R_{s-1})} * \dots * \pi_{\lambda(R_1)}$, where for a weight $\lambda$, $\pi_\lambda$ is the path $t \mapsto \lambda t$. It is almost obvious that $\pi e_j T = e_{\alpha_j} \pi T$ and that $\pi f_j T = f_{\alpha_j} \pi T$, we need just to use the various definitions of $e_j$, $f_j$ and of $e_{\alpha_j}$, $f_{\alpha_j}$. Now let see an example.

EXAMPLE 3.1. Let $T$ be the tableau $1, 123, 45, 135$ of shape $(4, 4, 2, 2) \setminus (3, 1, 1)$ and let $j = 3$. This tableau corresponds to the path

$$\pi(T) = \pi_{\omega_1 - \omega_2 + \omega_3 - \omega_4 + \omega_5} * \pi_{-\omega_3 + \omega_5} * \pi_{\omega_3} * \pi_{\omega_1}.$$

Then $h_3(T)$ is the map $(0, 1, 2, 3, 4) \mapsto (0, 1, 0, 1, 1)$. Hence $t_0 = 2$ and $t_1 = 0$. So $f_3(T) = 1, 124, 45, 135$ and $e_3(T) = 0$.

In the sequel we will follow the notation introduced in the Problem 3.1 where we fix $n < m$. Our first step is to investigate whose tableaux $T = R_1, R_2$ correspond to paths $\pi_{\omega_m} * \eta \in \Pi^+$, with $\eta \in \mathbb{B}\pi_{\omega_n}$. Clearly the tableau corresponding to the path $\pi_{\omega_m}$ is the tableau with just one row $R_2 = 1 \cdots m$. It is evident from the definition of the map $\lambda$ that $R_1$ must be of the following type $1 \cdots s\ m+1 \cdots h$, for some $h$ and $s$ and we have $\pi(T) = \pi_{\omega_m} * \pi_{\omega_h - \omega_m + \omega_s}$. We call a tableau of this kind a *maximal tableau.* Note that just one maximal tableau is standard, namely the tableau $T = 1 \cdots n\,,1 \cdots m$.

Now our next step is to extend the map $\sigma$ of Section 2 to non standard skew tableaux. Let $T = R_1\,, R_2$ be any such tableau of shape $(m\,, m) \setminus (m-n)$ and let $t$ be any positive integer. We define the following sets

$$H_t(T) = \{R \text{ row of length } n \mid R \subset R_2\,,\ R_1 \leq R_2(\infty)^t\}$$

where by $R_1(\infty)^t$ we mean the row obtained by adding to $R_1$ $t$-times the new symbol $\infty$ to the right and declaring $r < \infty$ for any integer $r$. Notice that $H_0(T) = H(T)$ as already defined in Section 2. But notice also that $H_0(T)$ is void if $T$ is non standard.

Let us see a simple example taking $T = 46\,, 1345$. Then $H_0(T) = \varnothing$, $H_1(T) = \{14\,, 15\,, 34\,, 35\}$ and for any $t \geq 2$ we have $H_t(T) = \{xy \mid x < y \text{ with } x, y \in \{1\,, 3\,, 4\,, 5\}\}$.

It is evident that in general $H_t(T)$ is non void if $t \gg 0$ (take $t = n$) and that $H_t(T) \subset H_{t+1}(T)$.

Now consider the minimum $t$ such that $H_t(T) \neq \varnothing$. We call such $t$ the *index* of the tableau $T$ and denote it $k(T)$. Trying to follow what we have already seen in Section 2 we come to the following proposition

Proposition 3.1. *Let $T = R_1\,, R_2$ be a skew tableau of shape $(m\,, m) \setminus (m-n)$ and let $k = k(T)$ be its index. Set $i_1 = \min\{i \mid r_{2,i} \geq r_{1,1}\,,\ i > k\}$, $i_2 = \min\{i \mid r_{2,i} \geq r_{1,2}\,,\ i > i_1\}$ and so on till $i_{n-k} = \min\{i \mid r_{2,i} \geq r_{1,n-k}\,,\ i > i_{n-k-1}\}$. Then $i_1\,, i_2\,, \ldots\,, i_{n-k}$ are well defined and $\overline{R_2} = r_{2,1} r_{2,2} \cdots r_{2,k} r_{2,i_1} \cdots r_{2,i_{n-k}}$ is the minimum element of $H_k(T)$.*

Proof. First notice that $k = k(T)$ implies $r_{2,m-n+1} r_{2,m-n+2} \cdots r_{2,m} \in H_k(T)$ and hence $i_1\,, \ldots\,, i_{n-k}$ are well defined. Let $R' = r'_1 \cdots r'_n \in H_k(T)$, then $R' \subset R_2$ and, for $h = 1\,, \ldots\,, n-k$, we have $r'_{k+h} \geq r_{1,h}$. Hence $r_{k+h'} \geq r_{2,i_h}$ and so $\overline{R_2} \leq R'$. □

Now we see the main definition and theorem of this section.

Definition 3.1. *Let $T = R_1\,, R_2$ be a skew tableau of shape $(m\,, m) \setminus (m-n)$ and let $k = k(T)$. Define the swapping of $T$ as the tableau $\sigma(T) = \overline{R_1}\,, \overline{R_2}$ of shape $(m\,, n)$ with $\overline{R_2} = \min H_k(T)$ and $\overline{R_1} = [(R_1 \cup R_2) \setminus \overline{R_2}]$.*

Theorem 3.4.

1. *If $f_j(T) \neq 0$ then $k(f_j(T)) = k(T)$, if $e_j(T) \neq 0$ then $k(e_j(T)) = k(T)$,*
2. *$\sigma(f_j(T)) = f_j(\sigma(T))$, $\sigma(e_j(T)) = e_j(\sigma(T))$,*
3. *if $T$ is maximal then $\sigma(T)$ is maximal.*

Proof. We will prove 1 and 2 togheter for $f_j$. Then they hold for $e_i$ too, since if $f_j(T) \neq 0$ then $e_j(f_j(T)) = T$. Set $T' = f_j(T) = R'_1\,, R'_2$, $S = \sigma(T) = S_1\,, S_2$, $S' = f_j(S) = S'_1\,, S'_2$ and $k = k(T)$.

First suppose that $f_j(T) = 0$. We have to show $f_j(S) = 0$. We note here, once at all, that $\sigma$ preserves the multiplicities of the entries of the tableaux. So we suffice to exclude the following cases («1» means `true`, «0» means `false`):

| | $j\in R_1$ | $j+1\in R_1$ | $j\in R_2$ | $j+1\in R_2$ | $j\in S_1$ | $j+1\in S_1$ | $j\in S_2$ | $j+1\in S_2$ |
|---|---|---|---|---|---|---|---|---|
| A | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| B | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| C | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |

Cases A, B. These are impossible since $S_2 \not\subset R_2$.
Case C. Consider $S_2' = S_2\{j+1 \longleftarrow j\}$. $S_2(\infty)^k \geq R_1$, $j+1 \notin R_1$ imply $S_2'(\infty)^k \geq R_1$ and we have also $S_2' \subset R_2$. So $S_2' \in H_k(T)$, $S_2' \lneq S_2$. This is impossible since $S_2 = \min H_k(T)$.

Now suppose that $f_j(T) \neq 0$. We have $f_j(S) \neq 0$ except in the following situation

| $j\in R_1$ | $j+1\in R_1$ | $j\in R_2$ | $j+1\in R_2$ | $j\in S_1$ | $j+1\in S_1$ | $j\in S_2$ | $j+1\in S_2$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |

and this is impossible since $S_2 \not\subset R_2$.

Now we have to prove $f_j(S) = \sigma(T')$. In the following table we have listed all the possibilities of $j, j+1$ in $T$ and in $S$ taking into account the multiplicities invariance and that $S_2 \subset R_2$ («x» means `true` or `false`, and has a fixed value for each line).

| | $j\in R_1$ | $j+1\in R_1$ | $j\in R_2$ | $j+1\in R_2$ | $j\in S_1$ | $j+1\in S_1$ | $j\in S_2$ | $j+1\in S_2$ |
|---|---|---|---|---|---|---|---|---|
| A | $x$ | 0 | 1 | 0 | $x$ | 0 | 1 | 0 |
| B | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| C | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
| D | 1 | 0 | $x$ | $x$ | 1 | 0 | $x$ | $x$ |
| E | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 |
| F | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |

Notation: if $S$ is a standard row we write

$$\overline{S} = \begin{cases} s_j(S) = S\{j \longleftarrow j+1\} & \text{if } \nu_j(S) =+1 \\ S & \text{otherwise} \end{cases}$$

$$\underline{S} = \begin{cases} s_j(S) = S\{j+1 \longleftarrow j\} & \text{if } \nu_j(S) = -1 \\ S & \text{otherwise.} \end{cases}$$

Note that $\overline{S}$ and $\underline{S}$ are still standard rows, that $\overline{S} \geq S$, $\underline{S} \leq S$, $\underline{(\overline{S})} = S = \overline{(\underline{S})}$. Moreover is easy to see that $S \geq T$ implies $\overline{S} \geq \overline{T}$, $\underline{S} \geq \underline{T}$. In cases A, B, C, D, E we procede in this way. We prove

$$S \in H_k(T) \Rightarrow \overline{S} \in H_k(T')$$

$$S \in H_k(T') \Rightarrow \underline{S} \in H_k(T)$$

for any $k$. This implies that $k(T') = k(T)$. Then, using $\min H_k(T') = \overline{S_2}$, we suffice to verify that $S'_2 = \overline{S_2}$.

Case A.
We have $R'_1 = R_1$, $R'_2 = R_2\{j \longleftarrow j+1\}$, $S'_1 = S_1$ and $S'_2 = S_2\{j \longleftarrow j+1\}$.

Let $S \in H_k(T)$, then $S(\infty)^k \geq R_1 \Rightarrow \overline{S}(\infty)^k \geq R'_1 = R_1$. Moreover if $\nu_j(S) =+1$ then $\overline{S} = S\{j \longleftarrow j+1\} \subset R'_2 = R_2\{j \longleftarrow j+1\}$ since $S_2 \subset R_2$. Otherwise if $\nu_j(S) \neq +1$ then $j \notin S$ since $j+1 \notin R_2$, $S \subset R_2$. So $\overline{S} = S \subset R'_2$. In any case $\overline{S} \in H_k(T')$.

Let $S \in H_k(T')$. Suppose $\nu_j(S) = -1$, so we have $j \notin S$, $j+1 \in S$. Then $\underline{S}(\infty)^k \geq R_1 = \underline{R'_1}$ since $S(\infty)^k \geq R'_1$ and $j+1 \notin R'_1$. Moreover $\underline{S} \subset \underline{R'_2} = R_2$ since $S \subset R'_2$. So $\underline{S} \in \overline{H}_k(R_1, R_2)$.
Suppose $\nu_j(S) \neq -1$, so we have $\underline{S}(\infty)^k = S(\infty)^k \geq R_1 = R'_1$. Further $S \subset R'_2$ implies $j \notin S$, so $j+1 \notin S$ too, hence $\underline{S} = S \subset R_2$ and $\underline{S} \in H_k(T)$.

Till now we have proved that $k(T') = k(T)$. But we have also $\overline{S_2} = S_2\{j \longleftarrow j+ +1\} = S'_2$ and this complete this case.

Case B.
We have $R'_1 = R_1$, $R'_2 = R_2\{j \longleftarrow j+1\}$, $S'_1 = S_1\{j \longleftarrow j+1\}$ and $S'_2 = S_2$.

Let $S \in H_k(T)$. We have $S(\infty)^k \geq R_1$ that implies $\overline{S}(\infty)^k \geq \overline{S}(\infty)^k \geq \overline{R_1} = R_1 = R'_1$. If $\nu_j(S) =+1$ then $\overline{S} \subset \overline{R_2}$ since $S \subset R_2$. Otherwise if $\nu_j(S) \neq 0$ then $j \notin S$ since $S \subset R_2$ forces $j+1 \notin S$. So $\overline{S} = S \subset R'_2$ and hence $\overline{S} \in H_k(T)$.

Let $S \in H_k(T')$. Suppose $\nu_j(S) = -1$. We have $j \notin S$, $j+1 \in S$. Further $\underline{S}(\infty)^k \geq R_1 = R'_1$ since $S(\infty)^k \geq R'_1$ and $j+1 \notin R_1$. We have also $\underline{S} \subset \underline{R'_2} = R_2$ since $S \subset R'_2$. So we deduce $\underline{S} \in H_k(T)$.
Suppose $\nu_j(S) = -1$. We have $\underline{S}(\infty)^k = S(\infty)^k \geq R_1 = R'_1$ since $S(\infty)^k \geq R'_1$. Moreover $S \subset R'_2$ implies $j \notin S$ and so $j+1 \notin S$ too since $\nu_j(S) \neq -1$. So $\underline{S} = S \subset \underline{R'_2} = R_2$. Hence we have $\underline{S} \in H_k(T)$.

This proves $k(T') = k(T)$ and then this case is proved since $\overline{S_2} = S_2 = S'_2$.

Case C, D, E.
These cases are very similar to the previous ones (or more easy) so the details are omitted.

Case F.
In this case we claim that $H_k(T) = H_k(T')$ for any $k$. Let $S \in H_k(T)$. We have $S(\infty)^k \geq R_1$ hence $\overline{S}(\infty)^k \geq \overline{R_1} = R'_1$. But $S \subset R_2$ implies $j \notin R_2$ and hence $\overline{S} = S$. Further it is clear that $S \subset R'_2 = R_2$. So we deduce $S \in H_k(T')$.
Let $S \in H_k(T)$. We have $S(\infty)^k \geq R'_1 > R_1$ and $S \subset R_2 = R'_2$. So it is clear that $S \in H_k(T)$.
This proves our claim and also that $k(T) = k(T')$. So $\min H_{k(T')}(T') = \min H_{k(T)}(T) = = S_2 = S'_2$ and hence $\sigma(T') = S'$.
This finish the proof of statement 1 and 2.

The statement 3 is very easy since we can directly compute $T$ and $\sigma(T)$ in the case $T$ maximal. Let $T = 1\ 2\ \cdots m-1\ m\,,\, 1\ 2\ \cdots s-1\ s\ m+1\ m+2\ \cdots\ h-1\ h$. Hence $k(T) = n - s$ and we have $\min H_{k(T)}(T) = 1\ 2\ \cdots\ n-1\ n$ using the Proposition 3.1. So for multiplicity reason we have $\sigma(T) = 1\ 2\ \cdots n-1\ n\,,\, 1\ 2\ \cdots s-1\ s\ n+ + 1\ n+2\ \cdots\ h-1\ h$ and this tableau is maximal. So the proof of the theorem is complete. □

Corollary 3.1. *The map $\sigma$ is invertible.*

Proof. This is clear since it is invertible on maximal tableaux using the computation in the proof of the statement 3 of the theorem above. □

It is possible to define a more symmetric form of $\sigma$. If $T = R_1\,,\, R_2$ then we have $\sigma(T) = \max G_k(T)\,,\, \min H_k(T)$ where $k = k(T)$, for $t \geq 0$ we define $H_t(T)$ as above and $G_t(T)$ is the set of all rows $R$ of length $m$ such that $R \supset R_1$ and $R \leq R_2(\infty)^t$. Moreover the index of $T$ can also be defined as the minimum $t$ such that $G_t(T) \neq \varnothing$. This kind of formulas can be used to define $\sigma^{-1}$ too: if $S = S_1\,,\, S_2$ has shape $(m\,,\, n)$ then $\sigma^{-1}(S) = \max G_k(S)\,,\, \min H_k(S)$ where for $t \geq 0$ we define $G_t(S) = = \{R \text{ row of length } n \mid R \subset R_1\,,\ R \leq R_2(\infty)^t\}$ and $H_t(S) = \{R \text{ row of length } m \mid R \supset \supset R_2\,,\ R_1 \leq R(\infty)^t\}$ and where $k = k(S)$ is the *index* of $S$ defined as the minimum $t$ such that $H_t(S) \neq \varnothing$. Also in this case $k(S)$ is the minimum $t$ such that $G_t(S) \neq \varnothing$ as well.

Corollary 3.2. *The index is invariant under the swap map.*

Proof. The computation in the proof of statement 3 of the theorem above gives $k(\sigma(T)) = k(T)$ for $T$ maximal. Now for general $T$ it suffices to use the statement 1 and 2. □

Corollary 3.3. *The map $T \mapsto \sigma(T)$ combinatorially describes the path isomorphism $\mathbb{B}\pi_{\omega_n} * \mathbb{B}\pi_{\omega_m} \longrightarrow \mathbb{B}\pi_{\omega_m} * \mathbb{B}\pi_{\omega_n}$ of Problem 3.1.*

The following corollary gives the answer for a generalization of Problem 3.1 to path basis of type $\mathbb{B}\pi_{\omega_{n_1}} * \cdots * \mathbb{B}\pi_{\omega_{n_p}}$.

Corollary 3.4. *Consider the set of tableaux $T = R_1\,,\, \ldots\,,\, R_p$ with $p$ rows and define the following maps*

$$\tau_i(T) = R_1\,,\, \ldots R_{i-1}\,,\, S_i\,,\, S_{i+1}\,,\, R_{i+2}\,,\, \ldots\,,\, R_p$$

*where $1 \leq i \leq n-1$ and $(S_i\,,\, S_{i+1}) = \sigma(R_i\,,\, R_{i+1})$. Then the maps $\tau_i$ define an $\mathcal{A}$-action of the group $\mathfrak{S}_p$ of permutations of $p$ symbols.*

Proof. That the $\tau_i$ define an action can be seen looking at the swapping of maximal tableaux with three rows. That this action commutes with root operators $e_j$, $f_j$ is an easy consequence of the theorem above. □

This corollary can be used to describe standard tableaux of any shape. Let $T = R_1, \ldots, R_p$ be a tableau with rows of length $n_1, \ldots, n_p$ such that $R_i \leq R_{i+1}$ for $i = 1, \ldots, p-1$. We call such tableaux *weak standard*. Let $\sigma \in \mathfrak{S}_p$ be a permutation such that $\sigma(n_i) \geq \sigma(n_{i+1})$ for $i = 1, \ldots, p-1$. Then we say that $T$ is *standard* if $\sigma(T)$ is standard in the usual sense. It is known that $T$ is standard if and only if $\tau(T)$ is weak standard for all permuations $\tau$ (see [4]).

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