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Corestriction of central simple algebras and families of Mumford-type

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Geometria algebrica. — *Corestriction of central simple algebras and families of Mumford-type.* Nota di Federica Galluzzi, presentata (*) dal Corrisp. C. Procesi.

Abstract. — Let $\mathcal{M}$ be a family of Mumford-type, that is, a family of polarized complex abelian fourfolds as introduced by Mumford in [9]. This family is defined starting from a quaternion algebra A over a real cubic number field and imposing a condition to the corestriction of such A. In this paper, under some extra conditions on the algebra A, we make this condition explicit and in this way we are able to describe the polarization and the complex structures of the fibers. Then, we look at the non simple CM-fibers and we give a method to construct a family of Mumford-type starting from such a fiber.

Key words: Abelian varieties; CM-type; Corestriction.

Riassunto. — *Corestrizione di algebre semplici centrali e famiglie di tipo Mumford.* Si prendono in considerazione le famiglie di varietà abeliane complesse di dimensione quattro definite da Mumford in [9]. Per la costruzione di queste famiglie si parte da un'algebra di quaternioni A con centro su un campo cubico totalmente reale che abbia corestrizione isomorfa a un'algebra di matrici. In questo lavoro, imponendo delle condizioni aggiuntive sull'algebra A, si sviluppa questa richiesta sulla corestrizione e si riescono così a fornire esempi espliciti di tali famiglie. Si studiano poi delle fibre di tipo CM e infine si dà un metodo per costruire una famiglia di tipo Mumford a partire da una data fibra di tipo CM.

Introduction

A family of Mumford-type is a 1-dimensional family of polarized abelian fourfolds as defined by Mumford in [9].

The generic fibers of these families gave the first example of abelian varieties not characterized by their endomorphism algebra and having a "small" Mumford-Tate group. More precisely, their Mumford-Tate group is smaller than the one of the generic abelian fourfold. Kuga proved the Hodge Conjecture for such varieties in [6, 2.2.2] and Hazama showed that there are exceptional classes in $H^4(X \times X)$, if X denotes such a variety (see [3, 5.2]). The Hodge Conjecture for a product of varieties of Mumford-type is still unknown. Nevertheless, their Hodge structure can be investigated studying the representations of their Mumford-Tate groups (see [2, 3]).

In this paper we present, for the first time, explicit examples of such families. To do this, we have to look at the original definition of Mumford trying to make it explicit.

We write an abelian variety of complex dimension g as (V, Λ, J, E), where $V \cong \mathbb{Q}^{2g}$ and Λ, J, E are the lattice, the complex structure and the polarization respectively.

The construction of the family starts from a quaternion algebra A over a real cubic field such that the corestriction of A is isomorphic to $M_8(\mathbb{Q})$ (for definition of corestriction see Section 2.5). This is just the condition we want to understand in order to produce explicit examples of families af Mumford-type.

(*) Nella seduta del 14 maggio 1999.

We assume some extra conditions on the algebra A. In this way we can give the data (V, Λ, J, E) for a fiber in a familiy of Mumford-type. Then, we construct families with a certain given fourfold of CM-type as a fiber (see Section 5 for definition of varieties of CM-type). This is a natural request since families of Mumford-type are characterized by having such fibers (see [9, Theorem 3]).

Finally, we give a family of Mumford-type having as a CM-fiber the variety $Y \times C$ where Y is the Jacobian of the hyperelliptic curve defined by $y^2 = x^7 - 1$ and C is an elliptic curve of CM-type with $\mathrm{End}^o(C) = \mathbb{Q}(\sqrt{-7})$.

These results could be a starting point for studying the geometry of such varieties in details. For example, is still unknown if a general abelian variety of Mumford-type can be isogenous to a jacobian.

The paper is organized as follows.

In Section 1 we give notations and definitions about complex abelian varieties and we recall the details of Mumford's construction. In Section 2 we introduce quaternion and cyclic algebras and we define the corestriction. If K is a Galois number field and A is central simple algebra over K, the corestriction $\mathrm{Cor}_{K/\mathbb{Q}}(A)$ over $\mathbb{Q}$ is defined as the $\mathbb{Q}$-subalgebra of $A^{\otimes 3}$ of invariants

$$\mathrm{Cor}_{K/\mathbb{Q}}(A) = (A^{\otimes 3})^{\mathrm{Gal}(K/\mathbb{Q})}$$

under a certain action of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ (for the definition of this action see Section 2.7). The corestriction $\mathrm{Cor}_{K/\mathbb{Q}}(A)$ is a central simple algebra over $\mathbb{Q}$. The example of Mumford starts from a quaternion algebra A over a totally real cubic number field K and it is based on the following hypothesis on $\mathrm{Cor}_{K/\mathbb{Q}}(A)$:

$$\mathrm{Cor}_{K/\mathbb{Q}}(A) \cong M_8(\mathbb{Q}).$$

In Section 3, under some mild restrictions on the algebra A, we will give $V \cong \mathbb{Q}^8$ as the subspace of L^8 fixed by a certain action of $\mathrm{Cor}_{K/\mathbb{Q}}(A)$, where L is a quadratic extension of K contained in A:

THEOREM 3.1. *There exist a matrix $R \in M_8(L)$ and an endomorphism $\tau \in \mathrm{End}(M_8(L))$ such that*

i) $\mathrm{Cor}_{K/\mathbb{Q}}(A) \cong \{M \in M_8(L) : R\tau(M)R^{-1} = M\}$,

ii) *there exists an isomorphism $\Psi : \mathrm{Cor}_{K/\mathbb{Q}}(A) \xrightarrow{\cong} \mathrm{End}_{\mathbb{Q}}(V)$, where V is the $\mathbb{Q}$-subspace of L^8 of dimension* 8 *given by:*

$$V := \{x \in L^8 : R\tau(x) = x\}.$$

To prove this result we first use the fact that A embeds, as a K-algebra, in $M_2(L)$ (see Proposition 2.9). Thus, $A^{\otimes 3} \hookrightarrow M_2(L)^{\otimes 3}$. We then define an action of $\mathrm{Gal}(L/\mathbb{Q})$ on $M_2(L)^{\otimes 3}$ such that

$$(M_2(L)^{\otimes 3})^{\mathrm{Gal}(L/\mathbb{Q})} \cong \mathrm{Cor}_{K/\mathbb{Q}}(A) \qquad \text{(see Lemma 3.5).}$$

Finally, we give an explicit isomorphism (see Lemma 3.6) $M_2(L)^{\otimes 3} \cong M_8(L)$ proving *i*) and *ii*) of 3.1.

Thanks to this result, in Section 4 we can give explicitly (V, Λ, J, E) for a general fiber of the family.

In Section 5 we find a non simple CM-fiber X isogenous to a product $Y \times C$ where Y, C are a simple threefold and an elliptic curve of CM-type respectively. This result suggests how to construct explicit examples. We prove in fact that if one gives a simple abelian threefold Y of CM-type and an elliptic curve C also of CM-type, with some assumptions on the CM-field of $Y,$ then it is possible to construct a family of Mumford-type with $Y \times C$ as a fiber (Theorem 5.5). Finally, in Section 5.5 we construct a family of Mumford-type having as a CM-fiber the variety $Y \times C$ where Y is the Jacobian of the hyperelliptic curve defined by $y^2 = x^7 - 1$ and C is an elliptic curve of CM-type with $\mathrm{End}^o(C) = \mathbb{Q}(\sqrt{-7})$.

1. Families of Mumford-type

1.1. *Complex abelian varieties*.

A *complex polarized abelian varieties* of dimension g is $X = (V, \Lambda, J, E)$ where V is a $\mathbb{Q}$-vector space of dimension $2g,$ Λ is a lattice in $V,$ J is the complex multiplication, that is, an $\mathbb{R}$-linear endomorphism of $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$, such that $J^2 = -I.$ Finally, $E : V \times V \longrightarrow \mathbb{Q}$ is a symplectic form satisfying the Riemann's conditions:

$$E(Jx, Jy) = E(x, y) \ , \quad E(x, Jx) \geq 0 \, .$$

We observe that to give J is equivalent to give a homomorphism of real algebraic groups $h : S^1 \to GL(V_{\mathbb{R}})$, since we can put $h(i)(v) := J(v)$, $\forall v \in V_{\mathbb{R}}$.

1.2. *Families of Mumford-type.*

We recall now the construction of Mumford. Let A be a quaternion algebra over a totally real cubic number field K such that its corestriction $\mathrm{Cor}_{K/\mathbb{Q}}(A)$ satisfies

$$\text{M1)} \qquad \mathrm{Cor}_{K/\mathbb{Q}}(A) \cong M_8(\mathbb{Q}) \, ,$$

(see Section 2.5 for the definition of corestriction). The $\mathbb{Q}$-vector space $V \cong \mathbb{Q}^8$ is defined by the condition

$$\mathrm{End}_{\mathbb{Q}}(V) \cong M_8(\mathbb{Q}).$$

For Λ we take any lattice $V.$ To define J and E we first ask the algebra A to verify

$$\text{M2)} \qquad A \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H} \oplus \mathbb{H} \oplus M(2, \mathbb{R}).$$

Let consider now the algebraic group

$$G := \{x \in A^* : N(x) = 1\} \, ,$$

where N is the canonical norm of the algebra A (see Definition 2.2). From M2) follows that $G(\mathbb{R}) \cong SU(2) \times SU(2) \times SL(2, \mathbb{R})$. The representation

$$\alpha : G \longrightarrow GL(V) \, , \quad x \longmapsto x \otimes x \otimes x$$

becomes over $\mathbb{R}$:

$$SU(2) \times SU(2) \times SL(2, \mathbb{R}) \to SO(4) \times SL(2, \mathbb{R})$$

and leaves invariant a unique (up to scalars) symplectic form E on $V_{\mathbb{R}}$. Finally, let us consider the map

$$\begin{array}{rcl} h_0 : S^1 & \longrightarrow & G(\mathbb{R}) \cong SU(2) \times SU(2) \times SL(2, \mathbb{R}) \\ e^{i\theta} & \longmapsto & \left(I, I, \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right). \end{array}$$

The composition

$$h : S^1 \longrightarrow Sp(V_{\mathbb{R}}, E) \subset GL(V_{\mathbb{R}}), \quad h := \alpha \circ h_0$$

gives the complex multiplication.

Mumford defines a family from (V, Λ, h, E) in the following way. Let K^0 be the connected component of the centralizer of $h(i)$ in $G(\mathbb{R})$:

$$K^0 = \{g \in G(\mathbb{R}) : \alpha(g)\, h(i) = h(i)\, \alpha(g)\}^0.$$

We obtain the map

$$\begin{array}{rcl} \mathbf{X} = G^0(\mathbb{R})/K^0 & \longrightarrow & Sp(V_{\mathbb{R}})/U_h \\ x = gK^0 & \longmapsto & \alpha(g)(h(i)\, U_h)\alpha(g)^{-1} \end{array}$$

where

$$U_h = \{\phi \in Sp(V_{\mathbb{R}}, E) : \phi\, h(i) = h(i)\, \phi\}.$$

Thus to any point $x = gK^0$ in the bounded symmetric domain $G^0(\mathbb{R})/K^0$, we can associate a polarized abelian variety

$$X_x := (V, \Lambda, \alpha(g) h \alpha(g)^{-1}, E).$$

Moreover, if $\Gamma \subset G$ is an arithmetic subgroup preserving Λ, then $\gamma \in \Gamma$ induces an isomorphism $X_x \cong X_{\gamma x}$, where the action γx is the one induced by G. The quotient $\Gamma/\mathbf{X}$ is a smooth quasi projective algebraic variety and we have a family of abelian varieties $\mathcal{M} = \{X_x : x \in \Gamma\backslash G^0(\mathbb{R})/K^0\}$ which can be glued in an analytic space fibered over $\Gamma\backslash G^0(\mathbb{R})/K^0$, (see [9, 8, 6]).

2. Quaternion algebras and corestriction

In this section we recall some basic facts about quaternion algebras and we get some results which allow us to understand better the construction of Mumford.

2.1. *Quaternion algebras.*

Definition 2.1. *Let K be a number field, a* quaternion algebra *over K is a central simple* *algebra with centre K such that*

$$A = K \oplus K\epsilon_1 \oplus K\epsilon_2 \oplus K\epsilon_3 \quad \textit{with} \begin{cases} \epsilon_1^2 = d, \\ \epsilon_2^2 = e, \\ \epsilon_3 = \epsilon_1\epsilon_2 = -\epsilon_2\epsilon_1 \end{cases} \qquad d, e \in K.$$

We denote it by $A = (d, e)_K$. If $L = K(\epsilon_1)$ and $x \in L$ then $A = L \oplus L\epsilon_2$ with $x\epsilon_2 = \epsilon_2\bar{x}$ where "-" indicates the non trivial K-automorphism of L.

Definition 2.2. *The algebra A has a* canonical involution

$$\iota : A \longrightarrow A, \quad a = a_0 + a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 \longmapsto \iota(a) = a_0 - a_1\epsilon_1 - a_2\epsilon_2 - a_3\epsilon_3$$

which defines a norm map

$$N : A \longrightarrow K, \quad a \longmapsto a\,\iota(a).$$

A quaternion algebra can be a skew field or a matrix algebra: there is the following [1, Theorem 1-5, p. 22]:

Theorem 2.3. *For $\sqrt{d} \notin K$ the quaternion algebra $A = (d, e)_K$ is a skew field if and only if the equation*

$$dx^2 + ey^2 - z^2 = 0$$

has no non-trivial solutions in K. If $A = (d, e)_K$ is not a skew field, then is isomorphic to $M_2(K)$.

Example 2.4. If $K = \mathbb{R}$, then $\quad A = (d, e)_K \cong \begin{cases} \mathbb{H} & \text{if } d, e < 0 \\ M_2(\mathbb{R}) & \text{otherwise.} \end{cases}$

2.2. *Cyclic algebras.*

We recall now some definitions and results about cyclic algebras. We will use these results later in proving certain isomorphisms between central simple algebras and matrix algebras.

Definition 2.5. *Let K be a number field and let L/K be a cyclic extension of degree n.* A cyclic algebra *is a central simple algebra over K containing L.*

Let now $\mathrm{Gal}(L/K)$ the Galois group of L over K and let σ be a generator of $\mathrm{Gal}(L/K)$. We can choose an L-basis of A as follows: $e_0 = 1$, $e_i = e^i$ $(i = 1, \ldots, n-1)$, where e is an element of A inducing σ. Moreover, $e^n = a \in L$ and since $e = \sigma(a) = = ae = e^{n+1} = ea$, we have $\sigma(a) = a$, so $a \in K$. The multiplication law for A is given by $e\lambda = \sigma(\lambda)e \quad \lambda \in L$. Thus A is determined up to isomorphism by L/K, a and σ. We write $A = (L/K, \sigma, a)$.

Denote with $N_{L/K}$ the reduced norm of L over K. There is the following Theorem (see [13, 8.12.4 and 8.12.6])

Theorem 2.6. *The algebra $(L/K, \sigma, a)$ is isomorphic to a matrix algebra if and only if $a \in N_{L/K}(L^*)$.*

There is an important result concerning tensor products which will be useful later:

THEOREM 2.7. *Let $(L/K, \sigma, a)$ and $(L/K, \sigma, b)$ be two cyclic algebras. Thus,*

$$(L/K, \sigma, a) \otimes (L/K, \sigma, b) \cong (L/K, \sigma, ab) \otimes M_2(K).$$

PROOF. See [13, Theorem 8.12.4].

REMARK 2.8. A quaternion algebra $A = (d, e)_K$ is always a cyclic algebra. If $\sqrt{d} \notin K$ it contains the quadratic subfield $L = K(\sqrt{d})$ and $A = (L/K, \sigma, e)$ where σ is the non trivial K-automorphism of L.

2.3. *Matrix algebras.*

From now on K denote a number field, $L = K(\delta)$, $\delta^2 = d \in K$ and $A = (d, e)_K$ is a quaternion algebra as in Definition 2.1. With "$^-$" we denote the K-linear involution of L. We have the following

PROPOSITION 2.9. *There is an injective homomorphism of K-algebras :*

$$R : A \longrightarrow M_2(L), \quad a = x + y\epsilon_2 \longmapsto \begin{pmatrix} x & ey \\ \bar{y} & \bar{x} \end{pmatrix}.$$

The algebra $R(A) \cong A$ is the K-subalgebra in $M_2(L)$ generated by the images of ϵ_1 and ϵ_2:

$$R(\epsilon_1) = \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix}, \quad R(\epsilon_2) = \begin{pmatrix} 0 & e \\ 1 & 0 \end{pmatrix}.$$

PROOF. By definition of A, we can identify A with L^2 via $x + y\epsilon_2 \mapsto (x, y)$. The injectivity of the homomorphism R is straightforward. □

REMARK 2.10. The norm $N(a)$ of an element $a \in A$ coincides with the determinant $\det(R(a))$.

2.4. *Galois actions.*

If E/F is a Galois field extension, we denote with $\mathrm{Gal}(E/F)$ the Galois group of E over F and with $Tr_{E/F}$, $N_{E/F}$ the reduced trace and the reduced norm of E over F respectively. The subalgebra $M_2(K) \subset M_2(L)$ is the fixed point set of the map induced by the K-linear involution of L

$$L \longrightarrow L, \quad x = a + b\delta \longmapsto \bar{x} = a - b\delta \quad a, b \in K.$$

From now on we denote it by

$$\rho : M_2(L) \longrightarrow M_2(L), \quad M = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \longmapsto \rho(M) = \overline{M} = \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{z} & \bar{t} \end{pmatrix}.$$

Now we define a Galois action on $M_2(L)$ whose fixed points set is isomorphic to $A = (d, e)_K$.

LEMMA 2.11. *Let $A = (d, e)_K$. Let $L = K(\sqrt{d})$, $n = \begin{pmatrix} 0 & e \\ 1 & 0 \end{pmatrix} \in M_2(L)$. Then $(d, e)_K \cong M_2(L)^{\phi} := \{M \in M_2(L) : \phi(M) = M\}$, with $\phi(M) = n\overline{M}n^{-1}$.*

PROOF. We have

$$\phi(\begin{pmatrix} u & v \\ z & t \end{pmatrix}) = \begin{pmatrix} \bar{t} & e\bar{z} \\ e^{-1}\bar{v} & \bar{u} \end{pmatrix}.$$

The fixed points are the matrices

$$\begin{pmatrix} x & y \\ \bar{y}/e & \bar{x} \end{pmatrix} \quad (x, y \in L).$$

which are just the elements of $R(A)$. $\square$

In case $e = N_{L/K}(L^*)(u)$ for some $u \in L$ it can be seen that the action we just defined is equivalent to ρ, the standard one.

2.5. *The corestriction of a central simple algebra.*

In this section we introduce the corestriction of a central simple algebra which is a crucial object in the construction of Mumford. We follow [11].

2.6. *The twist of an algebra.*

Let K be a Galois extension of $\mathbb{Q}$ and let A be a central simple K-algebra. For any σ in the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ let A^σ: be the ring A endowed with a K-algebra structure given by

$$\lambda \cdot a := \sigma^{-1}(\lambda)a \quad a \in A^\sigma, \ \lambda \in K.$$

PROPOSITION 2.12. *Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ and $d, e \in K$. Let $A = (d, e)_K$, then*

$$A^\sigma \cong (\sigma(d), \sigma(e))_K.$$

PROOF. Let $A = (d, e)_K$. Any element of A can be written as $a = a_0\epsilon_0 + \cdots + a_3\epsilon_3$ and $a_i \in K$. The multiplication in A^σ is given by

$$\epsilon_1^2 = e\epsilon_0 = \sigma(e) \cdot \epsilon_0, \quad \epsilon_2^2 = \sigma(d) \cdot \epsilon_0, \quad \epsilon_3 := \epsilon_1\epsilon_2 = -\epsilon_2\epsilon_1$$

and these are the relations defining the quaternion algebra $(\sigma(d), \sigma(e))_K$. $\square$

2.7. *The corestriction.*

Let $G = \mathrm{Gal}(K/\mathbb{Q}) = \{\sigma_1, \ldots, \sigma_n\}$. Let Z_A be the K-algebra defined by $Z_A := A^{\sigma_1} \otimes_K \cdots \otimes_K A^{\sigma_n}$. We define now an action $\widetilde{\sigma} : Z_A \to Z_A$ of G on Z_A:

$$\widetilde{\sigma}(a_1 \otimes \cdots \otimes a_n) := b_1 \otimes \cdots \otimes b_n, \quad \text{with} \quad b_i = a_j \ \text{if} \ \sigma_{i|K} = (\sigma\sigma_j)_{|K}.$$

DEFINITION 2.13. *The* corestriction *of* A *is the $\mathbb{Q}$-algebra of invariants*:

$$Cor_{K/\mathbb{Q}}(A) := Z_A^{\mathrm{Gal}(K/\mathbb{Q})}.$$

REMARK 2.14. The corestriction $Cor_{K/\mathbb{Q}}(A)$ is a central simple algebra.

We prove now a general result that will be useful later.

THEOREM 2.15. *Let K be an extension of $\mathbb{Q}$ and let V be a K-vector space of dimension n on which G acts σ-linearly*:

$$\sigma(\lambda v) = (\sigma\lambda)(\sigma v) \qquad \lambda \in K,\ v \in V,\ \sigma \in G.$$

Then the space on invariants V^G is a $\mathbb{Q}$-vector space and $V^G \otimes_{\mathbb{Q}} K = V$. In particular, $\dim_K V = \dim_{\mathbb{Q}} V^G$.

The space V^G is a $\mathbb{Q}$- vector space by definition of σ-linearity. Fix now a basis of V : $V \overset{\psi}{\cong} K^n$. We define two different actions of G on K^n:

i) $\sigma \cdot (x_1, \dots, x_n) := (\sigma(x_1), \dots, \sigma(x_n))$, so $(K^n)^G = \mathbb{Q}^n$
ii) $\sigma(x_1, \dots, x_n) := \psi(\sigma(\psi^{-1}(x_1, \dots, x_n)))$.

We can write $\sigma\underline{x} = A_\sigma(\sigma \cdot \underline{x})$ with $A_\sigma \in GL(n, K)$, $\underline{x} \in K^n$. The group G acts in an obvious way on $GL(n, K)$: we write ${}^\sigma M$ for the action on a matrix $M \in GL(n, K)$. It turns out that $\{A_\sigma\}_{\sigma \in G}$ is a 1-cocycle in the Galois cohomology of G on $GL(n, K)$. In fact a straightforward computation shows that $A_{\sigma\rho} = A_\sigma\, {}^\sigma A_\rho$. Moreover, the following result holds

$$H^1(G, GL(n, K)) = \{1\}$$

(see [12, Chapt. X, Prop. 3]). Thus the 1-cocycle $\{A_\sigma\}_{\sigma \in G}$ is a coboundary, that is, there exists a matrix $B \in GL(n, K)$ such that $A_\sigma = {}^\sigma B B^{-1}$, $\forall \sigma \in G$.

One has

$$(B\, {}^\sigma B^{-1})\sigma\underline{x} = \sigma \cdot \underline{x}$$

and this means that the two actions are equivalent. Thus, $V^G \otimes_{\mathbb{Q}} K \cong V$. □

COROLLARY 2.16. *Let A be a central simple K-algebra with K a Galois extension of $\mathbb{Q}$ and let* $\mathrm{Cor}_{K/\mathbb{Q}}(A)$ *be the corestriction of A. There is an isomorphism*

$$\mathrm{Cor}_{K/\mathbb{Q}}(A) \otimes_{\mathbb{Q}} K \cong Z_A.$$

3. THE CORESTRICTION FOR A FAMILY OF MUMFORD-TYPE

In this section, under some assumptions on the quaternion algebra $A = (d, e)_K$, we will make the construction of Mumford explicit by giving an isomorphism

$$\mathrm{Cor}_{K/\mathbb{Q}}(A) \xrightarrow{\cong} M_8(\mathbb{Q}).$$

From now on we consider the case when K is an extension of $\mathbb{Q}$ of degree 3. Consider the quaternion algebra $A = (d, e)_K$, with the assumptions

Q0) $L = K(\sqrt{d})$ is a cyclic extension of $\mathbb{Q}$ of degree 6,
Q1) $d \in \mathbb{Q}$, $d < 0$,
Q2) $N_{K/\mathbb{Q}}(e) = w\rho(w)$ for some $w \in L$, where $\rho \in \mathrm{Gal}(L/\mathbb{Q})$ is the element of order 2.

We write $\mathrm{Gal}(L/\mathbb{Q}) = \{\mathrm{id}, \tau, \dots, \tau^5\}$, $\mathrm{Gal}(K/\mathbb{Q}) = \{\mathrm{id}, \sigma, \sigma^2\}$, $\mathrm{Gal}(L/K) = \{\mathrm{id}, \rho\}$.

We will prove the following:

THEOREM 3.1. *Under the assumptions* Q0), Q1), Q2) *there exist a matrix* $R \in M_8(L)$ *and an endomorphism* $\tau \in \mathrm{End}(M_8(L))$, *such that*

i) $\mathrm{Cor}_{K/\mathbb{Q}}(A) \cong \{M \in M_8(L) \,:\, R\tau(M)R^{-1} = M\}$,

ii) *there exists an isomorphism* $\Psi : \mathrm{Cor}_{K/\mathbb{Q}}(A) \xrightarrow{\cong} \mathrm{End}_{\mathbb{Q}}(V)$, *where* V *is the* $\mathbb{Q}$*-subspace of* L^8 *of dimension* 8 *given by* :

$$V := \{x \in L^8 \,:\, R\tau(x) = x\}.$$

3.1. *Remarks.*

The K-algebra Z_A defined in Section 2.6 is isomorphic to a matrix algebra under our assumptions. In fact, by 2.6, 2.7 and 2.12 follows

PROPOSITION 3.2. *In case* $A = (d, e)_K$ *with* $d \in \mathbb{Q}$ *and* $N_{K/\mathbb{Q}}(e) = w\overline{w}$ *for some* $w \in \in \mathbb{Q}(\sqrt{d})$, *we have* $Z_A = M_8(K)$.

Note that this result doesn't assure that also $\mathrm{Cor}_{K/\mathbb{Q}}(A)$ is a matrix algebra. Proving the Theorem 3.1 we will give an explicit isomorphism which shows that the corestriction is isomorphic to $M_8(\mathbb{Q})$.

In order to prove Theorem 3.1 we need some definitions and lemmas.

By the assumption Q1) of Section 3, $d \in \mathbb{Q}$, hence $\sigma(d) = d$ for all $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. Then

$$\mathrm{Gal}(L/\mathbb{Q}) \cong \mathrm{Gal}(L/\mathbb{Q}(\sqrt{d})) \times \mathrm{Gal}(L/K) \cong \{\mathrm{id}, \sigma, \sigma^2\} \times \{\mathrm{id}, \rho\}.$$

We will identify the subgroup $\mathrm{Gal}(L/\mathbb{Q}(\sqrt{d}))$ of $\mathrm{Gal}(L/\mathbb{Q})$ with the quotient $\mathrm{Gal}(K/\mathbb{Q})$. We will write

$$\bar{x} := \rho(x) \quad \text{thus} \quad \tau(x) = (\sigma\rho)(x) = (\rho\sigma)(x) = \sigma(\bar{x}) = \overline{\sigma(x)},$$

note that τ is a generator of $\mathrm{Gal}(L/\mathbb{Q})$.

LEMMA 3.3. *The group* $\mathrm{Gal}(L/\mathbb{Q})$ *acts on* $M_2(L)^{\otimes 3}$.

PROOF. Define the action of the generator ρ of $\mathrm{Gal}(L/K)$ in the following way

$$\rho \cdot (m_1 \otimes m_2 \otimes m_3) := n\overline{m}_1 n^{-1} \otimes \sigma(n)\overline{m}_2\sigma(n^{-1}) \otimes \sigma^2(n)\overline{m}_3\sigma^2(n^{-1}), \quad n = \begin{pmatrix} 0 & e \\ 1 & 0 \end{pmatrix}.$$

Then, define the action of $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ by $\sigma \cdot (m_1 \otimes m_2 \otimes m_3) := \sigma(m_3) \otimes \sigma(m_1) \otimes \sigma(m_2)$.

We have to show that the action of σ commutes with the action of ρ. First of all we note that $\rho\sigma = \sigma\rho$ $(\in \mathrm{Gal}(L/\mathbb{Q}))$, thus the compositions

$$\sigma \cdot (\rho \cdot (m_1 \otimes m_2 \otimes m_3)) = \sigma \cdot (n\overline{m}_1 n^{-1} \otimes \sigma(n)\overline{m}_2\, \sigma(n^{-1}) \otimes \sigma^2(n)\overline{m}_3\, \sigma^2(n^{-1})) =$$

$$= n(\sigma\rho)(m_3)n^{-1} \otimes \sigma(n)(\sigma\rho)(m_1)\sigma(n^{-1}) \otimes \sigma^2(n)(\sigma\rho)(m_2)\sigma^2(n^{-1})$$

and

$$\rho \cdot (\sigma \cdot (m_1 \otimes m_2 \otimes m_3)) = \rho \cdot (\sigma(m_3) \otimes \sigma(m_1) \otimes \sigma(m_2)) =$$

$$= n(\rho\sigma)(m_3)n^{-1} \otimes \sigma(n)(\rho\sigma)(m_1)\sigma(n^{-1}) \otimes \sigma^2(n)(\rho\sigma)m_2\sigma^2(n^{-1})$$

coincide. $\square$

Lemma 3.4.

$$Z_A = A \otimes_K A^{\sigma} \otimes_K A^{\sigma^2} \cong (M_2(L)^{\otimes 3})^{\rho}.$$

Proof. Since $d \in \mathbb{Q}$, by 2.11 and 2.12 it follows that

$$A^{\sigma^i} \cong (d, e)_K^{\sigma^i} = (d, \sigma^i(e))_K \cong \{m \in M_2(L) : \sigma^i(n)\, \overline{m}\, \sigma^i(n^{-1}) = m\}.$$

Thus there is an embedding of simple K-algebras: $Z_A \hookrightarrow (M_2(L)^{\otimes 3})^{\rho}$. On the other hand by 2.15 we have $\dim_K(Z_A) = \dim_K(M_2(L)^{\otimes 3})^{\rho}$. $\square$

From the previous results we derive the following

Lemma 3.5. *There is an isomorphism of $\mathbb{Q}$-algebras*

$$\mathrm{Cor}_{K/\mathbb{Q}}(A) \cong (M_2(L)^{\otimes 3})^{\mathrm{Gal}(L/\mathbb{Q})}.$$

Proof. The action of $\mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L/K) \cong \mathrm{Gal}(K/\mathbb{Q})$ on $(M_2(L)^{\otimes 3})^{\rho} \cong Z_A$ given in Lemma 3.3 coincides with the action of $\mathrm{Gal}(K/\mathbb{Q})$ on Z_A which defines $\mathrm{Cor}_{K/\mathbb{Q}}(A)$. $\square$

We write now the action of $\mathrm{Gal}(K/\mathbb{Q})$ on $(M_2(L)^{\otimes 3})^{\rho}$ in terms of matrix actions in $M_8(L)$. To do this we need the condition $Q2$).

Lemma 3.6. i) *There exists a matrix $Q \in M_8(L)$ with $Q\overline{Q} = I$ such that the action of ρ is given by*

$$\rho : M_8(L) \longrightarrow M_8(L), \quad M \longmapsto Q\overline{M}Q^{-1}.$$

ii) *There exists a matrix $P \in M_8(L)$ such that the action of σ is given by*

$$\sigma : M_8(L) \longrightarrow M_8(L), \quad M \longmapsto P\sigma(M)P^{-1}.$$

Proof. Consider the standard basis of L^8, $\{f_1, \dots, f_8\}$. As a standard basis for the vector space $(L^2)^{\otimes 3}$ we take $e_{111}, e_{221}, e_{212}, e_{122}, e_{222}, e_{112}, e_{121}, e_{211}$, with $e_{ijk} := e_i \otimes e_j \otimes e_k$, where $\{e_1, e_2\}$ is the standard basis of L^2. Then the isomorphism

$$L^2 \otimes L^2 \otimes L^2 \cong L^8, \quad e_{111} \mapsto f_1, \ e_{221} \mapsto f_2, \dots, e_{221} \mapsto f_8$$

induces an isomorphism

$$M_2(L)^{\otimes 3} \xrightarrow{\cong} M_8(L).$$

The action of $\mathrm{Gal}(L/\mathbb{Q})$ on $M_2(L)^{\otimes 3}$ defines, via this isomorphism, an action of $\mathrm{Gal}(L/\mathbb{Q})$ on $M_8(L)$ and $\mathrm{Cor}_{K/\mathbb{Q}}(A) \cong M_8(L)^{\mathrm{Gal}(L/\mathbb{Q})}$ (see Lemma 3.5).

To prove *i*), we note that the action of ρ on $M_2(L)^{\otimes 3}$ can be written in this way:

$$\rho \cdot (m_1 \otimes m_2 \otimes m_3) = T(\overline{m}_1 \otimes \overline{m}_2 \otimes \overline{m}_3)T^{-1}, \quad T := n \otimes \sigma(n) \otimes \sigma^2(n).$$

The matrix T defines a L-linear map

$$T : (L^2)^{\otimes 3} \longrightarrow (L^2)^{\otimes 3}, \quad x_1 \otimes x_2 \otimes x_3 \longmapsto (nx_1) \otimes (\sigma(n)x_2) \otimes (\sigma^2(n)x_3)$$

which acts on the basis e_{ijk} of $(L^2)^{\otimes 3}$ by:

$$\begin{array}{lclllcl}
e_{111} & \longmapsto & e_{222} & \quad & e_{222} & \longmapsto & e\sigma(e)\sigma^2(e)e_{111} \\
e_{221} & \longmapsto & e\sigma(e)e_{112} & & e_{112} & \longmapsto & \sigma^2(e)e_{221} \\
e_{212} & \longmapsto & e\sigma^2(e)e_{121} & & e_{121} & \longmapsto & \sigma(e)e_{212} \\
e_{122} & \longmapsto & \sigma(e)\sigma^2(e)e_{211} & & e_{211} & \longmapsto & ee_{122}.
\end{array}$$

Thus the matrix in $M_8(L)$ corresponding to $T \in M_2(L)^{\otimes 3}$ is

$$B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$$

where B_1, B_2 are 4×4 blocks given by:

$$B_1 = \begin{pmatrix} N_{K/\mathbb{Q}}(e) & 0 & 0 & 0 \\ 0 & \sigma^2(e) & 0 & 0 \\ 0 & 0 & \sigma(e) & 0 \\ 0 & 0 & 0 & e \end{pmatrix},$$

$$B_2 = N_{K/\mathbb{Q}}(e)B_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e\sigma(e) & 0 & 0 \\ 0 & 0 & e\sigma^2(e) & 0 \\ 0 & 0 & 0 & \sigma(e)\sigma^2(e) \end{pmatrix}.$$

The action of ρ on $M \in M_8(L)$ is thus given by: $M \longmapsto B\overline{M}B^{-1}$. If we replace B by a matrix Q such that $B^{-1}Q = \lambda I$ for some $\lambda \in L$, we obtain the same action:

$$Q\overline{M}Q^{-1} = (\lambda B)\overline{M}(\lambda B)^1 = B\overline{M}B^{-1}.$$

Now we use the assumption Q2) of Section 3, $N_{K/\mathbb{Q}}(e) = w\overline{w}$ to define

$$Q := \begin{pmatrix} 0 & \overline{w}^{-1}B_1 \\ wB_1^{-1} & 0 \end{pmatrix},$$

and we have $B^{-1}Q = \begin{pmatrix} 0 & N_{K/\mathbb{Q}}(e)^{-1}B_1 \\ B_1^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \overline{w}^{-1}B_1 \\ wB_1^{-1} & 0 \end{pmatrix} = \begin{pmatrix} \overline{w}^{-1} & 0 \\ 0 & \overline{w}^{-1} \end{pmatrix}.$

It can be easily seen that $Q\overline{Q} = I$.

As for *ii*), define $S \in \mathrm{End}((L^2)^{\otimes 3})$ by

$$S : (L^2)^{\otimes 3} \longrightarrow (L^2)^{\otimes 3}, \quad x_1 \otimes x_2 \otimes x_3 \longmapsto x_3 \otimes x_1 \otimes x_2.$$

Thus we have:

$$(S(m_1 \otimes m_2 \otimes m_3)S^{-1})(x_1 \otimes x_2 \otimes x_3) = (S(m_1 \otimes m_2 \otimes m_3))(x_2 \otimes x_3 \otimes x_1) =$$

$$= S\left((m_1 x_2) \otimes (m_2 x_3) \otimes (m_3 x_1)\right) =$$

$$= (m_3 x_1) \otimes (m_1 x_2) \otimes (m_2 x_3) =$$

$$= \left(m_3 \otimes m_1 \otimes m_2\right)(x_1 \otimes x_2 \otimes x_3).$$

In $\mathrm{End}((L^2)^{\otimes 3})$ we have $S(m_1 \otimes m_2 \otimes m_3)S^{-1} = m_3 \otimes m_1 \otimes m_2$. Now,

$\sigma \cdot (m_1 \otimes m_2 \otimes m_3) = \sigma(m_3) \otimes \sigma(m_1) \otimes \sigma(m_2) = S(\sigma(m_1) \otimes \sigma(m_2) \otimes \sigma(m_3))S^{-1}$.

The matrix $P \in M_8(L)$ corresponding to S is easy to determine using explicitly the action of σ: $Se_{111} = e_{111}$, hence $Pf_1 = f_1$, $Se_{221} = e_{122}$ hence $Pf_2 = f_4$ etc. Thus

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \quad P_{11} = P_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad P_{21} = P_{12} = 0. \qquad \square$$

3.2. *Proof of Theorem 3.1.*

As for *i*), we proved in Lemma 3.5 that

$$\mathrm{Cor}_{K/\mathbb{Q}}(A) \cong (M_2(L)^{\otimes 3})^{\mathrm{Gal}(L/\mathbb{Q})}$$

so the corestriction of A will be isomorphic to the $\mathbb{Q}$-algebra of the invariants in $M_8(L)$ under the action of ρ and σ as in Lemma 3.6. The Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is cyclic and it is generated by $\tau = \rho\sigma$, so it suffices to take the invariants under the action of that element. Let now $R := PQ$, with P, Q as in Lemma 3.6 *i*), 3.6 *ii*). The map

$$\tau : M_8(L) \longrightarrow M_8(L), \quad M \longmapsto R\sigma(\overline{M})R^{-1}$$

defines an action of $\mathrm{Gal}(L/\mathbb{Q})$ on $M_8(L)$ such that

$$(M_8(L))^{\mathrm{Gal}(L/\mathbb{Q})} \cong \mathrm{Cor}_{K/\mathbb{Q}}(A).$$

Now to prove part *ii*) it suffices to find a $\mathbb{Q}$-subspace of L^8 which is invariant under the action on $\mathrm{Cor}_{K/\mathbb{Q}}(A)$. Consider the $\mathbb{Q}$-subspace

$$V := \{x \in L^8 \, : \, R\sigma(\bar{x}) = x\}.$$

One has $\mathrm{Cor}_{K/\mathbb{Q}}(A) = \{M \in M_8(L) : R\tau(M) = MR\}$ and $V = \{x \in L^8 : R\tau(x) = x\}$. Thus $\mathrm{Cor}_{K/\mathbb{Q}}(A)$ acts on V:

$$R\tau(Mx) = R\tau(M)\tau(x) = MR\tau(x) = Mx, \quad (M \in \mathrm{Cor}_{K/\mathbb{Q}}(A)\,,\ x \in V).$$

This gives a map $\mathrm{Cor}_{K/\mathbb{Q}}(A) \longrightarrow \mathrm{End}_{\mathbb{Q}}(V) \cong M_8(\mathbb{Q})$. Since $\mathrm{Cor}_{K/\mathbb{Q}}(A)$ is a simple algebra, this map is injective and it is surjective for dimension reasons. □

We derive the following result on the real points of the corestriction, which we will be useful later in studying complex structures of our families.

Corollary 3.7. $\mathrm{Cor}_{K/\mathbb{Q}}(A)(\mathbb{R}) := \mathrm{Cor}_{K/\mathbb{Q}}(A) \otimes_{\mathbb{Q}} \mathbb{R} \cong \{M \in M_8(\mathbb{C}) : Q\overline{M}Q^{-1} = M\}$.

Proof. We note that $\mathrm{Cor}_{K/\mathbb{Q}}(A) \otimes_{\mathbb{Q}} \mathbb{R} = (\mathrm{Cor}_{K/\mathbb{Q}}(A) \otimes_{\mathbb{Q}} K) \otimes_K \mathbb{R}$. On the other hand, by 2.16 and 3.4 we have $\mathrm{Cor}_{K/\mathbb{Q}}(A) \otimes_{\mathbb{Q}} K = Z_A \cong (M_8(L))^{\rho}$.

Thus,

$$\begin{aligned}(\mathrm{Cor}_{K/\mathbb{Q}}(A) \otimes_{\mathbb{Q}} K) \otimes_K \mathbb{R} &\cong (M_8(L))^{\rho} \otimes_K \mathbb{R}\\ &\cong \{M \in M_8(L \otimes_K \mathbb{R}) : Q\overline{M}Q^{-1} = M\}\\ &\cong \{M \in M_8(\mathbb{C}) : Q\overline{M}Q^{-1} = M\}. \quad \square\end{aligned}$$

4. Explicit description

With the results of the previous sections we can explicite the construction of Section 1.2. From now on we fix an embedding $L \hookrightarrow \mathbb{C}$ and an embedding $K \hookrightarrow \mathbb{R}$. Thus we can identify $\mathrm{Gal}(K/\mathbb{Q}) = \{\mathrm{id}\,, \sigma\,, \sigma^2\}$ with $\mathrm{Hom}(K, \mathbb{R})$ and $\mathrm{Gal}(L/\mathbb{Q}) = \{\mathrm{id}\,, \tau\,, \dots, \tau^5\}$ with $\mathrm{Hom}(L, \mathbb{C})$. Over $\mathbb{C}$, we indicate with "-" the complex conjugation, which is the $\mathbb{R}$-linear extension of the K-linear involution of L, also indicated with "-". We study now the condition M2) of Section 1.2 on the $\mathbb{R}$-points of a quaternion algebra $A = (d, e)_K$ satisfying the assumptions Q0), Q1), Q2) of Section 3:

$$\text{M2)} \quad A \otimes_K \mathbb{R} \cong \mathbb{H} \oplus \mathbb{H} \oplus M(2\,, \mathbb{R}).$$

Proposition 4.1. *Let $A = (d, e)_K$ be a quaternion algebra satisfying* Q0)-Q2) *of Section 3. Thus A satisfies the condition* M2) *if and only if*

$$\text{Q3)} \quad e < 0\,,\ \ \sigma(e) < 0\,,\ \ \sigma^2(e) > 0\,, \quad (e \in K \subseteq \mathbb{R}).$$

Proof. The condition M2) means that the K-algebra $A = (d, e)_K$ is isomorphic to the Hamilton's quaternions when extended to $\mathbb{R}$ for two of the embeddings $\{\mathrm{id}\,, \sigma\,, \sigma^2\}$ and it is isomorphic to a matrix algebra for the last one. Recalling that $d \in \mathbb{Q} < 0$ and Theorem 2.3, we need the condition Q3). □

Let

$$A = (d, e)_K = K \oplus K\epsilon_1 \oplus K\epsilon_2 \oplus K\epsilon_3$$

be a quaternion algebra which satisfies the assumptions Q0)-Q3) of Section 3 and Proposition 4.1. Under these assumptions we proved that A satisfies conditions M1), M2) of 1.2. Thus we can construct a family of Mumford-type

$$\mathcal{M} = \{X_x : x \in \Gamma \backslash G^0(\mathbb{R})/K^0\}.$$

We want to study now the abelian varieties $X_x = (V, \Lambda, \alpha(g)h\alpha(g)^{-1}, E)$ belonging to $\mathcal{M}$, where V is the $\mathbb{Q}$-vector space V (of dimension eight) we found in Theorem 3.1. In particular, we are going to determine the polarization E and the complex structures given by $\alpha(g)h\alpha(g)^{-1}$.

4.1. *The vector space V.*

In order to describe explicitly families of Mumford-type, we first parametrize $V \subset \subset L^8$. Let $F = \mathbb{Q}(\sqrt{d}) = L^{<\sigma>}$, the fixed field of the subgroup $<\sigma> \subset \mathrm{Gal}(L/\mathbb{Q})$. We write $N := N_{K/\mathbb{Q}}(e) = w\overline{w}$.

Proposition 4.2. *There exists an isomorphism of $\mathbb{Q}$-vector spaces*:

$$\Phi : F \oplus L \xrightarrow{\cong} V,\ (f, l) \longmapsto \left(w\bar{f}, \frac{\sigma^2(e)}{\overline{w}}\tau(l), \frac{\sigma(e)}{\overline{w}}\tau^3(l), \frac{e}{\overline{w}}\tau^5(l), f, \tau^4(l), l, \tau^2(l) \right).$$

Proof. First of all we note that $P : L^8 \to L^8$ is given by:

$$P : (x_1, y_1, y_2, y_3, x_2, z_1, z_2, z_3) \longmapsto (x_1, y_2, y_3, y_1, x_2, z_2, z_3, z_1),$$

and that

$$Q : (x_1, y_2, y_3, y_1, x_2, z_2, z_3, z_1) \mapsto$$
$$\mapsto \left(\frac{N}{\overline{w}}x_2, \frac{\sigma^2(e)}{\overline{w}}z_2, \frac{\sigma(e)}{\overline{w}}z_3, \frac{e}{\overline{w}}z_1, \frac{w}{N}x_1, \frac{w}{\sigma^2(e)}y_2, \frac{w}{\sigma(e)}y_3, \frac{w}{e}y_1 \right).$$

We have $QP\tau(x) = x$ if and only if the following equations hold:

$$x_1 = \frac{N}{\overline{w}}\tau(x_2), \qquad x_2 = \frac{w}{N}\tau(x_1),$$

$$y_1 = \frac{\sigma^2(e)}{\overline{w}}\tau(z_2), \quad y_2 = \frac{\sigma(e)}{\overline{w}}\tau(z_3), \quad y_3 = \frac{e}{\overline{w}}\tau(z_1),$$

$$z_1 = \frac{w}{\sigma^2(e)}\tau(y_2), \quad z_2 = \frac{w}{\sigma(e)}\tau(y_3), \quad z_3 = \frac{w}{e}\tau y_1.$$

We can solve the last six equations by successive substitutions and we find that they are consistent:

$$y_2 = \frac{\sigma(e)}{\overline{w}}\tau^3(z_2), \quad y_3 = \frac{e}{\overline{w}}\tau^5(z_2), \quad z_1 = \tau^4(z_2), \quad z_3 = \tau^2(z_2),$$

so the solutions are parametrized by $z_2 \in L$. As for the first two equations, using $w\overline{w} = N$ we obtain $x_1 = w\tau(x_2)$, $x_2 = \sigma(x_2)$. The solutions are $(x_1, x_2) = (w\bar{x}_2, x_2)$ with $x_2 \in F$ (recall that $\tau(x_2) = \bar{x}_2$ for $x_2 \in F$). □

4.2. *The polarization.*

In 1.2 we considered the symplectic form $E = E_1 \otimes E_2 \otimes E_3$ over $\mathbb{C}$ on the space $V_1 \otimes V_2 \otimes V_3$. The $\mathbb{Q}$-vector space V we found in 3.1 is contained in L^8. In the Proof of Lemma 3.6 we chose the basis $\{e_{ijk}\}$ for $L^8 \cong L^2 \otimes L^2 \otimes L^2$ and w.r.t. this basis (considered as a L-basis) of L^8 the form is, up to a scalar multiple, the standard one

$$E = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Let us consider now the isomorphism of 4.2 $\Phi : F \oplus L \overset{\cong}{\to} V$ and the symplectic form $\Phi^* E$ on $F \oplus L$ given by $(\Phi^* E)(x, y) := E(\Phi(x), \Phi(y))$. This form, up to a scalar multiple, gives us the polarization on $F \oplus L$. With respect to the $\mathbb{Q}$-basis of $F \oplus L$, given by

$$\begin{aligned} v_1 &= (1, 0), & v_5 &= (0, e^2), \\ v_2 &= (\delta, 0), & v_6 &= (0, \delta), \\ v_3 &= (0, 1), & v_7 &= (0, \delta e), \\ v_4 &= (0, e), & v_8 &= (0, \delta e^2), \end{aligned}$$

we find by an easy computation

$$\Phi^* E = c \cdot \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \quad \text{with} \quad c = 2\delta w$$

(recall that $N_{K/\mathbb{Q}}(e) = w\overline{w}$) where

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } E_2 = c' \cdot \begin{pmatrix} 0 & E' \\ -{}^tE' & 0 \end{pmatrix} \text{ with } E' = \begin{pmatrix} T(e) & T(e_1) & T(e_2) \\ T(e_1) & T(e_2) & T(e_3) \\ T(e_2) & T(e_3) & T(e_4) \end{pmatrix}$$

with

$$\begin{aligned} c' &= \frac{1}{w\overline{w}} \in \mathbb{Q}, & T(x) &:= Tr_{K/\mathbb{Q}} = x + \sigma(x) + \sigma^2(x), \\ e_1 &= e\sigma^2(e), & e_2 &= e\sigma^2(e^2), \\ e_3 &= e^3\sigma(e), & e_4 &= e\sigma^2(e^4). \end{aligned}$$

Thus the polarization on $F \oplus L$ is given by $\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$. We denote this form also with E.

4.3. *Complex structures on $V_{\mathbb{R}}$.*

In this section we find the complex structures on $V_{\mathbb{R}}$ given by the homomorphisms $\alpha(g)h\alpha(g)^{-1} \in Sp(V_{\mathbb{R}}, E) \subset \mathrm{Cor}_{K/\mathbb{Q}}(A)(\mathbb{R})$. Before doing this we recall the construction of Mumford: he introduces on the 8-dimensional space $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$ the complex structures $\alpha(g)\, h\, \alpha(g)^{-1}$ where $g \in G(\mathbb{R})$ and h is the homomorphism

$$\begin{array}{ccccc} h: S^1 & \longrightarrow & SU(2) \times SU(2) \times SL(2, \mathbb{R}) & \xrightarrow{\alpha} & Sp(V_{\mathbb{R}}, E) \subset \mathrm{Cor}_{K/\mathbb{Q}}(A)(\mathbb{R}) \\ e^{i\theta} & \longmapsto & \left(I_2, I_2, \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) & \longmapsto & I_2 \otimes I_2 \otimes \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \end{array}$$

Note that in this construction Mumford uses standard basis, while we used the basis fixed in the Proof of Lemma 3.6 in all the explicit constructions of the $\mathbb{Q}$-vector space $V \subset L^8$ and the algebra $\mathrm{Cor}_{K/\mathbb{Q}}(A) \cong M_8(\mathbb{Q})$. So, now we choose the same basis to find $h(S^1)$ explicitly.

Proposition 4.3. *The image of $I_2 \times I_2 \times SL(2, \mathbb{R})$ in $Sp(V_{\mathbb{R}}, E)$ is given by the group*

$$\mathcal{H} = \{M = I_2 \otimes I_2 \otimes M_1 \in M_2(\mathbb{C}^2)^{\otimes 3} : \ M_1 = \begin{pmatrix} a & b \\ \bar{b}/\sigma^2(e) & \bar{a} \end{pmatrix}, \ \det M_1 = 1\}.$$

Moreover, $M_1 \in SU(1, 1) \cong SL(2, \mathbb{R})$.

Proof. We assume the isomorphism $M_2(L)^{\otimes 3} \cong M_8(L)$. Consider the basis we fixed in the Proof of Lemma 3.6 and write a matrix

$$M = I_2 \otimes I_2 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2((\mathbb{C}^2)^{\otimes 3})$$

as an 8×8 matrix w.r.t. this basis. This matrix belongs to $\mathrm{Cor}_{K/\mathbb{Q}}(A)(\mathbb{R})$ if and only if it satisfies the condition of Corollary 3.7

$$\mathrm{Cor}_{K/\mathbb{Q}}(A) \otimes_{\mathbb{Q}} \mathbb{R} \cong \{M \in M_8(\mathbb{C}) : Q\overline{M}Q^{-1} = M\}$$

and it is easy to see that this happens if and only if it belongs to $\mathcal{H}$. Moreover, since $\sigma^2(e) > 0$, the matrix

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -\sigma^2(e) \end{pmatrix}$$

defines a hermitian form of type $(1, 1)$ and the matrix $M_1 = \begin{pmatrix} a & b \\ \bar{b}/\sigma^2(e) & \bar{a} \end{pmatrix}$ is such that ${}^t\overline{M_1} H M_1 = H$. Thus, $M_1 \in SU_H \cong SU(1, 1) \cong SL(2, \mathbb{R})$.

5. CM-fibers

We are going to study some special fibers of a family of Mumford-type, the CM-fibers, where CM stands for "complex multiplication". A good reference for this topic is [7].

5.1. *Abelian varieties of CM-type.*

A field L is a CM-field if it is an imaginary quadratic extension of a totally real number field. If we embed $L \hookrightarrow \mathbb{C}$, we can consider the conjugation map, $x \mapsto \bar{x}$,

$x \in L$. We denote also with "-" the only automorphism in $Aut(L/\mathbb{Q})$ which gives rise to the complex conjugation after the embedding of L in $\mathbb{C}$. Let L a CM-field of degree $2g$ over $\mathbb{Q}$, and let $S = Hom(L, \mathbb{C})$ the set of complex embeddings of L. A CM type for L is a subset Σ of S such that $S = \Sigma \cup \overline{\Sigma}$.

Definition 5.1. A simple abelian variety X is of CM-type if there exists a field L with $L \hookrightarrow \mathrm{End}(X) \otimes \mathbb{Q}$ and $[L, \mathbb{Q}] \geq 2 \dim X$. In this case $\dim(L) = 2 \dim X$, $L \cong \mathrm{End}(X) \otimes \mathbb{Q}$, and L is a CM-field, see [7, 1.3.1]. An abelian variety is of CM-type if it is isogenous to a product of simple abelian varieties of CM-type.

The study of these varieties makes sense here since the presence of such fibers characterize families of Mumford-type (see [9, Theorem 3]).

5.2. *A non simple fiber.*

Let us come back now to our explicit construction. By 4.2 we have that

$$V \cong F \oplus L \quad \text{with} \quad F = \mathbb{Q}(\sqrt{d}), \; L = K(\sqrt{d})$$

and the fields L, F are CM-fields. We show explicitly that there is a complex structure J on $V_{\mathbb{R}}$ such that the abelian variety $X = (V, \Lambda, J, E)$ is of Mumford-type and is isogenous to a non simple abelian variety of CM-type.

Proposition 5.2. *Consider a family of Mumford-type $\mathcal{M}$ with the notations and the assumptions of Section 4. The complex structure*

$$J := I_2 \otimes I_2 \otimes i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

gives a non simple CM-fiber $X = (V, \Lambda, J, E)$ which is isogenous to the product of two abelian varieties: $X \sim Y \times C$, where Y is a threefold of CM-type with $\mathrm{End}^0(Y) \supseteq L$ and C is an elliptic curve of CM-type with $\mathrm{End}^0(C) = F$.

Proof. By Proposition 4.3 we know that $J \in \mathcal{H} \subseteq M_8(\mathbb{C})$. In particular, if we write the matrix J w.r.t. the basis fixed in the Proof of Lemma 3.6 we obtain

$$J = \mathrm{diag}(i, i, -i, -i, -i, -i, i, i).$$

Consider now the isomorphism of 4.2, $\Phi : F \oplus L \xrightarrow{\cong} V$ and the $\mathbb{Q}$-vector spaces $V^1 := \Phi(L) \subset L^6$ and $V^2 := \Phi(F) \subset L^2$. The action of L over $V^1_{\mathbb{R}} = \Phi(L \otimes_{\mathbb{Q}} \mathbb{R})$ commutes with the one of $J_{|V^1_{\mathbb{R}}}$. In fact, w.r.t. the fixed basis, both act by diagonal matrices on $V^1_{\mathbb{R}}$:

$$\Phi(\lambda l) = \Delta(\lambda)\Phi(l) \text{ with } \Delta(\lambda) \text{ a diagonal matrix.}$$

Similarly, the action of F over $V^2_{\mathbb{R}}$ commutes with the one of $J_{|V^2_{\mathbb{R}}}$ and this means that $J(V^1_{\mathbb{R}}) \subseteq V^1_{\mathbb{R}}$. Thus, we get $X = (V, \Lambda, J, E) \sim Y \times C$ where

$$Y = (V^1, J_{|V^1}, E_{|V^1}), \quad C = (V^2, J_{|V^2}, E_{|V^2}). \quad \square$$

5.3. *The threefold Y.*

Now we want to investigate the structure of the CM-threefold $Y = (V^1, J_{|V^1}, E_{|V^1})$ with $\mathrm{End}^0(Y) = L$. If we restrict the polarization E and the complex multiplication J to V^1, (with the basis fixed in the Proof of Lemma 3.6) we obtain that the polarization of Y is

$$E_{|Y} = E_{|V^1} = E_2$$

and that the complex structure is

$$J_{|Y} = J_{|V^1} = i\,\mathrm{diag}(1\,,-1-1\,,-1\,,1\,,1).$$

We get the following

Proposition 5.3. *The CM type for Y is $\Sigma = (\mathrm{id}\,,\tau\,,\tau^2)$. In particular, Y is simple.*

Proof. The isomorphism induced by the one of 4.2 is

$$L \longrightarrow V^1 \cong \mathbb{Q}^6\,, \quad l \longmapsto \left(\frac{\sigma^2(e)}{\overline{w}}\tau(l)\,,\ \frac{\sigma(e)}{\overline{w}}\tau^3(l)\,,\ \frac{e}{\overline{w}}\tau^5(l)\,,\ \tau^4(l)\,,\ l,\ \tau^2(l)\right)$$

and we see directly that the CM type for Y is the one of the statement. A classical result for varieties of CM-type (see for example [7, 1.3.5.]) says that Y is simple if the subgroup

$$H = \{\tau^i \in \mathrm{Gal}(L/\mathbb{Q}) : \tau^i \circ \Sigma = \Sigma\}$$

is the identity. Thus, Y is simple, as one can verify directly. □

Since Y is of CM-type, the polarization $E_{|Y}$ of Y has the form

$$E_{|Y}(x\,,y) = Tr_{L/\mathbb{Q}}(\alpha x\bar{y})$$

for a suitable $\alpha = \delta\eta \in L\,,\eta \in K$ (see for example [10, 210-213]). In this case we have the following

Theorem 5.4. *Let Y be a threefold of CM-type as in 5.3. Then the polarization of Y has the form*

$$E_{|Y}(x\,,y) = Tr_{L/\mathbb{Q}}(\delta\eta x\bar{y}) \quad \textit{with} \quad \eta = \sigma(e).$$

Proof. If we consider the L-basis of L^6 given by the vectors $\{v_3\,,\ldots\,,v_8\}$ of 4.2 we obtain that, w.r.t. this basis, $E_{|Y} = E_2$ (see 4.2). Using the properties of the reduced trace we can write the matrix E_2 in this way

$$dE_2 = \begin{pmatrix} 0 & E' \\ -{}^tE' & 0 \end{pmatrix} \quad \text{with } E' = \begin{pmatrix} T(\sigma(e)) & T(e\sigma(e)) & T(e^2\sigma(e)) \\ T(e\sigma(e)) & T(e^2\sigma(e)) & T(e^3\sigma(e)) \\ T(e^2\sigma(e)) & T(e^3\sigma(e)) & T(e^4\sigma(e)) \end{pmatrix},$$

in fact, recalling the definitions of 4.2 we have for example

$$T(e_4) = T(e\sigma^2(e^4)) = T(e^4\sigma(e)).$$

We can obtain the remaining entries of the matrix by similar remarks. Consider now the $\mathbb{Q}$-basis of L (as in 4.2) given by

$$\{1\,,\, e\,,\, e^2\,,\, \delta\,,\, \delta e\,,\, \delta e^2\}.$$

From the form of E_2 we have

$$E_2(e^i\,,\, e^j) = 0\,, \quad E_2(e^i\,,\, \delta e^j) = T(e^{i+j}\sigma(e)) \quad 0 \le i\,, j\,, \le 2.$$

On the other hand,

$$Tr_{L/\mathbb{Q}}(\delta\,\eta\, e^i e^j) = Tr_{L/\mathbb{Q}}(\delta\,\eta\, e^{i+j}) = 0\,,$$

$$Tr_{L/\mathbb{Q}}(\delta\,\eta\, e^i \delta e^j) = dTr_{K/\mathbb{Q}}(\eta\, e^{i+j}).$$

We recall (see 1.2) that in a family of Mumford-type the polarization is taken up to constants (in this case d), thus we get the statement. □

5.4. *Families of Mumford-type with a given CM-variety as a fiber.*

The results of the previous sections also give a method to produce a family of Mumford-type with a given fiber of CM-type. In fact, we have the following

THEOREM 5.5. *If one has the following data*

i) $K = \mathbb{Q}(e)$ *a totally real cubic number field with* $e > 0$, $\sigma(e) < 0$, $\sigma^2(e) < 0$, *with* $< \sigma >= \mathrm{Gal}(K/\mathbb{Q})$,
ii) $L = \mathbb{Q}(e\,,\,\delta)$ *a cyclic CM-field of degree six over* $\mathbb{Q}$ *with* $\delta = \sqrt{d}$, $d < 0 \in \mathbb{Q}$,
iii) $N_{K/\mathbb{Q}}(e) = w\overline{w}$ *with* $w \in L$,
iv) Y *a threefold of CM-type with* $\mathrm{End}^0(Y) = L$ *and polarization given by* $E(x\,,\,y) = $ $= Tr_{L/\mathbb{Q}}(\sigma(e)\delta xy)$,
v) C *an elliptic curve of CM-type with* $\mathrm{End}^0(C) = F = \mathbb{Q}(\delta)$.

Then $Y \times C$ is a fiber in a family of Mumford-type with quaternion algebra $A =$ $= (d\,,\, e)_K$.

5.5. *An example.*

We give now an example of CM-variety that can be viewed as a fiber in a family of Mumford-type. To do this, we recall how to construct a variety of CM-type starting form a CM-field L. Let L be a CM-field of degree $2g$ and let Σ be a CM type for L. Let $\mathcal{O} \cong \mathbb{Z}^{2g}$ the ring of integers of L. The set $\Sigma(\mathcal{O}) = \{(\ldots\,,\sigma(l)\,,\ldots)_{\sigma\in\Sigma} \in \mathbb{C}^{\Sigma} : l \in L\}$ is a lattice in $\mathbb{C}^{\Sigma} \cong \mathbb{C}^g$. We define X_Σ to be the complex torus

$$X_\Sigma := \mathbb{C}^{\Sigma}/\Sigma(\mathcal{O}).$$

It can be proved that X_Σ is an abelian variety (see [10, pp. 210-213]). Let now $\zeta = \zeta_7$ be a primitive 7-th root of unity and let L be the field $\mathbb{Q}(\zeta)$. The minimum polynomial of ζ is

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$$

The field $K := \mathbb{Q}(\zeta + \zeta^{-1}) \subset L$ is a totally real cubic number field, the minimum polynomial of $\zeta + \zeta^{-1}$ is $x^3 + x^2 - 2x - 1 = 0$ and one has $[L : K] = 2$, thus the field L is a CM-field of degree six. The embeddings of L in $\mathbb{C}$ are given by

$$\begin{array}{cccc} \tau_l : & L & \longrightarrow & \mathbb{C} \\ & \tau_l(\zeta) & \longmapsto & e^{2\pi i l/7} \end{array} \qquad l = 1, \dots, 6.$$

As we fixed the embeddings, we can write $\mathrm{Gal}(L/\mathbb{Q}) = \{\tau_1, \dots, \tau_6\}$ and $\mathrm{Gal}(K/\mathbb{Q}) = \{\tau_1, \tau_2, \tau_3\}$. Consider now the hyperelliptic curve defined by $y^2 = x^7 - 1$, with genus $3 = (7-1)/2$. The Jacobian Y of this curve is an abelian threefold of CM-type which is isogenous to $X_\Sigma = \mathbb{C}^3/\Sigma(\mathcal{O}_L)$ where $\Sigma = \{\tau_1, \tau_2, \tau_3\}$ and $\mathcal{O}_L$ is the ring of integers of L. Let $\delta = \sqrt{-7}$. The field L can be written also as $L = K(\delta)$ and δ can be written as

$$\delta = \zeta + \zeta^2 + \zeta^4 - (\zeta^3 + \zeta^5 + \zeta^6).$$

We define a polarization $E_{|Y}$ on Y as

$$E_{|Y}(x, y) = Tr_{L/\mathbb{Q}}(\alpha x \bar{y}) \quad \alpha = \zeta - \zeta^{-1}.$$

This polarization is also principal, as one can compute directly on the basis $\{1, \zeta, \dots, \zeta^5\}$ of $\mathcal{O}_L$. We can write $\alpha = \delta\eta$, with

$$\eta = 1/7(\zeta - \zeta^{-1})[\zeta + \zeta^2 + \zeta^4 - \zeta^3 - \zeta^5 - \zeta^6].$$

One can sees directly that $\mathrm{Gal}(L/K)$ fixes η, so $\eta \in K$.

THEOREM 5.6. *Let C be an elliptic curve of CM-type with $\mathrm{End}^0(C) = \mathbb{Q}(\delta)$ and let be Y the Jacobian of the hyperelliptic curve defined by $y^2 = x^7 - 1$. Thus, $Y \times C$ is a fiber in the family $\mathcal{M}$ constructed with the algebra*

$$A = (-7, \tau^2(\eta))_K, \quad <\tau> = \mathrm{Gal}(K/\mathbb{Q}).$$

PROOF. We have to verify conditions *i*)-*v*) of Theorem 5.5 for L, K, δ, η as defined above. Conditions *i*), *ii*), *iv*), *v*) are verified by our construction of the fields L, K.

It suffices now to prove that $N_{K/\mathbb{Q}}(\tau^2(\eta)) = w\overline{w}$ for some $w \in L$, where, if $w = a + b\sqrt{-7}$, $\overline{w} = a - b\sqrt{-7}$. But $N_{K/\mathbb{Q}}(\tau^2(\eta)) = N_{K/\mathbb{Q}}(\eta)$ and an easy computation shows that

$$N_{K/\mathbb{Q}}(\eta) = 1/49 = (1/7)(\overline{1/7}). \quad \square$$

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