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Equicontinuous families of operators generating mean periodic maps

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Analisi funzionale. — *Equicontinuous families of operators generating mean periodic maps.*
Nota di Valentina Casarino, presentata (*) dal Socio E. Vesentini.

Abstract. — The existence of mean periodic functions in the sense of L. Schwartz, generated, in various ways, by an equicontinuous group U or an equicontinuous cosine function C forces the spectral structure of the infinitesimal generator of U or C. In particular, it is proved under fairly general hypotheses that the spectrum has no accumulation point and that the continuous spectrum is empty.

Key words: Mean periodicity; Equicontinuous groups; Schwartz spectrum.

Riassunto. — *Famiglie equicontinue di operatori che generano mappe medio-periodiche.* Si dimostra che l'esistenza di funzioni medio-periodiche nel senso di L. Schwartz, generate, in diversi modi, da un gruppo U o da una funzione coseno C equicontinui condiziona la struttura dello spettro del generatore infinitesimale di U e di C. In particolare, si dimostra sotto ipotesi piuttosto generali che lo spettro è privo di punti di accumulazione e che lo spettro continuo è vuoto.

A continuous complex-valued solution of the integral equation

$$\int f(t-u)\, d\mu(u) = 0\,,$$

where μ is a measure with compact support on $\mathbb{R}$, not identically vanishing, is said to be a mean periodic function. This definition, which extended a previous one due to J. Delsarte, was introduced by L. Schwartz [21], who during the fifties, together with J.P. Kahane [11-13], analyzed and proved the main properties of mean periodic functions.

What happens when f is generated, in a weak sense, by an equicontinuous family of operators acting on a locally convex space $\mathcal{E}$? More specifically, which constraints are imposed on a group U or on a cosine function C by the existence of an element $x_0 \in \mathcal{E}$ and of a functional $\lambda_0 \in \mathcal{E}'$, such that $\int < U(t-u)x_0\,, \lambda_0 > d\mu(u) = 0$ or $\int < C(t-u)x_0\,, \lambda_0 > d\mu(u) = 0$ for all $t \in \mathbb{R}$, for some compactly supported measure μ?

This question forms part of a broader project: the description of the spectral structure of the infinitesimal generator of a family of operators, giving rise to maps, which are, in some generalized sense, periodic. At the light of this fact, in the case of an equicontinuous group of class $\mathcal{C}_0$ $U : \mathbb{R} \to \mathcal{L}(\mathcal{E})$, the relationship between almost periodicity and mean periodicity for an application of type $\varphi(t) =< U(t)x_0\,, \lambda_0 >$, $t \in \mathbb{R}$, is, first of all, analyzed. Then, taking into account the notion of spectrum of a mean periodic map introduced by L. Schwartz, a connectionship is proved between the spectrum of φ and the spectrum of the infinitesimal generator X of U. Although this result resembles an analogous one obtained under assumptions on the almost periodic behaviour of φ [24], the nature of constraints imposed on $\sigma(X)$ in the framework of

(*) Nella seduta del 23 aprile 1999.

mean periodicity is different than under hypotheses on the almost periodic behaviour of U. If U generates mean periodic applications, then $\sigma(X)$ admits, under fairly general assumptions, no finite accumulation point. Moreover, the existence of mean periodic maps generated by U forces also the continuous spectrum of X, which, in some cases, results to be empty.

Analogous results are obtained also for an equicontinuous cosine function of class $\mathcal{C}_0$, generating at least a mean periodic application. In particular, in this case the set of zeros of the Fourier transform of μ, where μ is the measure appearing in the convolution equation, satisfies a particular condition of symmetry.

Although the theory of mean periodic maps on a half-line is in principle quite different from the theory of mean periodic functions on the entire real line, some of the results obtained for maps generated by a group can be extended to the case of functions associated to a semigroup. The case is also considered of maps, arising from a semigroup, which are asymptotically mean periodic with respect to a weak topology on $L^2(0\,,+\infty)$, as suggested by Beurling [3].

It has to be remarked that also from a technical point of view the study of mean periodicity and that of almost periodicity is different; if in the analysis of a mean periodic function f more significative are properties of the Fourier transform of μ, μ being a measure for which $\int f(t-u)\,d\mu(u) = 0\,,\ t \in \mathbb{R}$, in the analysis of almost periodic applications generated by a group the main tool is represented by ergodic theorems.

In the appendix, some extensions of mean ergodic theorems to the case of semigroups acting on a locally convex space are presented. E. Vesentini proved in [24] a version of mean ergodic theorem in the dual space $\mathcal{E}'$ of a Banach space $\mathcal{E}$; in particular, given a uniformly bounded semigroup T acting on $\mathcal{E}$, he showed the existence of a projector $R_{i\theta}$ defined on a subspace $\mathcal{H}'_{i\theta} \subseteq \mathcal{E}'$, such that

$$\ker(X^{\odot} - i\theta I) \oplus \overline{\mathcal{R}(X^{\odot} - i\theta I)} \subseteq \mathcal{H}'_{i\theta}\,,$$

where $X^{\odot}$ represents the infinitesimal generator of the adjoint semigroup $T^{\odot}$. Here it is proved, in the more general case in which $\mathcal{E}$ represents a locally convex space and T is an equicontinuous semigroup on $\mathcal{E}$, that, if $\mathcal{E}$ is $\odot$-reflexive with respect to T, then $\mathcal{H}'_{i\theta}$ coincides with $\ker(X^{\odot} - i\theta I) \oplus \overline{\mathcal{R}(X^{\odot} - i\theta I)}$, and it is shown with an example that, if $\mathcal{E}$ is not $\odot$-reflexive, the inclusion proved by E. Vesentini may be strict.

1. Mean periodic functions

Let $\mathcal{C}$ denote the space of all complex-valued functions, continuous on $\mathbb{R}$; $\mathcal{C}$, endowed with the topology of uniform convergence on compacta of $\mathbb{R}$, is a Hausdorff locally convex linear topological space; its dual space $\mathcal{C}'$ is the space of complex-valued Borel regular measures with compact support on $\mathbb{R}$.

If $f \in \mathcal{C}$, the symbol $\tau(f)$ will denote the closure, with respect to the topology of $\mathcal{C}$, of the linear subspace generated by the translates of f, *i.e.* of

$$\bigvee \{f(\cdot + u) :\ u \in \mathbb{R}\}. \tag{1.1}$$

A function f in $\mathcal{C}$ is said to be *mean periodic* if $\tau(f) \neq \mathcal{C}$.

As a consequence of Hahn-Banach theorem, a function f in $\mathcal{C}$ is mean periodic if, and only if, there exists a non-zero measure μ in C', such that

$$\int f(t-u)\, d\mu(u) = 0 \quad \text{for every } t \in \mathbb{R}. \tag{1.2}$$

In other words, a function f is mean periodic if, and only if, it is a continuous solution of a homogeneous integral equation $f * \mu = 0$, μ being a non identically vanishing measure in $\mathcal{C}'$ (1).

A closed subspace in $\mathcal{C}$, which is translation-invariant, of finite dimension $n \geq 1$ and not representable as a sum of two such subspaces, is called a *simple subspace*.

The problem of harmonic analysis consists, as it is well known, in studying the simple subspaces contained in $\tau(f)$. J.P. Kahane proved [13] that, if f is mean periodic, the only simple subspaces in $\tau(f)$ are those generated by the translates of a «polynomial-exponential», *i.e.* by the translates of $P(t)\, e^{i\theta t}$, $P(t)$ being a polynomial and θ a complex number.

The problem of spectral synthesis is: Is f the limit of finite sums $\sum_{j=1}^{n} f_j$, f_j belonging, for every $j \in \{1, \dots, n\}$, to simple subspaces contained in $\tau(f)$?

L. Schwartz solved this problem, by showing that, if f is mean periodic, then f can be approximated by finite sums of polynomial-exponentials belonging to $\tau(f)$ (and therefore $\tau(f)$ is the closed span of polynomial-exponentials contained in it).

Define now

$$f^{+}(t) = \begin{cases} 0 & \text{if } t < 0 \\ f(t) & \text{if } t \geq 0 \end{cases}$$

and

$$f^{-}(t) = \begin{cases} f(t) & \text{if } t < 0 \\ 0 & \text{if } t \geq 0. \end{cases}$$

If f is mean periodic, then it results $\left(f^{-} + f^{+}\right) * \mu = 0$ for some non-zero measure $\mu \in \mathcal{C}'$. Set

$$g = f^{-} * \mu = -f^{+} * \mu;$$

g is compactly supported and therefore its Fourier transform

$$\mathcal{G}(\zeta) = \int e^{-it\zeta}\, g(t)\, dt$$

is an entire function.

(1) This definition was introduced by L. Schwartz and it differs slightly from the original one of M. Delsarte [8], who considered the case $d\mu = k(u)\, du$, with k bounded.

Let $\mathcal{M}(\zeta)$ denote the Fourier transform of μ, *i.e.* $\mathcal{M}(\zeta) = \int e^{-it\zeta}\, d\mu(t)$. Consider the meromorphic function

$$\mathcal{F}(\zeta) = \frac{\mathcal{G}(\zeta)}{\mathcal{M}(\zeta)}. \tag{1.3}$$

J.P. Kahane [11] showed that $\mathcal{F}$ coincides with the *Carleman transform* (which will be defined in § 3) of f. J.P. Kahane also proved the following

LEMMA 1.1. *$P(t)e^{i\theta t}$ belongs to $\tau(f)$ if, and only if, for every non-zero measure $\mu \in \mathcal{C}'$, with $f * \mu = 0$, it results:*

$$\mathcal{M}(\theta) = \mathcal{M}^{(1)}(\theta) = \ldots = \mathcal{M}^{(n)}(\theta) = 0\,,$$

n being the degree of P.

As a consequence of this fact, he proved also that $P(t)\, e^{i\theta t}$ belongs to $\tau(f)$ if, and only if, θ is a pole of $\mathcal{F}(\zeta)$ of order at least equal to n.

The *spectrum* $S(f)$ of a mean periodic function f is defined as the set of all poles of $\mathcal{F}(\zeta)$, each counted with its order of multiplicity.

Since the monomial-exponentials $t^n e^{i\theta t}$ in $\tau(f)$ form a basis of $\tau(f)$, f (which belongs to $\tau(f)$) admits a formal development with respect to this basis

$$f(t) \sim \sum a(\theta\,, p) t^p e^{i\theta t}\,, \qquad \theta \in S(f);$$

this series is said to be the *Fourier series* of f.

J.P. Kahane investigated the case of bounded mean periodic functions; if f is bounded, he showed that the spectrum of f is real and simple, and that the coefficients $a(\theta)$ in the Fourier series of f are bounded.

Some of the results obtained by Schwartz and Kahane for mean periodic functions were extended by P. Koosis to maps, defined on a half-line. Let $\mathcal{C}_+$ denote the space of all complex-valued functions, continuous on $\mathbb{R}_+$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$; if $f \in \mathcal{C}_+$, the symbol $\tau_+(f)$ will denote the closure, with respect to the same topology, of the linear subspace generated by the positive translates of f, *i.e.* of

$$\bigvee \{f(\cdot + u) :\ u \in \mathbb{R}_+\}. \tag{1.4}$$

A function f in $\mathcal{C}_+$ is said to be *asymptotically mean periodic* if $\tau_+(f) \neq \mathcal{C}_+$.
In particular, a function f in $\mathcal{C}_+$ is asymptotically mean periodic if, and only if, there exists a non-zero measure μ, with compact support in $(-\infty\,, 0]$, such that $\int_{-\infty}^{0} f(t - -u)\, d\mu(u) = 0$ for every $t \geq 0$.

The behaviour of asymptotically mean periodic applications differs strongly from that of mean periodic maps; in particular, as a celebrated example of Beurling illustrates [3], it may happen that $\tau_+(f)$ contains no polynomial-exponential, unless a suitable weak topology on $\mathcal{C}_+$ is given.

Anyway, also in case $\tau_+(f)$ contains some polynomial-exponential, f may be not uniformly approximable on the compacta of $\mathbb{R}_+$by finite linear combinations of polynomial-exponentials belonging to $\tau_+(f)$. Indeed, it may happen that only a translate of f, $f(\cdot + u_0)$, for some sufficiently large $u_0 > 0$, is approximable by means of polynomial-exponentials.

Recall finally that, if $f : \mathbb{R} \to \mathbb{C}$ ($f : \mathbb{R}_+ \to \mathbb{C}$) is an almost periodic (an asymptotically almost periodic) function, then the limit

$$\lim_{t \to +\infty} \int_0^t e^{-i\theta s} f(s)\, ds$$

exists for all real values of θ and vanishes always, with the possible exception of an at most countable set of values of θ. The values of θ such that the limit above doesn't vanish are said to be the *frequencies* of f; moreover, the set of all frequencies of f constitutes the *Fourier spectrum* of f (which will be denoted by the symbol $\sigma_{\mathcal{F}}$).

2. Mean periodicity and almost periodicity for maps associated to a group

Let $\mathcal{E}$ be a locally convex linear topological space, with topology defined by the family of continuous seminorms $\mathcal{P}$, satisfying the separation axiom; elements of $\mathcal{P}$ will be denoted by p_α or q_α, $\alpha \in A$; moreover, let $\mathcal{L}(\mathcal{E})$ and $\mathcal{E}'$ be, respectively, the space of all linear continuous operators on $\mathcal{E}$ and the topological dual of $\mathcal{E}$; if $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$, the symbol $< x, \lambda >$ will denote the value of λ in x.

A net $\{x_i : i \in I, \geq\}$ in $\mathcal{E}$ is a *Cauchy net* if for every neighbourhood U of 0 there is some $i_0 \in A$ such that $x_i - x_j \in U$ whenever both i and j follow i_0 in the order $\geq$. A *Cauchy sequence* is a sequence which is a Cauchy net.

The space $\mathcal{E}$ is said to be *complete* if every Cauchy net in $\mathcal{E}$ converges to some element in $\mathcal{E}$.

$\mathcal{E}$ is said to be *sequentially complete* if every Cauchy sequence in $\mathcal{E}$ converges to some element in $\mathcal{E}$.

An application $T : [0, +\infty) \to \mathcal{L}(\mathcal{E})$ is said to be an *equicontinuous semigroup of class* $\mathcal{C}_0$ if it represents a one-parameter family of bounded linear operators in $\mathcal{L}(\mathcal{E})$ such that:

(1) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ and $T(0) = I$;

(2) $\lim_{t \to t_0} T(t)x = T(t_0)x$ for every $t_0 \geq 0$ and $x \in \mathcal{E}$;

(3) for every continuous semi-norm p on $\mathcal{E}$, there exists a continuous semi-norm q on $\mathcal{E}$ such that $p\big(T(t)x\big) \leq q(x)$ for every $t \geq 0$ and $x \in \mathcal{E}$.

According to the definition of Yosida, an equicontinuous semigroup T of class $\mathcal{C}_0$ is called an *equicontinuous group of class* $\mathcal{C}_0$ if there exists an equicontinuous semigroup $\hat{T}$ of class $\mathcal{C}_0$ satisfying the following condition:
the family of operators $\{U(t)\}$, defined by $U(t) = T(t)$ for $t \geq 0$ and $U(-t) = \hat{T}(t)$

for $t \geq 0$, has the group property:

$$U(t+s) = U(t)U(s) \text{ for every } t, s \in \mathbb{R} \text{ and } U(0) = I.$$

If T is an equicontinuous semigroup of class $\mathcal{C}_0$ on $\mathcal{E}$, the family of operators $\{T'(t) : t \geq 0\}$, where $T'(t)$ denotes the dual operator of $T(t)$ for every $t \geq 0$, satisfies the semigroup property, is equicontinuous with respect to $t \geq 0$, but it is not of class $\mathcal{C}_0$, in general. Let now $\mathcal{E}$ be sequentially complete and such that its dual space $\mathcal{E}'$ is also sequentially complete. Under these assumptions, K. Yosida showed [25] that the semigroup $\{T^{\odot}(t)\}$, defined by $T^{\odot}(t) = T'(t)_{|\mathcal{E}^{\odot}}$ for $t \geq 0$, where $\mathcal{E}^{\odot}$ represents the closure of $\mathcal{D}(X')$ in the strong topology of $\mathcal{E}'$, is an equicontinuous semigroup of class $\mathcal{C}_0$. Its infinitesimal generator $X^{\odot}$ is, moreover, the largest restriction of X' with domain and range in $\mathcal{E}^{\odot}$.

Let the symbol (H1) denote the following assumption on the space $\mathcal{E}$:

(H1) $\mathcal{E}$ is a locally convex Hausdorff sequentially complete linear topological space, such that $\mathcal{E}'$ is sequentially complete.

Frequent use will be made of the following result about linear functionals acting on locally convex spaces:

Proposition 2.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be locally convex spaces, with topologies defined, respectively, by the families of semi-norms $\{p_\alpha\}$, $\alpha \in A$, and $\{s_\beta\}$, $\beta \in B$. Let $L : \mathcal{E} \to \mathcal{F}$ be a linear functional.*

L is continuous if, and only if, for every $\beta \in B$ there exist $\alpha_1, \ldots, \alpha_n \in A$ and a constant C such that

$$s_\beta(Lx) \leq C \sum_{k=1}^{n} p_{\alpha_k}(x) \quad \textit{for every } x \in \mathcal{E}.$$

According to a result of J.P. Kahane, every mean periodic function, which is uniformly continuous and bounded, is almost periodic in the sense of Bohr; there exist, however, bounded mean periodic functions, which are not almost periodic [12].

It will now be shown that in the case of functions, generated by an equicontinuous group of linear bounded operators acting on a locally convex space, mean periodicity always entails almost periodicity.

Lemma 2.2. *Let $U : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ be an equicontinuous group of class $\mathcal{C}_0$ on a Hausdorff locally convex space $\mathcal{E}$. If the complex-valued map, defined on $\mathbb{R}$, $< U(\cdot)x_0, \lambda_0 >$ is mean periodic for some $x_0 \in \mathcal{E}$ and $\lambda_0 \in \mathcal{E}'$, then it is almost periodic.*

Proof. Let $x_0 \in \mathcal{E}$ and $\lambda_0 \in \mathcal{E}'$ be such that the map $\varphi(t) =< U(t)x_0, \lambda_0 >$ is mean periodic. First of all, the function φ is bounded; indeed, by Proposition 2.1, there exist $C > 0$ and $\alpha_1, \ldots, \alpha_n \in A$ such that

$$|\varphi(t)| \leq C \sum_{k=1}^{n} p_{\alpha_k}\big(U(t)x_0\big).$$

Equicontinuity of U implies that, for every semi-norm p_{α_k}, $k \in \{1, \ldots, n\}$, there exists a semi-norm q_{α_k}, $k \in \{1, \ldots, n\}$, for which

$$p_{\alpha_k}\big(U(t)x_0\big) \leq q_{\alpha_k}(x_0) \quad \text{for every } t \in \mathbb{R},$$

so that

$$\sup_{t\in\mathbb{R}} |\varphi(t)| \leq C \sum_{k=1}^{n} q_{\alpha_k}(x_0).$$

The application φ is, moreover, uniformly continuous, since the same argumentations as above yield:

$$|\varphi(t+h) - \varphi(t)| = | < U(t+h)x_0 - U(t)x_0 \,, \lambda_0 > | \leq C \sum_{k=1}^{n} p_{\alpha_k}\big(U(t+h)x_0 - U(t)x_0\big) =$$

$$= C \sum_{k=1}^{n} p_{\alpha_k}\big(U(t)\,(U(h)x_0 - x_0)\big) \leq C \sum_{k=1}^{n} q_{\alpha_k}\big(U(h)x_0 - x_0\big).$$

Since the last term converges to zero, uniformly with respect to t, when $h \to 0$, φ is uniformly continuous. Thus φ is almost periodic in the sense of Bohr. $\square$

Viceversa, starting from an equicontinuous group U on $\mathcal{E}$ of class $\mathcal{C}_0$, it is possible to build almost periodic functions, which are not mean periodic. Of course, these maps are necessarily uniformly continuous and bounded. Some preliminaries are necessary to build such maps.

In [19] it is shown that, if X is the infinitesimal generator of a uniformly bounded semigroup acting on a Banach space, then the following inclusion holds:

$$\mathrm{p}\sigma(X) \cap i\mathbb{R} \subseteq \mathrm{p}\sigma(X^{\odot}) \cap i\mathbb{R}. \tag{2.1}$$

The proof of J. van Neerven holds, without substantial modifications, for an equicontinuous semigroup of class $\mathcal{C}_0$ acting on a space satisfying (H1).

Lemma 2.3. *Let $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{E})$ be an equicontinuous semigroup of class $\mathcal{C}_0$ on a space $\mathcal{E}$ fulfilling assumption* (H1). *Let X be the infinitesimal generator of T. Then*:

$$\big(\mathrm{p}\sigma(X) \cup \mathrm{r}\sigma(X)\big) \cap i\mathbb{R} = \mathrm{p}\sigma(X^{\odot}) \cap i\mathbb{R}. \tag{2.2}$$

Proof. (2.2) follows from (2.1) and from inclusions

$$\mathrm{r}\sigma(X) \subseteq \mathrm{p}\sigma(X^{\odot}) \subseteq \mathrm{p}\sigma(X) \cup \mathrm{r}\sigma(X). \quad \square$$

Lemma 2.4. *Let $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{E})$ be an equicontinuous semigroup of class $\mathcal{C}_0$ on a space $\mathcal{E}$ satisfying* (H1). *If there are $x_0 \in \mathcal{E}$ and $\lambda_0 \in \mathcal{E}'$ such that the application $t \mapsto < T(t)x_0 \,, \lambda_0 >$ is non-constant, asymptotically almost periodic, then the set of all frequencies of this function is contained in $\frac{1}{i}\big[\mathrm{p}\sigma(X^{\odot}) \cap i\mathbb{R}\big]$.*

Proof. The same proof of Theorem 3 in [24], provided that the version of mean ergodic theorem for semigroups on a locally convex space presented the Appendix is used, shows that the set of all frequencies is contained in $\frac{1}{i}\left[\left(\mathrm{p}\sigma(X) \cup \mathrm{r}\sigma(X)\right)\right) \cap i\mathbb{R}\right]$. Equality (2.2) yields then the thesis. $\square$

Let U be an equicontinuous group of class $\mathcal{C}_0$ with generator X, acting on a space $\mathcal{E}$ fulfilling (H1). Suppose that $\mathrm{p}\sigma(X^{\odot})$ is discrete and the map $\varphi(t) = < U(t)x_0\,, \lambda_0 >$ is almost periodic for some $x_0 \in \mathcal{E}$ and $\lambda_0 \in \mathcal{E}'$. Let $\{\theta_1\,, \theta_2\,, \dots\}$ be an ordering of the set $\frac{1}{i}\mathrm{p}\sigma(X^{\odot})$. The Fourier series associated to φ is:

$$\sum_{n=1}^{+\infty} a_n\, e^{i\theta_n \cdot}\,, \tag{2.3}$$

where $a_n = a_n(x_0\,, \lambda_0) \in \mathbb{C}$ for every $n \in \mathbb{N}$ (2). If the set $\{\theta_1\,, \theta_2\,, \dots\}$ is linearly independent over the rationals numbers, then, as a consequence of a convergence theorem that H.Bohr proved at the end of his book [4], the Fourier series (2.3) converges uniformly on $\mathbb{R}$ to the map φ.

Suppose now $a_n \neq 0$ for a non finite set of $n \in \mathbb{N}$ and that θ_n converges to some θ_0, when $n \to \infty$. For every n, such that a_n is different from 0, $e^{i\theta_n t}$ belongs to $\tau(\varphi)$; thus, if there exists a non zero measure $\mu \in \mathcal{C}'$ such that $\varphi * \mu = 0$, from Lemma 1.1 it follows $\mathcal{M}(\theta_n) = 0$ for every $n \in \mathbb{N}$, yielding a contradiction, since $\mathcal{M}(\zeta)$ is an entire function.

Example. Fix $\theta_0 \in \mathbb{R} \setminus \{0\}$. Let α be a trascendent number in $(0\,, 1)$.

Let U be the unitary group in the Hilbert space l^2 generated by the self-adjoint operator defined on the standard basis $\{e_n : n \in \mathbb{Z}\}$ of l^2 by

$$Xe_n = \operatorname{sign}(n)\, i\, \theta_0 \left(1 + \alpha^{|n|}\right) e_n\,,$$

if $n \neq 0$, and by $Xe_0 = 0$.

The group is almost periodic, the point spectrum of X (or of $X^{\odot}$) is the set $\{i(\operatorname{sign} n)\theta_0(1 + \alpha^{|n|}) : n \in \mathbb{Z}\}$, which is linearly independent over $\mathbb{Q}$. Thus, for every $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}^{\odot}$ the Fourier series (2.3) converges uniformly on $\mathbb{R}$ to the almost periodic application $t \mapsto < U(t)x, \lambda >$; neverthless, no such map is mean periodic, since the set of its frequencies has a finite limite point.

3. Spectral properties of groups generating mean periodic maps

Let f belong to $L^\infty(\mathbb{R})$. Define

$$\mathcal{F}^+(\zeta) = -\int_0^{+\infty} f(t)\, e^{-i\zeta t} dt \text{ if } \Im m\zeta < 0 \text{ and}$$

$$\mathcal{F}^-(\zeta) = \int_{-\infty}^{0} f(t)\, e^{-i\zeta t} dt \text{ if } \Im m\zeta > 0.$$

(2) In particular, if the linear topological space $\mathcal{E}$ is of Baire and U is weakly almost periodic, by Proposition A3 in Appendix the Fourier coefficient a_n can be written as $< x, \mathcal{R}_{i\theta_n}\lambda >$ for every $n \in \mathbb{N}$, where $\mathcal{R}_{i\theta_n}$ represents a projection operator in the dual space $\mathcal{E}'$.

$\mathcal{F}^+$ and $\mathcal{F}^-$ are defined and holomorphic on their domains. The *Carleman transform* (3) of f is defined by

$$\mathcal{F}(\zeta) = \begin{cases} \mathcal{F}^+(\zeta) & \text{if } \Im m\zeta < 0 \\ \mathcal{F}^-(\zeta) & \text{if } \Im m\zeta > 0. \end{cases}$$

It is easy to prove that the meromorphic function defined by formula (1.3) coincides with the Carleman transform of f.

Consider now the set

$$\rho_C(f) = \{\theta \in \mathbb{R} : \text{ there exists } r_0 > 0 \text{ such that } \mathcal{F}(\zeta) \text{ can be analytically extended to } B(\theta\,,\, r_0)\}.$$

The *Carleman spectrum* of f is defined by

$$\sigma_C(f) = \mathbb{R} \setminus \rho_C(f).$$

Moreover, a tempered distribution T_f is associated to every $f \in L^\infty(\mathbb{R})$ by

$$< \psi\,,\, T_f >= \int_{\mathbb{R}} f(t)\psi(t)dt \text{ for every } \psi \in \mathcal{S},$$

$\mathcal{S}$ denoting the Schwartz space. The Fourier transform of T_f is defined by

$$< \psi\,, \mathcal{F}T_f >=< \mathcal{F}\psi\,,\, T_f > \text{ for every } \psi \in \mathcal{S}.$$

Finally, the support of T_f is defined by

$$\operatorname{supp}(T_f) = \{\theta \in \mathbb{R} : \text{ for every } \varepsilon > 0 \text{ there exists } \psi \in \mathcal{S} \text{ such that } \operatorname{supp}\psi \subseteq (\theta - \varepsilon\,, \theta + \varepsilon) \text{ and } < \psi\,,\, T_f > \neq 0\}.$$

Katznelson showed [14] that the support of $\mathcal{F}T_f$ coincides with the Carleman spectrum of f $\sigma_C(f)$. Moreover, if f is a bounded mean periodic function, L. Schwartz proved ([21, p. 907]) that the spectrum of f $S(f)$ coincides with the support of the Fourier transform associated, in the sense of distributions, to f, so that

$$S(f) = \operatorname{supp}(\mathcal{F}T_f) = \sigma_C(f). \tag{3.1}$$

Given, finally, a complex-valued function f in $L^\infty(\mathbb{R})$, let $\tau_{w^*}(f)$ denote the weak-star closure of the smallest translation-invariant subspace of $L^\infty(\mathbb{R})$, containing f. The weak-star spectrum of f is defined by

$$\sigma_{w^*}(f) = \{\theta \in \mathbb{R} : e^{i\theta t} \in \tau_{w^*}(f)\}.$$

(3) The original definition of Carleman [5] was slightly different; in particular, given a function $f \in L^\infty(\mathbb{R})$, he defined a transform $\hat{f}$ by

$$\hat{f}(\zeta) = \begin{cases} \int_0^{+\infty} f(t)\, e^{-\zeta t}\, dt \text{ if } \Re e\zeta > 0 \\ -\int_{-\infty}^0 f(t)\, e^{-\zeta t}\, dt \text{ if } \Re e\zeta < 0\,. \end{cases}$$

For every complex-valued function $f \in L^\infty(\mathbb{R})$ Katznelson proved [14] that

$$\sigma_{w^*}(f) = \operatorname{supp}(\mathcal{F}T_f)\,,$$

and therefore the following equality holds for a bounded mean periodic function f:

$$S(f) = \sigma_{w^*}(f). \tag{3.2}$$

Katznelson also proved that, if $f \in L^\infty(\mathbb{R})$ is such that $\operatorname{supp}(\mathcal{F}T_f)$ has countable boundary, then f admits weak-star sinthesis, *i.e.* f belongs to the closure, with respect to the weak-star topology, of

$$\bigvee\{\, e^{i\theta t} : \theta \in \sigma_{w^*}(f)\,\}.$$

If, moreover, $\operatorname{supp}(\mathcal{F}T_f)$ itself is countable and f is uniformly continuous, then f admits norm spectral synthesis, *i.e.* f is almost periodic.

For a mean periodic application f, assumptions on supp $(\mathcal{F}T_f)$ are obviously satisfied, since supp $(\mathcal{F}T_f)$ coincides with $S(f)$, which is countable. Thus

Proposition 3.1. *Let f be a bounded mean periodic map.*
Then f admits weak-star spectral synthesis.
If, moreover, f is uniformly continuous, then f admits norm spectral synthesis.

Observe that the second part of the statement above is contained, for the special case of maps generated by an equicontinuous group, in Lemma 2.2.

Proposition 3.2. *Let $f : \mathbb{R} \to \mathbb{C}$ be a bounded mean periodic function.*
Then $S(f)$ has no finite accumulation point, if $f \neq 0$.
If, moreover, f is uniformly continuous, then the set of the frequencies of f (i.e. *the Fourier spectrum $\sigma_{\mathcal{F}}(f)$) is contained in $S(f)$.*

Proof. Let θ be an accumulation point of $S(f)$, if $f \neq 0$; then there exists $\{\theta_n\} \subset \subset S(f)$ such that $\theta_n \neq \theta_m$ if $n \neq m$ and $\theta_n \to \theta$. Thus $e^{i\theta_n t} \in \tau(f)$ for every $n \in \mathbb{N}$. Let $\mu \in \mathcal{C}'$ be non identically vanishing and such that $f * \mu = 0$. If $\mathcal{M}$ is the Fourier transform of μ, then $\mathcal{M}(\theta_n) = 0$ for all $n \in \mathbb{N}$ by Lemma 1.1; this yields a contradiction, $\mathcal{M}(\zeta)$ being an entire function.

If f is uniformly continuous, f is almost periodic and it can be proved that $\theta \in \mathbb{R}$ is a frequency of f if, and only if, $e^{i\theta t} \in \tau_{L^\infty}(f)$, where $\tau_{L^\infty}(f)$ denotes the norm closure of the smallest translation-invariant subspace of $L^\infty(\mathbb{R})$ containing f. If $e^{i\theta t} \in \tau_{L^\infty}(f)$, then obviously $e^{i\theta t} \in \tau(f)$. $\square$

Equality (3.1) shows that $S(f)$ is closed and therefore Proposition 3.2 entails that the closure of the set of all frequencies of f is a subset of $S(f)$. More precisely, W. Arendt and C.J.K. Batty have recently pointed out [1] that, if f is an almost periodic map, then the Carleman spectrum σ_C coincides with the closure of Fourier spectrum of f ([1, Proposition 2.3]). Thus, if f is uniformly continuous, bounded and mean periodic, then equality 3.1 and Proposition 3.2 entail that $\sigma_{\mathcal{F}}(f) = S(f)$.

It is now possible to relate the spectrum of a mean periodic function f generated by a group of linear bounded operators U, with the spectrum of the infinitesimal generator of U.

THEOREM 3.3. *Let* $U : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ *be an equicontinuous group of class* $\mathcal{C}_0$ *on a space* $\mathcal{E}$ *fulfilling* (H1).

1) *If there are* $x_0 \in \mathcal{E}$ *and* $\lambda_0 \in \mathcal{E}'$ *such that the function* $t \mapsto < U(t)x_0 , \lambda_0 >$ *is non-constant, mean periodic, then every real number* θ *belonging to* $S(< U(\cdot)x_0 , \lambda_0 >)$ *is contained in* $\frac{1}{i}\mathrm{p}\sigma(X^{\odot})$.

2) *If* $i\theta \in \mathrm{p}\sigma(X^{\odot})$ *for some* $\theta \in \mathbb{R}$, *then there exist* $x_0 \in \mathcal{D}(X)$ *and* $\lambda_0 \in \mathcal{D}(X^{\odot})$ *such that the function* $t \mapsto < U(t)x_0 , \lambda_0 >$ *is mean periodic, and* $\theta \in S(< U(\cdot)x_0 , \lambda_0 >)$.

PROOF. 1) Let $\theta \in \mathbb{R}$ belong to $S(< U(\cdot)x_0 , \lambda_0 >)$; then by (3.1) θ belongs to $\sigma_C(< U(\cdot)x_0 , \lambda_0 >)$; in view of what has been pointed out above, θ belongs to the Fourier spectrum of the almost periodic application $< U(t)x_0 , \lambda_0 >$. Then, as a consequence of Lemma 2.4 θ belongs, up to a factor $\frac{1}{i}$, to $\mathrm{p}\sigma(X^{\odot})$.

2) Let now $i\theta \in \mathrm{p}\sigma(X^{\odot})$ for some $\theta \in \mathbb{R}$; then there exists $\lambda_0 \in \mathcal{D}(X^{\odot})$ such that $X^{\odot}\lambda_0 = i\theta\lambda_0$. By the spectral mapping theorem in locally convex spaces, it holds $U^{\odot}(t)\lambda_0 = e^{i\theta t}\lambda_0$ for every $t \in \mathbb{R}$. Thus

$$e^{i\theta t} < x, \lambda_0 >=< x, U^{\odot}(t)\lambda_0 >=< U(t)x, \lambda_0 >$$

for every $t \in \mathbb{R}$ and $x \in \mathcal{E}$. Since the set $\{y \in \mathcal{E} : < y, \lambda_0 > \neq 0\}$ is open and non-empty, and since $\mathcal{D}(X)$ is dense in $\mathcal{E}$, then there exists $x_0 \in \mathcal{D}(X)$ such that $< x_0 , \lambda_0 > \neq 0$. Define now the map from $\mathbb{R}$ to $\mathbb{C}$ $t \mapsto < U(t)x_0 , \lambda_0 >= e^{i\theta t} < x_0 , \lambda_0 >$, which is either periodic (if $\theta \neq 0$) with period $\frac{2\pi}{\theta}$, or constant, if $\theta = 0$.

If $\theta = 0$, then $< U(\cdot)x_0 , \lambda_0 >$ is mean periodic and $0 \in S(< U(\cdot)x_0 , \lambda_0 >)$. If $\theta \neq 0$, f is mean periodic, since $\tau(f)$ contains only periodic maps, and therefore it cannot coincide with the space $\mathcal{C}$. Moreover, since $< U(\cdot)x_0 , \lambda_0 >= e^{i\theta\cdot} < x_0 , \lambda_0 >$, $e^{i\theta\cdot}$ belongs to $\tau(< U(\cdot)x_0 , \lambda_0 >)$, *i.e.* θ belongs to the spectrum of f. □

Equality (2.2) and Corollary A5 entail now the following

COROLLARY 3.4. *If* U *acts on a reflexive space fulfilling* (H1), *then the spectrum of any mean periodic function* $< U(\cdot)x_0 , \lambda_0 >$ *is contained in* $\frac{1}{i}\mathrm{p}\sigma(X)$.

Suppose now that the family

$$\{< U(\cdot)x, \lambda >: x \in \mathcal{D}(X) , \lambda \in \mathcal{D}(X^{\odot}) , < x, \lambda > \neq 0\} \tag{3.3}$$

is mean periodic: this means that there exists some non zero measure $\mu \in \mathcal{C}'$ such that $\int < U(t-u)x, \lambda > d\mu(u) = 0$ for all $t \in \mathbb{R}$ and for every $x \in \mathcal{D}(X)$, $\lambda \in \mathcal{D}(X^{\odot})$, with $< x, \lambda > \neq 0$.

An example of a group satisfying this condition is given [24] by a strongly continuous, eventually differentiable and weakly almost periodic group U on a Banach space $\mathcal{E}$. In this case, it results

$$< U(t)x, \lambda >= \sum_{n=1}^{N} a_n e^{i\theta_n t}$$

for every $x \in \mathcal{E}$, $\lambda \in \mathcal{E}'$, $t \in \mathbb{R}$ and some $a_n = a_n(x, \lambda) \in \mathbb{C}$, $n = 1, \ldots, N$, so that with respect to any measure μ, whose Fourier transform vanishes on $\{\theta_n : n = 1, \ldots, N\}$ family (3.3) is mean periodic.

Let $i\theta$ be an eigenvalue of $X^{\odot}$ with eigenvector λ_0, and let $x_0 \in \mathcal{D}(X)$ be such that $< x_0, \lambda_0 > \neq 0$. Then it results $< U(t)x_0, \lambda_0 >= e^{i\theta t} < x_0, \lambda_0 >$ for every $t \in \mathbb{R}$, and, moreover,

$$\int < U(t-u)x_0, \lambda_0 > d\mu(u) = \int e^{i\theta(t-u)} < x_0, \lambda_0 > d\mu(u) =$$

$$= e^{i\theta t} < x_0, \lambda_0 > \int e^{-i\theta u} d\mu(u) = e^{i\theta t} < x_0, \lambda_0 > \mathcal{M}(\theta) = 0,$$

implying that $\mathcal{M}(\theta) = 0$, and therefore $e^{i\theta t} \in \tau(< U(\cdot)x_0, \lambda_0 >)$. Since μ is compactly supported, its Fourier transform $\mathcal{M}$ is an entire function of exponential type, bounded on the real line; thus the set of zeros of $\mathcal{M}$ admits no finite accumulation point. Since every eigenvalue of $X^{\odot}$ belongs to the set of zeros of $\mathcal{M}$, the following result can be stated:

Theorem 3.5. *Let $U : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ be an equicontinuous group of class C_0 on a space $\mathcal{E}$ fulfilling* (H1).

If the family (3.3) *is mean periodic, then* $\mathrm{p}\sigma(X^{\odot})$ (*and therefore also* $\mathrm{p}\sigma(X)$) *has no finite accumulation point. Moreover, if $\mathcal{M}$ has no real zeros, then* $\mathrm{p}\sigma(X^{\odot}) = \emptyset$.

From properties of zeros of entire functions of exponential type it follows also that, if $\{\theta_n\}$ represents an ordering of $\frac{1}{i}\mathrm{p}\sigma(X^{\odot})$, (with $\theta_n \in \mathbb{R}$ for every $n \in \mathbb{N}$), then $\sum \frac{1}{|\theta_n|^2} < \infty$.

Recall that the continuous spectrum $\mathrm{c}\sigma(X)$ of a linear operator X, acting on a Fréchet space $\mathcal{E}$, is the set of all complex numbers ζ for which $\zeta I - X$ is injective, has a dense range, but $(\zeta I - X)^{-1}$ is not continuous.

Lemma 3.6. *Let X be a linear closed operator on a Fréchet space $\mathcal{E}$.*

If a complex number ζ belongs to $\mathrm{c}\sigma(X)$, *then there exists a seminorm q_0 such that for every seminorm p there exists $\{y_\nu{}^{(p)}\} \subset \mathcal{D}(X)$, satisfying $q_0(y_\nu{}^{(p)}) = 1$ for every ν and such that $\{p\big((\zeta I - X)y_\nu{}^{(p)}\big)\}$ tends to* 0 *when $\nu \to \infty$.*

Proof. Suppose that for every seminorm q there exists a constant $C_q > 0$ and a seminorm p_q for which

$$p_q\big((\zeta I - X)x\big) \geq C_q\, q(x) \tag{3.4}$$

for all $x \in \mathcal{D}(X)$ satisfying $q(x) \neq 0$.

It will be shown that under this condition the range of $\zeta I - X$ is closed. Let $y_\nu = (\zeta I - X)x_\nu$ tend to some $y \in \mathcal{E}$. For every seminorm q (3.4) entails

$$q(x_\nu - x_\mu) \leq \frac{1}{C_q} p_q\big((\zeta I - X)(x_\nu - x_\mu)\big) = \frac{1}{C_q} p_q(y_\nu - y_\mu),$$

for some $C_q > 0$ and some seminorm p_q, showing that $\{x_\nu\}$ is a Cauchy sequence. By the completeness of $\mathcal{E}$, there exists $x \in \mathcal{E}$, such that $x_\nu \to x$. Since the operator $\zeta I - X$ is closed, $x \in \mathcal{D}(X)$ and $(\zeta I - X)x = y$, showing that the range of $\zeta I - X$ is closed.

Since $\zeta \in c\sigma(X)$, $\mathcal{R}(\zeta I - X)$ is dense, and therefore it results $\mathcal{R}(\zeta I - X) = \mathcal{E}$. Thus $(\zeta I - X)^{-1}$ is a closed operator defined on $\mathcal{E}$; as a consequence of the closed graph theorem, it is continuous, yielding a contradiction.

Thus there exists a seminorm q_0 such that for every $\nu > 0$ and for every seminorm p there exists an element $x_\nu{}^{(p)} \in \mathcal{D}(X)$, fulfilling $q_0(x_\nu{}^{(p)}) \neq 0$ and $p\big((\zeta I - X)x_\nu{}^{(p)}\big) < \frac{1}{\nu} \cdot q_0(x_\nu{}^{(p)})$.

By setting $y_\nu{}^{(p)} = \dfrac{x_\nu{}^{(p)}}{q_0(x_\nu{}^{(p)})}$, one obtains finally the thesis. □

A complex number ζ, such that for every seminorm p there exists a sequence $\{y_\nu{}^{(p)}\} \subset \mathcal{D}(X)$ satisfying $q_0(y_\nu{}^{(p)}) = 1$ for every ν, for some seminorm q_0 (not depending on p), and such that $\{p\big((\zeta I - X)y_\nu{}^{(p)}\big)\}$ tends to 0, will be called *approximate eigenvalue* of X, in analogy to terminology holding in normed spaces. The set of all approximate eigenvalues will be called *approximate point spectrum* of X ($ap\sigma(X)$). The sequence $\{y_\nu{}^{(p)}\}$ will be said to be an *approximate eigenvector* of X, with respect to p.

The set of all complex numbers ζ such that $\overline{\mathcal{R}(\zeta I - X)} \neq \mathcal{E}$ is called the *compression spectrum* of X and denoted by $k\sigma(X)$.

The role played by these parts of $\sigma(X)$ is illustrated by the following proposition, well known in the framework of Banach spaces.

PROPOSITION 3.7. *Let X be a closed densely defined operator on a Fréchet space $\mathcal{E}$. Then*

$$\sigma(X) = k\sigma(X) \cup ap\sigma(X)\,,$$

the union being not necessarily disjoint.

PROOF. Let ζ belong to $\sigma(X) \setminus k\sigma(X)$.

If $\zeta I - X$ is not injective, then there is some $0 \neq x_0 \in \mathcal{D}(X)$ for which $Xx_0 = \zeta x_0$. Since $\mathcal{E}$ is Hausdorff, there exists a seminorm q_0 such that $q_0(x_0) \neq 0$. By choosing $y_\nu{}^{(p)} = \dfrac{x_0}{q_0(x_0)}$ for every seminorm p and for all ν, one shows that $\zeta \in ap\sigma(X)$.

Let now $\zeta I - X$ be injective. If condition (3.4) holds, then $\mathcal{R}(\zeta I - X)$ is closed and, therefore, it coincides with $\mathcal{E}$, showing that $\zeta I - X$ is invertible; this is absurd, since $\zeta \in \sigma(X)$. Now the same proof as in Lemma 3.6 shows that ζ belongs to $ap\sigma(X)$. □

Moreover, when the operator X generates a uniformly bounded semigroup on a Banach space, the description of the unitary spectrum of X (*i.e.* of $\sigma(X) \cap i\mathbb{R}$) further simplifies, as the following proposition, which is given for the sake of completeness and will not be used in the following, shows.

PROPOSITION 3.8. *Let X be the infinitesimal generator of a uniformly bounded semigroup of class C_0 on a Banach space $\mathcal{E}$. Then*

$$\sigma(X) \cap i\mathbb{R} = ap\sigma(X) \cap i\mathbb{R}.$$

Proof. Let $i\theta \in k\sigma(X)$, $\theta \in \mathbb{R}$. By rescaling, one may assume that $\theta = 0$. If X is not injective, then $0 \in \mathrm{p}\sigma(X)$, and therefore $0 \in ap\sigma(X)$.

If X is injective, suppose that there exists some constant $C > 0$ such that

$$||X x|| \geq C||x||$$

for every $x \in \mathcal{D}(X)$. This assumption, exactly as condition (3.4) (with $\zeta = 0$) in Lemma 3.6, entails that the range of X $\mathcal{R}(X)$ is closed.

It will now be shown that, if $\mathcal{R}(X)$ is closed, then

$$\mathcal{E} = \ker X \oplus \mathcal{R}X. \tag{3.5}$$

First of all, it will be proved that

$$\ker X' \cap \mathcal{R}X' \subseteq \ker X^{\odot} \cap \mathcal{R}X^{\odot}. \tag{3.6}$$

If $\lambda \in \ker X'$, then $X'\lambda = 0$, and therefore $\lambda \in \mathcal{D}(X^{\odot})$ and $X^{\odot}\lambda = 0$.

Let now $\mu \in \mathcal{D}(X')$ be such that $\lambda = X'\mu$. Since $X'\mu \in \mathcal{D}(X')$, μ belongs to $\mathcal{D}(X^{\odot})$ and therefore $\lambda = X^{\odot}\mu$, whence (3.6) follows.

From the closed range theorem of S. Banach it follows now that

$$\left(\ker X \oplus \mathcal{R}X\right)^{\perp} \subseteq \ker X^{\perp} \cap \mathcal{R}X^{\perp} = \mathcal{R}(X') \cap \ker(X') \subseteq \mathcal{R}(X^{\odot}) \cap \ker(X^{\odot}) = \{0\}\,,$$

since $X^{\odot}$ is the infinitesimal generator of a uniformly bounded semigroup, whence (3.5) follows.

Then $\ker X \neq \{0\}$, yielding a contradiction, since X is, by hypothesis, injective.

Thus such a constant C does not exist, and therefore it is possible to build a sequence of vectors $\{x_\nu\} \subseteq \mathcal{D}(X)$ of norm one and such that $||X x_\nu|| \to 0$ when $\nu \to +\infty$, showing that $0 \in ap\sigma(X)$. □

Corollary 3.9. *If X is the infinitesimal generator of a uniformly bounded group of class C_0 on a Banach space $\mathcal{E}$, then*

$$\sigma(X) = ap\sigma(X).$$

A spectral mapping theorem, relating the approximate point spectrum of a semigroup with that of its infinitesimal generator, will now be proved.

Lemma 3.10. *Let T be a semigroup of class C_0 on a Fréchet space $\mathcal{E}$, with infinitesimal generator X.*

Let T be equicontinuous, i.e. *for every seminorm p there is a seminorm q for which*

$$p(T(t)x) \leq q(x) \text{ for every } x \in \mathcal{E},\ t \geq 0. \tag{3.7}$$

If $\{y_\nu{}^{(p)}\}$ is an approximate eigenvector for X with respect to p with approximate eigenvalue ζ, then $\{y_\nu{}^{(q)}\}$ is an approximate eigenvector for $T(t)$, with respect to the same seminorm p, with approximate eigenvalue $e^{\zeta t}$, for every $t \geq 0$.

Proof. The proof is inspired to that of Proposition 2.1.6 in [19].

Let q be a seminorm for which (3.7) holds. By hypothesis, there exists a sequence $\{y_\nu{}^{(q)}\} \subset \mathcal{D}(X)$ satisfying $q_0(y_\nu{}^{(q)}) = 1$ for every ν, for some seminorm q_0, and such that $q\big((\zeta I - X) y_\nu{}^{(q)} \big)$ tends to 0. In [20] the following identity is proved for a strongly continuous semigroup on a Banach space $\mathcal{E}$:

$$\big(e^{\zeta t} I - T(t) \big)\, x = e^{\zeta t} \int_0^t e^{-\zeta s}\, T(s)(\zeta I - X) x\, ds \quad \text{for all } x \in D(x) \text{ and } t \geq 0.$$

It is easy to check that the same identity holds also for an equicontinuous semigroup acting on a locally convex space, provided that the right integral is considered as a Riemann integral. Indeed, the procedure of defining the Riemann integral for scalar-valued functions can be extended to maps with values in a locally convex sequentially complete space $\mathcal{E}$, by substituting absolute values of numbers with continuous seminorms on $\mathcal{E}$. It follows therefore

$$\begin{aligned} p\big((e^{\zeta t} I - T(t)) y_\nu{}^{(q)} \big) &= p\big(e^{\zeta t} \int_0^t e^{-\zeta s}\, T(s)(\zeta I - X) y_\nu{}^{(q)}\, ds \big) \leq \\ &\leq e^{2|\Re e \zeta| t} \int_0^t p\big(T(s)(\zeta I - X) y_\nu{}^{(q)} \big)\, ds \leq \\ &\leq e^{2|\Re e \zeta| t} \int_0^t q\big((\zeta I - X) y_\nu{}^{(q)} \big)\, ds = \\ &= e^{2|\Re e \zeta| t} \cdot t \cdot q\big((\zeta I - X) y_\nu{}^{(q)} \big) \to 0\,, \quad \nu \to \infty\,, \end{aligned}$$

the convergence being uniform on every interval $[0\,, t_0]$, $t_0 > 0$. □

A group $U : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ is said to be *uniformly weakly mean periodic* if there exists a non-zero, compactly supported measure μ, for which

$$\int < U(t-u)x, \lambda > d\mu(u) = 0$$

for every $t \in \mathbb{R}$ and for all $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$.

THEOREM 3.11. *Let $U : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ be an equicontinuous group of class $\mathcal{C}_0$ of linear bounded operators acting on a Fréchet space $\mathcal{E}$, with infinitesimal generator X.*
If U is uniformly weakly mean periodic, then $c\sigma(X) = \emptyset$.

PROOF. Let $i\theta$ belong to $c\sigma(X)$, for some $\theta \in \mathbb{R}$. As a consequence of Lemma 3.6, there exists a seminorm q_0, such that for every seminorm p there is a sequence $\{y_\nu{}^{(p)}\} \subset \subset \mathcal{D}(X)$, satisfying $q_0(y_\nu{}^{(p)}) = 1$ for every ν and such that $\{p\big((i\theta I - X) y_\nu{}^{(p)}\big)\}$ tends to 0.

Since U is equicontinuous, there is a seminorm q_1 for which

$$q_0(U(t)x) \leq q_1(x) \text{ for every } x \in \mathcal{E},\ t \in \mathbb{R}.$$

There exists also a sequence $\{y_\nu{}^{(q_1)}\} \subset \mathcal{D}(X)$ satisfying $q_0(y_\nu{}^{(q_1)}) = 1$ for every ν, and such that $q_1\big((\zeta I - Y) y_\nu{}^{(q_1)} \big)$ tends to 0.

For every ν consider the subspace of $\mathcal{E}$ $\mathcal{M}_\nu{}^{(q_1)} = \mathbb{C} \cdot y_\nu{}^{(q_1)}$ and define the linear functional $f_\nu{}^{(q_1)}$ on $\mathcal{M}_\nu{}^{(q_1)}$ defined by $< \alpha y_\nu{}^{(q_1)}, f_\nu{}^{(q_1)} >= \alpha$, for all $\alpha \in \mathbb{C}$. Since it holds

$$| < \alpha y_\nu{}^{(q_1)}, f_\nu{}^{(q_1)} > | = |\alpha| = |\alpha| \cdot q_0(y_\nu{}^{(q_1)}) = q_0(\alpha y_\nu{}^{(q_1)})\,,$$

as a consequence of the Hahn-Banach theorem for every ν there exists a functional ${\lambda_\nu}^{(q_1)} \in \mathcal{E}'$ for which

$$< \alpha {y_\nu}^{(q_1)}, {\lambda_\nu}^{(q_1)} > = \alpha \text{ for every } \alpha \in \mathbb{C} \text{ and for every } \nu$$
$$\text{and } | < x, {\lambda_\nu}^{(q_1)} > | \leq q_0(x) \text{ for all } x \in \mathcal{E}.$$

Since U is weakly uniformly mean periodic, for every ν and for every $t \in \mathbb{R}$ it results

$$\int < U(t-u){y_\nu}^{(q_1)}, {\lambda_\nu}^{(q_1)} > d\mu(u) = 0.$$

Choose now t such that $t - u \in [0, t_0]$, for some $t_0 > 0$. Then it results

$$\begin{aligned}\left|\int e^{i\theta(t-u)} d\mu(u)\right| &= \left|\int e^{i\theta(t-u)} < {y_\nu}^{(q_1)}, {\lambda_\nu}^{(q_1)} > d\mu(u) - \right.\\ &\quad \left. - \int < U(t-u){y_\nu}^{(q_1)}, {\lambda_\nu}^{(q_1)} > d\mu(u)\right| \leq \\ &\leq \int < e^{i\theta(t-u)}{y_\nu}^{(q_1)} - U(t-u){y_\nu}^{(q_1)}, {\lambda_\nu}^{(q_1)} > \big| d\mu(u) \leq \\ &\leq \int q_0\big(e^{i\theta(t-u)}{y_\nu}^{(q_1)} - U(t-u){y_\nu}^{(q_1)}\big)\, d\mu(u),\end{aligned}$$

and the last term tends to zero by Lemma 3.10, when ν tends to ∞. Thus $\int e^{i\theta(t-u)} d\mu(u) = 0$ for every $t \in \mathbb{R}$, implying that $\mathcal{M}(\theta) = 0$. By Lemma 1.1 θ belongs to $S(< U(\cdot)x, \lambda >)$, for any $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$, so that, by Theorem 3.3, $i\theta$ belongs to $\mathrm{p}\sigma(X^{\odot}) = \mathrm{p}\sigma(X) \cup \mathrm{r}\sigma(X)$, yielding a contradiction. □

If X is the infinitesimal generator of a strongly continuous semigroup acting on a Banach space, it has been proved [23] that $k\sigma(X) = \mathrm{p}\sigma(X^{\odot})$. The same proof holds when X generates an equicontinuous semigroup on a locally convex space fulfilling (H1). During the proof of Theorem 3.11, it has been proved that, if $i\theta$ belong to $ap\sigma(X)$, then $\mathcal{M}(\theta) = 0$, so that $i\theta \in \mathrm{p}\sigma(X^{\odot}) = k\sigma(X)$, *i.e.*, when U is a uniformly weakly mean periodic group, the spectrum of X reduces, at the light of Proposition 3.7, to $\mathrm{p}\sigma(X^{\odot})$. Thus Theorem 3.3 provides in this case a complete description of $\sigma(X)$ by means of the Schwartz spectrum of any mean periodic function associated to U.

R. Nagel proved [2, p. 91] that, if U is a uniformly bounded group in a Banach space $\mathcal{E}$ different from $\{0\}$, then the infinitesimal generator of U has a non-empty spectrum. The same proof works also when U is an equicontinuous group of linear bounded operators in a Fréchet space. Thus the following result can be stated:

Corollary 3.12 . *Under the assumptions of Theorem* 3.11, *if* μ *is a compactly supported measure such that*

$$\int < U(t-u)x, \lambda > d\mu(u) = 0$$

for all $t \in \mathbb{R}$ *and for every* $x \in \mathcal{E}$, $\lambda \in \mathcal{E}'$, *then* $\mathcal{M}$ *admits at least a real zero.*

PROOF. If $\mathcal{M}$ never vanishes on $\mathbb{R}$, then by Theorem 3.5 $\mathrm{p}\sigma(X) \cup \mathrm{r}\sigma(X) = \mathrm{p}\sigma(X^{\odot}) = \emptyset$. Proposition 3.6 entails that $\mathrm{c}\sigma(X)$, and therefore $\sigma(X)$, is empty, yielding a contradiction. □

4. Spectral properties of cosine functions generating mean periodic maps

Let $\mathcal{E}$ be a Hausdorff locally convex space. An equicontinuous cosine function C of class $\mathcal{C}_0$ of linear bounded operators on $\mathcal{E}$ is a one-parameter family $\{C(t)\}$, $t \in \mathbb{R}$, of bounded linear operators in $\mathcal{L}(\mathcal{E})$ such that:

(1) $C(t+s) + C(t-s) = 2C(t)C(s)$ for every $t, s \in \mathbb{R}$ and $C(0) = I$;

(2) $\lim_{t \to t_0} C(t)x = C(t_0)x$ for every $t_0 \in \mathbb{R}$ and $x \in \mathcal{E}$;

(3) for every continuous semi-norm p on $\mathcal{E}$, there exists a continuous semi-norm q on $\mathcal{E}$ such that $p\big(C(t)x\big) \leq q(x)$ for every $t \in \mathbb{R}$ and $x \in \mathcal{E}$.

The infinitesimal generator of C will be denoted by the symbol Y.
For the main properties of cosine functions on locally convex spaces the reader is referred to [9, 10].

Recall, in particular, that, if C is equicontinuous, the spectrum of Y is contained in the real negative semi-axis; moreover, in [6] the equality

$$\mathrm{p}\sigma(Y) \cup \mathrm{r}\sigma(Y) = \mathrm{p}\sigma(Y^{\odot})$$

has been proved for a uniformly bounded cosine function in a Banach space $\mathcal{E}$. The proof of the same equality in locally convex spaces satisfying (H1) is straightforward.

Exactly as in Lemma 2.2, one can prove that every mean periodic application $\varphi(\cdot) = < C(\cdot)x_0 , \lambda_0 >$, with $x_0 \in \mathcal{E}$ and $\lambda_0 \in \mathcal{E}'$, from $\mathbb{R}$ to $\mathbb{C}$, is almost periodic. In analogy to the case of groups, there are almost periodic functions of the form $< C(\cdot)x_0 , \lambda_0 >$, which are not mean periodic.
If Y represents the infinitesimal generator of an equicontinuous cosine function, the following notations will be used:

$$\pm\sqrt{-\mathrm{p}\sigma(Y)} = \{\zeta \in \mathbb{R} : -\zeta^2 \in \mathrm{p}\sigma(Y)\} \quad \text{and} \quad \pm\sqrt{\big(-\mathrm{p}\sigma(Y^{\odot})\big)} = \{\zeta \in \mathbb{R} : -\zeta^2 \in \mathrm{p}\sigma(Y^{\odot})\}.$$

Theorem 6.1 in [7], adapted to the case of equicontinuous cosine functions acting on a locally convex space, leads to the following result, analogous to Theorem 3.3:

THEOREM 4.1.
Let C be an equicontinuous cosine function of class $\mathcal{C}_0$ on a space $\mathcal{E}$ fulfilling (H1).

1) *If there are $x_0 \in \mathcal{E}$ and $\lambda_0 \in \mathcal{E}'$ such that the function $t \mapsto < C(t)x_0 , \lambda_0 >$ is non-constant, mean periodic, then every real number θ belonging to $S(< C(\cdot)x_0 , \lambda_0 >)$ is contained in $\pm\sqrt{\big(-\mathrm{p}\sigma(Y^{\odot})\big)}$.*

2) *Conversely, for every $-\theta^2 \in \mathrm{p}\sigma(Y^{\odot})$, with $\theta \in \mathbb{R}$, there are $x_0 \in \mathcal{D}(Y)$ and $\lambda_0 \in \mathcal{D}(Y^{\odot})$ such that the function $t \mapsto < C(t)x_0 , \lambda_0 >$ is mean periodic, and θ and $-\theta$ belong to $S(< C(\cdot)x_0 , \lambda_0 >)$.*

Proof. The proof is omitted, since it is very similar to that of Theorem 3.3. □

When $\mathcal{E}$ is reflexive, in the theorem above $\mathrm{p}\sigma(Y^{\odot})$ can be substituted by $\mathrm{p}\sigma(Y)$.

The family of functions from $\mathbb{R}$ to $\mathbb{C}$

$$\{< C(\cdot)x, \lambda >: x \in \mathcal{D}(Y), \lambda \in \mathcal{D}(Y^{\odot}), < x, \lambda > \neq 0\} \tag{4.1}$$

is said to be *mean periodic* if there exists some non zero measure $\mu \in \mathcal{C}'$ such that $\int < C(t-u)x, \lambda > d\mu(u) = 0$ for all $t \in \mathbb{R}$ and for every $x \in \mathcal{D}(Y)$, $\lambda \in \mathcal{D}(Y^{\odot})$, with $< x, \lambda > \neq 0$.

Proposition 4.2. *Let $C : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ be an equicontinuous cosine function of class $\mathcal{C}_0$ on a space $\mathcal{E}$ fulfilling* (H1).

If the family (4.1) *is mean periodic, then* $\mathrm{p}\sigma(Y^{\odot})$ *(and therefore also* $\mathrm{p}\sigma(Y)$*) has no finite accumulation point.*

Moreover, if the set of zeros of $\mathcal{M}$ contains no pair $(\theta, -\theta)$, θ being a real number different from 0, *then* $\mathrm{p}\sigma(Y^{\odot}) \subseteq \{0\}$.

Proof. Let $-\theta^2$ be an eigenvalue of $Y^{\odot}$ with eigenvector λ_0, and let $x_0 \in \mathcal{D}(X)$ be such that $< x_0, \lambda_0 > \neq 0$. Then by the spectral mapping theorem for cosine functions it results

$$\begin{aligned}\int < C(t-u)x_0, \lambda_0 > d\mu(u) &= \int \cos(\theta(t-u)) < x_0, \lambda_0 > d\mu(u) = \\ &= \frac{< x_0, \lambda_0 >}{2} \int \left(e^{i\theta(t-u)} + e^{-i\theta(t-u)}\right) d\mu(u) = \\ &= \frac{< x_0, \lambda_0 >}{2} \left[e^{i\theta t}\mathcal{M}(\theta) + e^{-i\theta t}\mathcal{M}(-\theta)\right] = 0\end{aligned}$$

for all $t \in \mathbb{R}$, implying that both $\mathcal{M}(\theta) = 0$ and $\mathcal{M}(-\theta) = 0$. Thus the set $\pm\sqrt{(-\mathrm{p}\sigma(Y^{\odot}))}$ is discrete, and therefore also $\mathrm{p}\sigma(Y^{\odot})$ consists only of isolated points. Finally, since the pair $(\theta, -\theta)$ belongs to the set of zeros of the entire function $\mathcal{M}(\theta)$, the thesis follows. □

Also in this case, if $\{\theta_n\}$ represents an ordering of $+\sqrt{(-\mathrm{p}\sigma(Y^{\odot}))}$ then $\sum \frac{1}{|\theta_n|^2} < < \infty$, *i.e.* the series of eigenvalues of Y, up to a change of sign, converges.

The behaviour of continuous spectrum will now be analyzed. The following spectral mapping lemma is analogous to Lemma 3.10, holding for semigroups.

Lemma 4.3. *Let C be a cosine function of class $\mathcal{C}_0$ on a Fréchet space $\mathcal{E}$, generated by Y.*
Let C be equicontinuous, i.e. *for every seminorm p there is a seminorm q for which*

$$p(C(t)x) \leq q(x) \quad \textit{for every } x \in \mathcal{E},\ t \in \mathbb{R}. \tag{4.2}$$

If $\{y_\nu{}^{(p)}\}$ is an approximate eigenvector for Y with respect to p with approximate eigenvalue ζ, then $\{y_\nu{}^{(q)}\}$ is an approximate eigenvector for $C(t)$, with respect to the same seminorm p, with approximate eigenvalue $\cosh\sqrt{\zeta}\,t$, *for every $t \in \mathbb{R}$.*

Proof. Let q be a seminorm for which (4.2) holds. By hypothesis, there exists a sequence $\{y_\nu{}^{(q)}\} \subset \mathcal{D}(X)$ satisfying $q_0(y_\nu{}^{(q)}) = 1$ for every ν, for a fixed seminorm q_0, and such that $q\big((\zeta I - Y)y_\nu{}^{(q)}\big)$ tends to 0.

For a strongly continuous cosine function acting on a Banach space, B. Nagy proved [17] the following formula for $0 \neq \zeta \in \mathbb{C}$, $x \in \mathcal{D}(Y)$ and $t \in \mathbb{R}$:

$$\frac{1}{\sqrt{\zeta}} \int_0^t \sinh \sqrt{\zeta}(t-s)\, C(s)(\zeta I - Y)x\, ds = \big(\cosh t\sqrt{\zeta} I - C(t) \big)x. \tag{4.3}$$

In the same framework, for $\zeta = 0$, $x \in \mathcal{D}(Y)$ and $t \in \mathbb{R}$ Sova proved [22] that

$$\int_0^t (t-s)\, C(s)\, Yx\, ds = \big(C(t) - I \big)x. \tag{4.4}$$

Exactly as for semigroups, it is easy to verify that these formulae hold also for an equicontinuous cosine function on a locally convex space. Thus, if $\zeta \neq 0$, one gets by (4.3):

$$p\big(C(t)y_\nu{}^{(q)} - \cosh(t\sqrt{\zeta})y_\nu{}^{(q)}\big) = p\left(\frac{1}{\sqrt{\zeta}} \int_0^t \sinh \sqrt{\zeta}(t-s)\, C(s)(\zeta I - Y)y_\nu{}^{(q)}\, ds \right) \leq$$
$$\leq \frac{|t|}{|\sqrt{\zeta}|}\, e^{2|\Re\sqrt{\zeta}|\cdot|t|}\, q\big((\zeta I - Y)y_\nu{}^{(q)}\big) \to 0.$$

If $\zeta = 0$, (4.4) entails :

$$p\big((C(t) - I)y_\nu{}^{(q)}\big) \leq t^2\, q(Yy_\nu{}^{(q)});$$

also in this case, the right expression tends to 0 for $\nu \to \infty$. In both cases, the convergence is uniform on the compact intervals of $\mathbb{R}$. □

In analogy to the case of groups, a cosine function $C : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ is said to be *uniformly weakly mean periodic* if there exists a non-zero, compactly supported measure μ, for which

$$\int < C(t-u)x, \lambda > d\mu(u) = 0$$

for every $t \in \mathbb{R}$ and for all $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$.

Theorem 4.4. *Let Y be the infinitesimal generator of an equicontinuous cosine function C of class $\mathcal{C}_0$ of linear bounded operators acting on a Fréchet space $\mathcal{E}$.*
If C is uniformly weakly mean periodic, then $c\sigma(Y) = \emptyset$.

Proof. Let $-\theta^2$ belong to $c\sigma(Y)$, for some $\theta \in \mathbb{R}$. As a consequence of Lemma 3.6, there exists a seminorm q_0, such that for every seminorm p there is a sequence $\{y_\nu{}^{(p)}\} \subset \mathcal{D}(Y)$, satisfying $q_0(y_\nu{}^{(p)}) = 1$ for every ν and such that $\{p\big((-\theta^2 I - Y)y_\nu{}^{(p)}\big)\}$ tends to 0.

Since C is equicontinuous, there is a seminorm q_1 for which

$$q_0(C(t)x) \leq q_1(x) \text{ for every } x \in \mathcal{E},\ t \in \mathbb{R}.$$

Exactly as in Theorem 3.11, for every ν consider the subspace of $\mathcal{E}$ $\mathcal{M}_\nu{}^{(q_1)} = \mathbb{C}\cdot y_\nu{}^{(q_1)}$ and consider the functional $f_\nu^{(q_1)}$ on $\mathcal{M}_\nu{}^{(q_1)}$ defined by $< \alpha y_\nu{}^{(q_1)}, f_\nu^{(q_1)} >= \alpha$.

For every ν extend $f_\nu^{(q_1)}$ by means of Hahn-Banach theorem to a functional $\lambda_\nu{}^{(q_1)} \in \in \mathcal{E}'$, for which

$$| < x, \lambda_\nu{}^{(q_1)} > | \leq q_0(x) \text{ for all } x \in \mathcal{E}.$$

From the weak uniform mean periodicity of C, it follows that

$$\int < C(t-u)y_\nu{}^{(q_1)}, \lambda_\nu{}^{(q_1)} > d\mu(u) = 0$$

for every ν and for every $t \in \mathbb{R}$. By choosing t such that $t - u \in [0, t_0]$, for some $t_0 > 0$, one gets

$$\begin{aligned}
\left|\int \cos\theta(t-u)d\mu(u)\right| &= \left|\int \cos\theta(t-u) < y_\nu{}^{(q_1)}, \lambda_\nu{}^{(q_1)} > d\mu(u) - \right.\\
&\quad \left. - \int < C(t-u)y_\nu{}^{(q_1)}, \lambda_\nu{}^{(q_1)} > d\mu(u)\right| \leq \\
&\leq \int \left| < \cos\theta(t-u)y_\nu{}^{(q_1)} - C(t-u)y_\nu{}^{(q_1)}, \lambda_\nu{}^{(q_1)} > \right| d\mu(u) \leq \\
&\leq \int q_0\big(\cos\theta(t-u)y_\nu{}^{(q_1)} - C(t-u)y_\nu{}^{(q_1)}\big)\, d\mu(u).
\end{aligned}$$

Since the last term tends to zero by Lemma 4.3, when ν tends to $+\infty$, it results $\int \cos\theta(t-u)d\mu(u) = 0$ for every $t \in \mathbb{R}$, implying that $\mathcal{M}(\theta) = 0$ and $\mathcal{M}(-\theta) = 0$. By Lemma 1.1 θ and $-\theta$ belong to $S(< C(\cdot)x, \lambda >)$, for any $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$, so that, by Theorem 4.1, $-\theta^2$ belongs to $\mathrm{p}\sigma(Y^{\odot}) = \mathrm{p}\sigma(Y) \cup \mathrm{r}\sigma(Y)$, yielding a contradiction. □

In Lemma 1.5.1 in [6] it has been proved that the spectrum of the generator of a uniformly bounded cosine function in a Banach space $\mathcal{E} \neq \{0\}$ is never empty. In analogy to the case of groups, this result can be extended, with only slight modifications, to the framework of equicontinuous cosine functions on Fréchet spaces. For the sake of completeness, a sketch of the proof is reported.

First of all, for $f \in L^1(\mathbb{R})$ define the operator

$$\hat{f}(C)x = \int_{\mathbb{R}} f(t)C(t)x\, dt, \quad x \in \mathcal{E},$$

which is well-defined and continuous, since for every seminorm p there exists a seminorm q for which

$$p\big(\hat{f}(C)x\big) \leq ||f||_{L^1(\mathbb{R})} \cdot q(x)$$

for all $x \in \mathcal{E}$. Exactly as in [6], this operator can be represented as

$$\hat{f}(C)x = \frac{1}{2\pi}\lim_{\delta\to 0}\int_{\mathbb{R}} \hat{f}(s)[(\delta - is)R\big((\delta - is)^2, X\big) + (\delta + is)R\big((\delta + is)^2, X\big)]x\, ds. \tag{4.5}$$

Suppose now $\sigma(Y)$ is empty. It follows from (4.5) that $\hat{f}(C) = 0$ for every $f \in L^1(\mathbb{R})$, whose Fourier transform has compact support, and therefore, by density, that $\hat{f}(C) = 0$

for all $f \in L^1(\mathbb{R})$. The proof paraphrases now that of [19, p. 48]. Let $f_0 \in L^1(\mathbb{R})$ be defined by $f_0(t) = e^{-t}$ for $t \geq 0$, $f_0(t) = 0$ for $t < 0$. It holds $1 \cdot R(1, X) = \hat{f}_0(C) = 0$, and therefore $\mathcal{E} = \overline{R(1, X)\mathcal{E}} = \{0\}$, whence the thesis follows.

In view of what has been proved, the following result can be stated:

Corollary 4.5. *Under the assumptions of Theorem* 4.4, *if* μ *is a compactly supported measure* μ *such that*

$$\int < C(t-u)x, \lambda > d\mu(u) = 0$$

for all $t \in \mathbb{R}$ *and for every* $x \in \mathcal{E}$, $\lambda \in \mathcal{E}'$, *then the set of zeros of* $\mathcal{M}$, *if it does not contain* 0, *contains at least one pair* $(\theta, -\theta)$, *for some real* θ.

Proof. If the set of zeros of $\mathcal{M}$ contains neither 0 nor any pair $(\theta, -\theta)$, θ belonging to $\mathbb{R}\setminus\{0\}$, then Proposition 4.2 entails that $\mathrm{p}\sigma(Y^{\odot}) = \emptyset$, and therefore $\mathrm{p}\sigma(Y) \cup \mathrm{r}\sigma(Y) = = 0$. Now Theorem 4.4, stating that $\mathrm{c}\sigma(Y)$ is empty, yields a contradiction. □

5. Asymptotically mean periodic functions

As recalled in § 1, J.P. Kahane proved that every mean periodic function belonging to $BUC(\mathbb{R})$ is almost periodic in the sense of Bohr. It will now be shown that an analogous result does not hold for asymptotically mean periodic maps in $BUC(\mathbb{R}_+)$. Consider the function:

$$f(t) = \begin{cases} e^{-1} & \text{if } t \in [0, 1) \\ e^{-t} & \text{if } t \geq 1. \end{cases} \tag{5.1}$$

f defined by (5.1) is bounded, uniformly continuous on $\mathbb{R}_+$ and it is asymptotically mean periodic, since $\tau_+(f) \neq \mathcal{C}_+$. Suppose f is asymptotically almost periodic (in the sense of Fréchet). Then there exist an almost periodic map g and a function $h \in \mathcal{C}_0(\mathbb{R}_+)$, such that

$$f(t) = g_{|\mathbb{R}_+}(t) + h(t), \quad t \geq 0.$$

Since $g(t) = e^{-t} - h(t)$ for every $t \geq 1$, $g(t) \to 0$ for $t \to +\infty$, and therefore g vanishes identically. Thus $f(t) = h(t)$ for every $t \geq 0$, yielding a contradiction.

As showed by P. Koosis [16], f is such that $A_f = 1$ (where A_f is defined as the infimum of all numbers $u \geq 0$, such that $f(\cdot + u)$ is uniformly approximable on the compact subsets of $\mathbb{R}_+$ by exponential-polynomials belonging to $\tau_+(f)$). It is not known if there exist asymptotically mean periodic functions f, belonging to $BUC(\mathbb{R}_+)$ and such that $A_f = 0$, which are not asymptotically almost periodic.

As recalled in § 1, a function $f : \mathbb{R}_+ \to \mathbb{C}$ is asymptotically mean periodic if, and only if, the function $g(t) = (f * \mu)(t)$, as well as μ, has compact support in $(-\infty, 0]$. Since the Fourier transforms $\mathcal{M}$ and $\mathcal{G}$ are both entire functions, the Fourier transform of $\mathcal{F}$, defined formally as $\mathcal{F}(\zeta) = \frac{\mathcal{G}(\zeta)}{\mathcal{M}(\zeta)}$, is meromorphic. By adopting the same procedure followed by L. Schwartz and J.P. Kahane in order to define the spectrum of a mean periodic map, it is now possible to define the Schwartz spectrum of the asymptotically mean periodic function f (denoted with the symbol $S(f)$) as the set of all poles of $\mathcal{F}(\zeta)$, each counted with its order of multeplicity.

It is also possible to prove that a complex number ζ is a pole of order at least $n+1$ for $\mathcal{F}$ if, and only if, $t^n\, e^{i\zeta t}$ belongs to $\tau_+(f)$.

However, almost all notions of spectrum introduced in § 3 for a bounded, uniformly continuous function defined on $\mathbb{R}$ make, unfortunately, no sense for a map f defined only on a half-line, and, therefore, the basic Theorem 3.3 can only partially be recovered, when the equicontinuous group U is substituted by a semigroup T.

Proposition 5.1. *Let $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{E})$ be an equicontinuous semigroup of class $\mathcal{C}_0$ on a space $\mathcal{E}$ fulfilling* (H1). *Let X be the infinitesimal generator of T.*
If $\zeta \in \mathrm{p}\sigma(X^{\odot})$ for some $\zeta \in \mathbb{C}$, then there exist $x_0 \in \mathcal{D}(X)$ and $\lambda_0 \in \mathcal{D}(X^{\odot})$ such that the application, from $\mathbb{R}_+$ to $\mathbb{C}$, $t \mapsto < T(t)x_0\,, \lambda_0 >$ is asymptotically mean periodic, and $-i\zeta \in S(< U(\cdot)x_0\,, \lambda_0 >)$.

Proof. Let $\lambda_0 \in \mathcal{D}(X^{\odot})$ be such that $X^{\odot}\lambda_0 = \zeta\lambda_0$. Take $x_0 \in \mathcal{D}(X)$ such that $< x_0\,, \lambda_0 > \neq 0$. Then the application

$$f:\ t \mapsto < T(t)x_0\,, \lambda_0 > = e^{\zeta t} < x_0\,, \lambda_0 >$$

is asymptotically mean periodic, since obviously $\tau_+(f) \neq \mathcal{C}_+$. Moreover, $e^{\zeta t}$ belongs to $\tau_+\big(<T(\cdot)x_0\,,\lambda_0>\big)$, whence $-i\,\zeta$ belongs to the Schwartz spectrum of $<T(\cdot)x_0\,,\lambda_0>$. □

During the proof of Proposition 5.1 it has been observed that, if ζ is an eigenvalue of $X^{\odot}$, then the map

$$t \mapsto < T(t)x_0\,, \lambda_0 > = e^{\zeta t} < x_0\,, \lambda_0 > ,$$

$\lambda_0 \in \mathcal{D}(X^{\odot})$ and $x_0 \in \mathcal{D}(X)$ being, respectively, such that $X^{\odot}\lambda_0 = \zeta\lambda_0$ and $< x_0\,, \lambda_0 > \neq$ $\neq 0$, is asymptotically mean periodic, and therefore there exists a measure μ, compactly supported in $(-\infty\,, 0]$, for which

$$\int_{-\infty}^{0} < T(t-u)x_0\,, \lambda_0 > d\mu(u) = 0 \quad \text{for every} \quad t \geq 0, \ \textit{i.e.}$$

$$e^{\zeta t} < x_0\,, \lambda_0 > \int_{-\infty}^{0} e^{-\zeta u}\, d\mu(u) = 0 \quad \text{for every } t \geq 0.$$

Since the support of μ is contained in $(-\infty\,, 0]$, this implies $\int_{-\infty}^{+\infty} e^{-\zeta u}\, d\mu(u) = 0$, *i.e.* $\mathcal{M}(-i\zeta) = 0$, $\mathcal{M}$ denoting the Fourier transform of μ [4]. In particular, if $\zeta = i\theta$ for some $\theta \in \mathbb{R}$, then $\mathcal{M}(\theta) = 0$. In analogy to § 3, the family of functions, defined on $\mathbb{R}_+$ and with values in $\mathbb{C}$,

$$\{< T(\cdot)x, \lambda >:\ x \in \mathcal{D}(X)\,, \lambda \in \mathcal{D}(X^{\odot})\,,\ < x, \lambda > \neq 0\} \tag{5.2}$$

(4) Observe that $\mathcal{L}(\zeta) = \int_0^{+\infty} e^{\zeta\, u}\, d\mu(-u)$ could also be regarded as a Laplace transform of the compactly supported measure μ.

is said to be asymptotically mean periodic if there exists some non zero measure $\mu \in \mathcal{C}'$, with compact support on $(-\infty\,,0]$, for which $\int_{-\infty}^{0} < T(t-u)x, \lambda > d\mu(u) = 0$ for all $t \geq 0$ and for every $x \in \mathcal{D}(X)\,,\ \lambda \in \mathcal{D}(X^{\odot})\,,$ with $< x, \lambda > \neq 0$.

$\mathcal{M}$ will denote the Fourier transform of μ. In view of what has been observed above, the following proposition, analogous to Theorem 3.5, can be stated:

Proposition 5.2. *Let T be an equicontinuous semigroup of class $\mathcal{C}_0$ on a space $\mathcal{E}$ satisfying* (H1). *If the family of functions* (5.2) *is asymptotically mean periodic, then* $\mathrm{p}\sigma(X^{\odot})$ (*and also* $\mathrm{p}\sigma(X)$) *is discrete. Moreover, if $\mathcal{M}$ has no real zeros, then* $\mathrm{p}\sigma(X^{\odot}) \cap i\mathbb{R} = \emptyset$.

By combining proposition above and the stability theorem by Arendt and Batty [19] holding in Banach spaces, one gets the following:

Proposition 5.3. *Let T be a uniformly bounded semigroup of class $\mathcal{C}_0$ on a Banach space $\mathcal{E}$. If family* (5.2) *is asymptotically mean periodic, $\mathcal{M}$ has no real zeros and $\sigma(X) \cap i\mathbb{R}$ is countable, then T is uniformly stable,* i.e. $\lim_{t\to+\infty} ||T(t)x|| = 0$ *for every $x \in \mathcal{E}$.*

Some partial results can be obtained, in the framework of semigroups acting on Fréchet spaces, also for the continuous spectrum of X.

Proposition 5.4. *Let T be an equicontinuous semigroup on a Fréchet space $\mathcal{E}$. Suppose T is uniformly weakly asymptotically mean periodic. Then* $\mathrm{c}\sigma(X)$ *has no finite accumulation point.*

Proof. By hypothesis, there exists some non-zero measure $\mu \in \mathcal{C}'$, with compact support on $(-\infty\,,0]$, for which $\int_{-\infty}^{0} < T(t-u)x, \lambda > d\mu(u) = 0$ for all $t \geq 0$ and for every $x \in \mathcal{E}\,,\ \lambda \in \mathcal{E}'$.
Let ζ belong to $\mathrm{c}\sigma(X)$. By arguing exactly as in Theorem 3.11, one gets $\int e^{\zeta(t-u)} d\mu(u) = 0$ for every $t \geq 0$, whence $\mathcal{M}(-i\zeta) = 0$. Paley-Wiener theorem yields finally the thesis. □

The same argumentations hold by substituting the topology of uniform convergence on compact subsets of $\mathbb{R}_+$ with a weak topology on $L^2(0\,,\infty)$, suggested by A. Beurling in [3]. For more details on this topology see [16].

For any $\varepsilon \geq 0$, let $L^{2,\varepsilon}$ denote the Hilbert space of measurable functions from $[0\,,+\infty)$ to $\mathbb{C}$, arising from the inner product

$$(f, g)_\varepsilon = \int_0^{+\infty} f(t)\, g(t)\, e^{-\varepsilon t}\, dt.$$

Set $L^{2,0} = L^2$. Let A be a subset of $L^2(0\,,+\infty)$. The symbol A_ε will denote the closure of A in $L^{2,\varepsilon}$ under $||\cdot||_\varepsilon$ (where $||f||_\varepsilon = \left(\int_0^\infty |f(t)|^2 e^{-\varepsilon t} dt\right)^{1/2}$ for every $f \in L^{2,\varepsilon}$). Set

$$A^B = \bigcap_{\varepsilon>0} \left(A_\varepsilon \cap L^2\right).$$

Let now f belong to $L^2(0, +\infty)$. Let S_f denote the set of all finite linear combinations of positive translates of f. Set $E_f = \overline{S_f}^B$. The function f is said to be *asymptotically (B)-mean periodic* if, and only if, $E_f \subsetneq L^2(0, +\infty)$.

As a consequence of Hahn-Banach theorem, a function $f \in L^2(0, +\infty)$ is asymptotically (B)-mean periodic if, and only if, there exist $\varepsilon > 0$ and a non identically vanishing function $k \in L^{2,\varepsilon}$ such that $(f(\cdot + t), k)_\varepsilon = 0$ for every $t \geq 0$.

In the following, any function $k \in L^{2,\varepsilon}$ will be extended to the entire real line, by setting $k(t) = 0$ for all $t < 0$. Define $K(t) = k(-t) \cdot e^{\varepsilon t}$. Then the application $g(t) = \big(f * K\big)(t)$ vanishes on $[0, +\infty)$. Since f belongs to $L^2(0, +\infty)$, one can define the Fourier transform $\mathcal{F}$ of f in the usual way. As a consequence of the relation

$$\mathcal{F}(\zeta) = \frac{\mathcal{G}(\zeta)}{\mathcal{K}(\zeta)},$$

Koosis proved, moreover, that $\mathcal{F}$ has a meromorphic extension to the complex plane and that its poles lie necessarily in the half-plane $\mathcal{I}m\zeta < 0$.

Also in this case, it is possible to define the Schwartz spectrum of an asymptotically (B)-mean periodic function f (denoted with the symbol $S(f)$) as the set of all poles of $\mathcal{F}(\zeta)$, each counted with its order of multeplicity.

It is also possible to prove that a complex number ζ is a pole of order at least $n+1$ for $\mathcal{F}$ if, and only if, $t^n\, e^{i\zeta t}$ belongs to $\tau_+(f)$.

Suppose now T be an equicontinuous semigroup on a locally convex space $\mathcal{E}$ fulfilling (H1). Let $< T(\cdot)x_0, \lambda_0 >$ belong to $L^2(0,\infty)$, for some $x_0 \in \mathcal{D}(X)$ and $\lambda_0 \in \mathcal{D}(X^\odot)$. If λ_0 is an eigenvector of $X^\odot$ with eigenvalue ζ, and $x_0 \in \mathcal{D}(X)$ is such that $< x_0, \lambda_0 > \neq 0$, then

$$\int_0^{+\infty} |< T(t)x_0, \lambda_0 >|^2 dt = |< x_0, \lambda_0 >|^2 \int_0^{+\infty} e^{2\Re e\zeta\cdot t}\, dt < +\infty,$$

entailing $\Re e\zeta < 0$.

Under the same assumptions, the application $< T(\cdot)x_0, \lambda_0 >$ is asymptotically (B)-mean periodic, and therefore there exist $\varepsilon > 0$ and $g \in L^{2,\varepsilon}$ such that, by setting $K(t) = k(-t)\cdot e^{\varepsilon t}$, it results:

$$\int_{-\infty}^{+\infty} < T(t-u)x_0, \lambda_0 > K(u)\, du = 0 \text{ for every } t \geq 0, \textit{ i.e.}$$

$$\int_{-\infty}^{+\infty} e^{-\zeta u} K(u)\, du = 0,$$

i.e. the Fourier transform $\mathcal{K}$ of K vanishes in the point $-i\zeta$, implying, as in [16], that the exponential monomial $e^{\zeta t}$ belongs to $E_{< T(\cdot)x_0, \lambda_0 >}$, and therefore that $-i\zeta$ belongs to the Schwartz spectrum of $< T(\cdot)x_0, \lambda_0 >$. Summing up, if $\zeta \in \mathrm{p}\sigma(X^\odot)$ for some $\zeta \in \mathbb{C}$, then there exist $x_0 \in \mathcal{D}(X)$ and $\lambda_0 \in \mathcal{D}(X^\odot)$ with $< x_0, \lambda_0 > \neq 0$ such that the function, belonging to $L^2(0, +\infty)$, $t \mapsto < T(t)x_0, \lambda_0 >$ is asymptotically (B)-mean periodic, and $-i\zeta \in S(< U(\cdot)x_0, \lambda_0 >)$.

In view of what has been observed above, zeros of $\mathcal{K}$, arising from eigenvalues either of X or of $X^{\odot}$, have positive imaginary parts.

From now on, family (5.2) will be said to be asymptotically (B)-mean periodic if there exist $\varepsilon > 0$ and $g \in L^{2,\varepsilon}$ such that $\left(< T(\cdot + t)x, \lambda > , g \right)_{\varepsilon} = 0$ for every $t \geq 0$, for all $x \in \mathcal{D}(X)$, $\lambda \in \mathcal{D}(X^{\odot})$ with $< x, \lambda > \neq 0$.

As a consequence of Paley-Wiener theorem, the following result can be stated:

Proposition 5.5. *Let T be an equicontinuous semigroup of class $\mathcal{C}_0$ on a space $\mathcal{E}$ satisfying* (H1). *If family* (5.2) *is asymptotically (B)-mean periodic, then both* $\mathrm{p}\sigma(X)$ *and* $\mathrm{p}\sigma(X^{\odot})$ *are contained in the left half-plane $\Pi_l = \{\zeta \in \mathbb{C} : \Re\zeta < 0\}$ and are discrete sets.*

Appendix A

Ergodic properties of equicontinuous semigroups

The main tool in the study of almost periodicity properties for a semigroup of linear bounded operators on a Banach space $\mathcal{E}$ is given by the mean ergodic theorem. Here some extensions to the case of an equicontinuous semigroup T of class $\mathcal{C}_0$ acting on a locally convex space $\mathcal{E}$ are briefly presented.

First of all, set

$$\mathcal{F}_{i\theta} = \left\{ x \in \mathcal{E} : \lim_{t \to +\infty} \frac{1}{t} \int_0^t e^{-i\theta s} T(s)x \, ds \text{ exists in } \mathcal{E} \right\},$$

where the integral has to be considered as a Riemann integral.

For every $x \in \mathcal{F}_{i\theta}$ define

$$P_{i\theta} x = \lim_{t \to +\infty} \frac{1}{t} \int_0^t e^{-i\theta s} T(s)x \, ds.$$

$P_{i\theta}$ is a linear bounded operator on $\mathcal{F}_{i\theta}$; in particular, for every continuous seminorm p on $\mathcal{E}$ it results

$$p(P_{i\theta} x) \leq q(x),$$

q being a seminorm for which equicontinuity property $p(T(t)x) \leq q(x)$, for every $t \geq 0$ and $x \in \mathcal{E}$, holds. Properties of $P_{i\theta}$ are listed in the following

Proposition A1. *Let T be an equicontinuous semigroup on a space $\mathcal{E}$, satisfying property* (H1). *Let $P_{i\theta}$ and $\mathcal{F}_{i\theta}$ be defined as above. Then*:

(1) $T(\tau)\mathcal{F}_{i\theta} \subset \mathcal{F}_{i\theta}$ *and* $T(\tau)P_{i\theta} = P_{i\theta}T(\tau) = e^{i\theta\tau}P_{i\theta}$ *for every* $\tau \geq 0$;

(2) $P_{i\theta}$ *is a linear projection in* $\mathcal{F}_{i\theta}$;

(3) $\overline{\mathcal{R}(X - i\theta I)} \subset \ker P_{i\theta}$;

(4) $\mathcal{R}P_{i\theta} = \ker(X - i\theta I)$ *and* $P_{i\theta\,|\ker(X - i\theta I)} = I$;

(5) $\mathcal{F}_{i\theta} = \ker(X - i\theta I) \oplus \overline{\mathcal{R}(X - i\theta I)}$ *and* $\ker P_{i\theta} = \overline{\mathcal{R}(X - i\theta I)}$.

Proof. A proof can be found in [6], under the additional hypothesis that $\mathcal{E}$ is complete. This assumption is there used only to prove the closure of $\ker(X - i\theta I) \oplus$

$\oplus\overline{\mathcal{R}(X - i\theta I)}$; it is easy to check that this fact can be proved under the only assumption that $\mathcal{E}$ is sequentially complete. □

Observe that (3) and (4) imply, in particular, that for every $\theta \in \mathbb{R}$

$$\ker(X - i\theta I) \cap \overline{\mathcal{R}(X - i\theta I)} = \{0\}. \tag{A1}$$

Let the locally convex space $\mathcal{E}$ be given a local countable basis and let $\mathcal{E}$ be weakly complete. If for some $x \in \mathcal{E}$ the limit

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t e^{-i\theta s} < T(s)x, \lambda > ds \tag{A2}$$

exists for every $\lambda \in \mathcal{E}'$, then there exists some $Q_{i\theta}x \in \mathcal{E}$ for which the limit (A2) equals $< Q_{i\theta}x, \lambda >$. Define

$$\mathcal{G}_{i\theta} = \left\{ x \in \mathcal{E} : \lim_{t\to+\infty} \frac{1}{t} \int_0^t e^{-i\theta s} < T(s)x, \lambda > ds \text{ exists for every } \lambda \in \mathcal{E}' \right\}.$$

Proposition A2. *Let T be an equicontinuous semigroup on a space $\mathcal{E}$, satisfying property* (H1). *Let $\mathcal{E}$ be endowed with a local countable basis and weakly complete. Let $Q_{i\theta}$ and $\mathcal{G}_{i\theta}$ be defined as above. Then*:
(1) $T(\tau)\mathcal{G}_{i\theta} \subset \mathcal{G}_{i\theta}$ *and* $T(\tau)Q_{i\theta} = Q_{i\theta}T(\tau) = e^{i\theta\tau}Q_{i\theta}$ *for every* $\tau \geq 0$;
(2) $Q_{i\theta}$ *is a linear projection in* $\mathcal{G}_{i\theta}$;
(3) $\overline{\mathcal{R}(X - i\theta I)} \subset \ker Q_{i\theta}$;
(4) $\mathcal{R}Q_{i\theta} = \ker(X - i\theta I)$ *and* $Q_{i\theta|\ker(X-i\theta I)} = I$;
(5) $\mathcal{G}_{i\theta} = \ker(X - i\theta I) \oplus \overline{\mathcal{R}(X - i\theta I)}$ *and* $\ker Q_{i\theta} = \overline{\mathcal{R}(X - i\theta I)}$.

Proof. The proof is very similar to that which holds in Banach spaces [24]. □

Let now $\mathcal{E}$ be a linear topological space of Baire. Then, as a consequence of Banach-Steinhaus theorem in linear topological spaces, $\mathcal{E}'$ is complete with respect to the weak-star topology, so that if $\lambda \in \mathcal{E}'$ is such that the limit (A2) exists for every $x \in \mathcal{E}$, then there exists $R_{i\theta}\lambda \in \mathcal{E}'$ for which (A2) equals $< x, R_{i\theta}\lambda >$. Define

$$\mathcal{H'}_{i\theta} = \left\{ \lambda \in \mathcal{E}' : \lim_{t\to+\infty} \frac{1}{t} \int_0^t e^{-i\theta s} < T(s)x, \lambda > ds \text{ exists for every } x \in \mathcal{E} \right\}.$$

Proposition A3. *Let T be an equicontinuous semigroup on a space $\mathcal{E}$, satisfying property* (H1). *Let, moreover, $\mathcal{E}$ be of Baire. Let $R_{i\theta}$ and $\mathcal{H'}_{i\theta}$ be defined as above. Then*:
(1) $T'(\tau)\mathcal{H'}_{i\theta} \subset \mathcal{H'}_{i\theta}$ *and* $T'(\tau)R_{i\theta} = R_{i\theta}T'(\tau) = e^{i\theta\tau}R_{i\theta}$ *for every* $\tau \geq 0$;
(2) $R_{i\theta}$ *is a linear projection in* $\mathcal{H'}_{i\theta}$;
(3) $\mathcal{R}R_{i\theta} = \ker(X^{\odot} - i\theta I)$;
(4) $\mathcal{H'}_{i\theta} = \mathcal{R}(R_{i\theta}) \oplus \ker(R_{i\theta})$ *and* $\mathcal{R}(R_{i\theta}) \cap \ker(R_{i\theta}) = \{0\}$;
(5) $\ker(X^{\odot} - i\theta I) \oplus \overline{\mathcal{R}(X^{\odot} - i\theta I)} \subset \mathcal{H'}_{i\theta}$ *and for every* $\lambda \in \ker(X^{\odot} - i\theta I) \oplus \overline{\mathcal{R}(X^{\odot} - i\theta I)}$ *it holds*

$$R_{i\theta}\lambda = \lim_{t\to+\infty} \frac{1}{t} \int_0^t e^{-i\theta s} T^{\odot}(s)\lambda \, ds.$$

Proof. Also in this case, the proof is very similar to that which holds in Banach spaces [24]. $\square$

The situation will now be investigated in which the locally convex space $\mathcal{E}$ is reflexive. The *evaluation mapping* $\mathcal{I}$ of $\mathcal{E}$ into its second adjoint $\mathcal{E}''$ is defined by

$$< \lambda\,, \mathcal{I}x >=< x\,, \lambda > \quad \text{for every } x \in \mathcal{E} \text{ and } \lambda \in \mathcal{E}'.$$

$\mathcal{E}$ is said to be *semi–reflexive* if $\mathcal{I}(\mathcal{E}) = \mathcal{E}''$. If, moreover, $\mathcal{I}$ is a topological isomorphism of $\mathcal{E}$ onto $\mathcal{E}''$, $\mathcal{E}$ is called *reflexive*.

An analogous proof to that holding in Banach spaces [23] shows that, if a locally convex space $\mathcal{E}$ fulfilling (H1) is reflexive, then X' is densely defined with respect to the strong topology of $\mathcal{E}'$ and, therefore, $\mathcal{E}' = \mathcal{E}^{\odot}$.

Let A be a non-void subset of $\mathcal{E}$. The polar set A° is defined by

$$A^\circ = \{\lambda \in \mathcal{E}' : |< x, \lambda >| \leq 1 \text{ for all } x \in \mathcal{E}\}.$$

Recall that a subset A of $\mathcal{E}$ is called *balanced* if $\alpha A \subseteq A$ for all $\alpha \in \mathbb{C}$, $|\alpha| \leq 1$. Some properties of polar sets will be used in the following [15]:
(1) if $A \subseteq B$, then $B^\circ \subseteq A^\circ$;
(2) if A and B are non void, convex and balanced subsets of $\mathcal{E}$, it results

$$(A + B)^\circ \subseteq (A \oplus B)^\circ \subseteq A^\circ \cap B^\circ;$$

(3) if $\mathcal{E}$ is reflexive and A is a non-void subset of $\mathcal{E}$, then $A^{\circ\circ}$ is the smallest convex, balanced, weakly closed subset in $\mathcal{E}$ which contains A.

Proposition A4. *Let $\mathcal{E}$ be a reflexive space satisfying* (H1). *Let X be the infinitesimal generator of an equicontinuous semigroup T of class C_0 on $\mathcal{E}$. Then*

$$\mathcal{E} = \ker X \oplus \overline{\mathcal{R}X}. \tag{A3}$$

Proof. Since $\ker X$ and $\mathcal{R}(X)$ (and therefore also $\overline{\mathcal{R}X}$) are convex and balanced subsets of $\mathcal{E}$, it results

$$\left(\ker X \oplus \overline{\mathcal{R}(X)}\right)^\circ \subseteq \ker X^\circ \cap \overline{\mathcal{R}X}^\circ. \tag{A4}$$

Let now $\lambda \in (\overline{\mathcal{R}X})^\circ$, *i.e.* $|< Xx, \lambda >| \leq 1$ for every $x \in \mathcal{D}(X)$. If $x \in \mathcal{D}(X)$ and $Xx \neq 0$, then for every $t > 0$ it results $|< X(tx)\,, \lambda >| \leq 1$, whence $|< Xx, \lambda >| \leq \frac{1}{t}$, and therefore $< Xx, \lambda >= 0$. Thus λ belongs to $\mathcal{D}(X')$ and $X'\lambda$=0, so that

$$(\overline{\mathcal{R}X})^\circ \subseteq \ker X'.$$

Let now $\mu \in \overline{\left(\mathcal{R}(X')\right)}^\circ$; thus $\mu \in \left(\mathcal{R}(X')\right)^\circ$, *i.e.* by definition $|< X'\lambda\,, \mu >| \leq 1$ for every $\lambda \in \mathcal{D}(X')$. Since $\mathcal{E}$ is reflexive, it results $|< x, X'\lambda >| \leq 1$ for some $x \in \mathcal{E}$ and for every $\lambda \in \mathcal{D}(X')$. This entails $< x, X'\lambda >= 0$, and therefore $< Xx, \lambda >= 0$ for every $\lambda \in \mathcal{D}(X')$; since X' is densely defined in $\mathcal{E}'$, then $x \in \ker X$. Now, property (1) yields

$$\ker X^\circ \subseteq \overline{\mathcal{R}(X')}^{\circ\circ}.$$

$\overline{\mathcal{R}(X')}$ is weakly closed, since it is convex, and therefore property (3) yields

$$\overline{\mathcal{R}(X')}^{\circ\circ} = \overline{\mathcal{R}(X')}.$$

Thus from (A4) it follows

$$\left(\ker X \oplus \overline{\mathcal{R}X}\right)^{\circ} \subseteq \ker X' \cap \overline{\mathcal{R}(X')}.$$

By applying (A1) to the operator X', it follows that $\ker X' \cap \overline{\mathcal{R}(X')} = \{0\}$; finally, an application of Hahn-Banach theorem yields the thesis. $\square$

By recalling that, if X generates an equicontinuous semigroup of class $\mathcal{C}_0$ on $\mathcal{E}$, then $X - i\theta I$ generates, for every $\theta \in \mathbb{R}$, an equicontinuous semigroup of class $\mathcal{C}_0$, one gets, under the same assumptions of Proposition A4, the equality

$$\ker(X - i\theta I) \oplus \overline{R(X - i\theta I)} = \mathcal{E}. \tag{A5}$$

(A5), combined with the definition of residual spectrum of a linear operator, immediately yields the following

Corollary A5. *Under the same assumptions of Proposition* A4, *it results*

$$\mathrm{r}\sigma(X) \cap i\mathbb{R} = \emptyset.$$

If, in particular, X generates an equicontinuous group of class $\mathcal{C}_0$, then $\mathrm{r}\sigma(X) = \emptyset$.

Consider now the mapping

$$j:\ \mathcal{E} \to \mathcal{E}^{\odot\prime},$$

defined by $< \lambda\,, jx > = < x, \lambda >$ for all $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}^{\odot}$. It holds $j\mathcal{E} \subseteq \mathcal{E}^{\odot\odot}$.
The space $\mathcal{E}$ is said to be $\odot$-*reflexive* with respect to T if $j\mathcal{E} = \mathcal{E}^{\odot\odot}$.
When $\mathcal{E}$ is reflexive, $\mathcal{E}'$ is also reflexive, and therefore from Proposition A4 it follows that

$$\mathcal{E}' = \ker(X' - i\theta I) \oplus \overline{\mathcal{R}(X' - i\theta I)};$$

in this case, $R_{i\theta}$ is defined on the whole space $\mathcal{E}'$.

It will now be proved that, if $\mathcal{E}$ is $\odot$-reflexive, then for all $\theta \in \mathbb{R}$ it holds

$$\mathcal{H}'_{i\theta} = \ker(X^{\odot} - i\theta I) \oplus \overline{\mathcal{R}(X^{\odot} - i\theta I)}; \tag{A6}$$

it will also be shown that in the non $\odot$-reflexive case the inclusion in (5) of Theorem A3 can be strict.

Theorem A6. *Let $\mathcal{E}$ be a space satisfying property* (H1).
Let $\mathcal{E}$ be $\odot$-reflexive. Then

$$\mathcal{H}'_0 = \ker X^{\odot} \oplus \overline{\mathcal{R}X^{\odot}}.$$

Proof. Suppose that $\ker X^{\odot} \oplus \overline{\mathcal{R}X^{\odot}} \subsetneq \mathcal{H}'_0$, *i.e.* that there exists some $\lambda_0 \in \mathcal{H}'_0$, $\lambda_0 \notin \ker X^{\odot} \oplus \overline{\mathcal{R}X^{\odot}}$. By the Hahn-Banach theorem there is some $\psi_0 \in \mathcal{E}''$, such that
(1) ψ_0 vanishes on $\ker X^{\odot} \oplus \overline{\mathcal{R}X^{\odot}}$;
(2) $< \lambda_0\,, \psi_0 > \neq 0$.

Since $\mathcal{E}^{\odot} \subseteq \mathcal{E}'$, it results $\mathcal{E}'' \subseteq \mathcal{E}^{\odot\prime}$, so that $\psi_0 \in \mathcal{E}^{\odot\prime}$. Indeed, $\psi_0 \in \ker(X^{\odot\odot})$, since (1) entails that $< X^{\odot}\lambda , \psi_0 >= 0$ for every $\lambda \in \mathcal{D}(X^{\odot})$, whence $\psi_0 \in \mathcal{D}(X^{\odot\prime})$ and, moreover, $X^{\odot\prime}\psi_0 = 0$. Thus $\psi_0 \in \mathcal{D}(X^{\odot\odot})$ and $X^{\odot\odot}\psi_0 = 0$, *i.e.* ψ_0 belongs to $\ker(X^{\odot\odot})$.

Since $X^{\odot\odot}$ is the largest restriction of $X^{\odot\prime}$ with domain and range contained in $\mathcal{E}^{\odot\odot}$ and $\mathcal{E}$ is $\odot$-reflexive, there exists some $x_0 \in \mathcal{E}$ for which $\psi_0 = j(x_0)$. It will now be shown that $x_0 \in \ker(X)$.

It results

$$j\left(\lim_{t\to 0+} \frac{1}{t}(T(t)x_0 - x_0)\right) = \lim_{t\to 0+} \frac{1}{t}(T^{\odot\odot}(t)jx_0 - jx_0) = X^{\odot\odot}\psi_0 = 0.$$

By applying j^{-1} to both members, one gets $Xx_0 = 0$ (and therefore $T(t)x_0 = x_0$ for every $t \geq 0$).

Observe now that (2) implies that $< x_0 , \lambda_0 >\neq 0$, since

$$< x_0 , \lambda_0 >=< \lambda_0 , j(x_0) >=< \lambda_0 , \psi_0 >\neq 0\,, \text{ whence}$$

$$\lim_{t\to +\infty} \frac{1}{t}\int_0^t < T(s)x_0 , \lambda_0 > ds =< x_0 , \lambda_0 >\neq 0.$$

Since $\lambda_0 \in \mathcal{H}'_0$, it holds also

$$\lim_{t\to +\infty} \frac{1}{t}\int_0^t < T(s)x_0 , \lambda_0 > ds =< x_0 , R_0\lambda_0 > .$$

Now, (3) of Theorem A3 states that $R_0\lambda_0 \in \ker(X^{\odot})$, and therefore

$$\lim_{t\to +\infty} \frac{1}{t}\int_0^t < T(s)x_0 , \lambda_0 > ds =< x_0 , R_0\lambda_0 >=< R_0\lambda_0 , j(x_0) >=< R_0\lambda_0 , \psi_0 >= 0\,,$$

since ψ_0 vanishes identically on $\ker(X^{\odot})$.

This yields a contradiction, and therefore the thesis follows. □

By applying the result above to the semigroup $\{e^{-i\theta t}T(t)\}$, generated by $X - i\theta I$, one gets finally (A6).

Let now $\mathcal{E} = \mathcal{C}_0(\mathbb{R})$ be the Banach space of all complex valued continuous functions on $\mathbb{R}$, vanishing at infinity, endowed with the sup-norm. Let T be the translation group on $\mathcal{E}$, *i.e.*

$$T(t)x(r) = x(r + t) \text{ for every } t, r \in \mathbb{R}.$$

$\mathcal{E}'$ is the space $\mathcal{M}(\mathbb{R})$ of all complex Radon measures, and it has been proved [18] that $\mathcal{E}^{\odot}$ coincides with the subspace of $\mathcal{M}(\mathbb{R})$ of all absolutely continuous measures. $\mathcal{E}$ is not $\odot$-reflexive, since $\mathcal{E}^{\odot\odot}$ coincides with the space $BUC(\mathbb{R})$ of all complex-valued uniformly continuous bounded functions on $\mathbb{R}$.

In this case, it results $\ker X^{\odot} \oplus \overline{\mathcal{R}X^{\odot}} \subsetneq \mathcal{H}'_0$. Consider, indeed, the Dirac measure δ_0 concentrated in 0. δ_0 belongs to $\mathcal{M}(\mathbb{R})$, but it does not belong to $\ker X^{\odot} \oplus \overline{\mathcal{R}X^{\odot}}$, since $\ker X^{\odot} \oplus \overline{\mathcal{R}X^{\odot}} \subseteq \mathcal{E}^{\odot}$ and δ_0 is a singular measure. Nonetheless, it results

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t < T(s)x\,,\delta_0 > ds = \lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)ds = 0$$

for every $x \in \mathcal{C}_0(\mathbb{R})$, whence δ_0 belongs to $\mathcal{H}'_0$. Indeed, if $x \in \mathcal{C}_0(\mathbb{R})$, then for every $\varepsilon > 0$ there is some $L_\varepsilon > 0$ such that $|x(s)| \le \varepsilon$ for all $|s| \ge L_\varepsilon$. Thus

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\,ds = \lim_{t\to+\infty} \frac{1}{t}\left(\int_0^{L_\varepsilon} + \int_{L_\varepsilon}^t\right) x(s)\,ds = 0\,,$$

since on $(0\,, L_\varepsilon)$ x is bounded and on $(L_\varepsilon\,, +\infty)$ it results $|x| \le \varepsilon$.

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