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On the alpine ski with dry friction and air resistance. Some optimization problems for it

Memoria (*) di ALDO BRESSAN

ABSTRACT. — In the present work, divided in three parts, one considers a real skis-skier system, Σ_R , descending along a straight-line l with constant dry friction; and one schematizes it by a holonomic system $\Sigma = A \cup U$, having any number $n \geq 4$ of degrees of freedom and subjected to (non-ideal) constraints, partly one-sided. Thus, *e.g.*, jumps and also «steps made with sliding skis» can be schematized by Σ . Among the n Lagrangian coordinates for Σ two are the Cartesian coordinates ξ and η of its center of mass, C , relative to the downward axis that includes l and to the upward axis that is normal to l in a vertical plane; the others are to be regarded as controls in that their values can be determined by the skier. Four alternative laws of air resistance, A2.5,1 to A2.5,4, are considered for Σ . They have increasing simplicity and according to all of them the resultant of air resistance, $m\mathcal{R}$, is parallel to l and independent of Σ 's possible asymmetries with respect to the vertical plane trough l . Briefly, $m\mathcal{R}$ is independent of the skier's configuration \mathcal{C}_U with respect to the skis, these being always supposed to be parallel to l ; according to A2.5,3 $m\mathcal{R}$ is a (possibly non-homogeneous) linear function of C 's velocity $\dot{\xi}$; and according to A2.5,4 $m\mathcal{R} \equiv 0$.

In Part 1, after some preliminaries, Σ 's dynamic equations are written in a suitable form by which, under the law A2.5,3, a control-free first integral can be deduced, notwithstanding controls can raise and lower C , which affects C 's velocity $\dot{\xi}$ because of dry friction. Given Σ 's initial conditions at $t = 0$, this first integral is a relation between ξ , η , $\dot{\xi}$, $\dot{\eta}$ and the present time t . In the case $m\mathcal{R} \equiv 0$ it can be integrated again. Thus ξ appears to be determined by η and t . The afore-mentioned results on Σ are simple; and here it is convenient to note that the present work does not aim at refined results; furthermore its Parts 2 and 3 are completely based on the afore-mentioned result valid for $m\mathcal{R} \equiv 0$; and they treat two problems on Σ that have a special interest for Σ_R , these problems being useful in connection with races and hence with tourism. This occurs in that, by a suitable device, the conclusions of the above treatment can be used to obtain good informations on Σ_R also in case $m\mathcal{R}$ is for Σ_R practically constant and large. At the end of Part 1, under an air resistance law more general than A2.5,4, one considers the possibility of rendering the (negative) work of dry friction in a given time interval $[0, T]$ arbitrarily small by means of «steps made with sliding skis»; and one shows that this fact has a negligible influence on the length $\xi(T) - \xi(0)$ of the ski-run's stretch covered by C in $[0, T]$. This result reasonably holds for Σ_R with a good approximation; thus it is «explained» why the above steps are not made in practice.

In Part 2, where the identity $m\mathcal{R} \equiv 0$ is assumed, the following is considered:

PROBLEM 9.1. Given $\bar{\xi} > 0$ and the initial conditions at $t = 0$, how can one minimize the time $\bar{t} (> 0)$ taken by C 's absciss ξ to cover the ski-run's stretch $[0, \bar{\xi}]$?

This Problem concerns alpine ski. On the other hand the few mathematical works on ski, that are known to the author but not related to his papers, deal with-ski jumps. Furthermore Problem 9.1 is different from all preceding ski problems treated by the author in that it involves dry friction, air resistance, and one-sided constraints. For the same reasons Σ cannot be regarded as a special case of some holonomic system to which the author has applied control theory. In conformity with this, instead of solving Problem 9.1 by this theory (Pontriagin's maximum principle), it is convenient to preliminarily consider the following

PROBLEM 6.1. Given $T > 0$ and the initial conditions at $t = 0$, how can one maximize the length $\xi(T) - \xi(0)$ of the ski-run's stretch $[0, \bar{\xi}]$ covered by C in the time T ?

For this problem ∞^∞ solutions are exhibited in $C^1 \cap PC^2$, so that they are much more regular than the solutions (in L^1) assured by the most known existence theorem in control theory (if applicable). Lastly the solutions of the above two problems are shown to be the same when a certain relation holds between T

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and $\bar{\xi}$. The optimal values of $\xi(T)$ and \bar{t} for Problem 6.1 and Problem 9.1 respectively can be expressed by means of the data, independently of Σ 's optimal motions. Various properties of these are considered; and it is shown that, generally, the skier can affect the values of $\xi(T)$ and \bar{t} very little.

Part 3 treats Σ 's motions without jumps, *i.e.*, the most common ones. *E.g.* some upper bounds for the afore-mentioned little influence of the skier are given. Furthermore for every $t \in [0, T[$ one exhibits some conditions equivalent to the possibility of extending a jumpless motion of Σ in $[0, t]$ to such a motion in $[0, T]$. One shows that this may be useful to implement the «skier U » as a robot, in order to compare jumpless motions of Σ_R , with the corresponding dynamic motions of Σ considered in Part 2 where the identity $m\mathcal{R} \equiv 0$ is assumed. The afore-mentioned device allows us to perform such a comparison in an interesting real case where $m\mathcal{R}$ keeps very near a possibly large constant value.

KEY WORDS: Ski; Holonomic systems; Optimization problems; Controls.

RIASSUNTO. — Nel presente lavoro, diviso in tre parti, si considera un sistema reale sci-sciatore, Σ_R , che scende lungo una traiettoria rettilinea l avente attrito costante; e lo si schematizza mediante un sistema olonomo $\Sigma = A \cup U$ a un imprecisato numero $n (\geq 4)$ di gradi di libertà e a vincoli (non lisci) in parte unilaterali; e così per Σ possono considerarsi, per es., salti e «passi fatti con sci scivolanti». Delle n coordinate Lagrangiane di Σ , due sono quelle Cartesiane ξ ed η del suo baricentro C , relative all'asse discendente contenente l e all'asse ascendente e ortogonale ad l in un piano verticale; le altre vanno riguardate come controlli, in quanto hanno valori determinabili istante per istante dallo sciatore.

Per Σ si considerano quattro leggi alternative per la resistenza dell'aria, A2.5,1-A2.5,4, di semplicità crescente. In tutte il risultante $m\mathcal{R}$ di questa resistenza è considerato parallelo a l e indipendente dalle eventuali asimmetrie di Σ rispetto al piano verticale per l . Brevemente, nelle A2.5,2-A2.5,4 $m\mathcal{R}$ è indipendente dalla configurazione \mathcal{C}_U dello sciatore rispetto agli sci, supposti sempre paralleli ad l . Secondo la A2.5,3 $m\mathcal{R}$ è lineare nella velocità ma non necessariamente di tipo viscoso; secondo la A2.5,4 è $m\mathcal{R} \equiv 0$.

Nella Parte 1, dopo i suaccennati preliminari, si scrivono le equazioni dinamiche di Σ in forma opportuna in modo che, ammessa la A2.5,3, si possa ricavare un certo integrale primo indipendente dai controlli, nonostante questi permettano di alzare e abbassare C , il che influisce sulla velocità $\dot{\xi}$ di C a causa dell'attrito. Date le condizioni iniziali, questo integrale primo è una relazione tra ξ , η , $\dot{\xi}$, $\dot{\eta}$ e il tempo t . Nel caso di resistenza dell'aria trascurabile, esso è ulteriormente integrabile; ξ risulta allora determinata da η e dal tempo t . I suddetti risultati sono semplici; e conviene qui notare che il presente lavoro non è di rifinitura; invece, per es., le sue Parti 2 e 3 sono completamente basate sul suaccennato risultato valido per $m\mathcal{R} \equiv 0$; e trattano due problemi di particolare interesse per Σ_R , in connessione con le gare e quindi col turismo. Ciò accade in quanto, in base a fatti ben noti sullo sci, un artificio permette di usare le conclusioni della detta trattazione, in modo da ottenere interessanti buone informazioni su Σ_R anche nel caso che, per Σ_R , $m\mathcal{R}$ sia praticamente costante e magari grande. Alla fine della Parte 1 si mostra, riferendosi a una legge di resistenza dell'aria più generale della A2.5,2, che nonostante il lavoro negativo dell'attrito in un dato intervallo di tempo $[0, T]$ possa ridursi piccolo a piacere mediante «passi fatti con sci scivolanti», ciò in sostanza non influisce affatto sulla lunghezza $\xi(T) - \xi(0)$ del tratto di pista «percorso» da C in $[0, T]$. Ragionevolmente questi risultati appaiono validi con buona approssimazione; e quindi resta «spiegato» perché in realtà i detti passi non si fanno.

Nella Parte 2, trascurando la resistenza dell'aria, si considera il problema seguente:

PROBLEMA 9.1. Dato $\bar{\xi} > 0$, come minimizzare il tempo $\bar{t} > 0$ in cui l'ascissa ξ di C descrive l'intervallo di pista $[0, \bar{\xi}]$, sotto assegnati dati iniziali?

Esso riguarda lo sci da discesa mentre gli unici pochissimi lavori sullo sci, a conoscenza dell'autore e che non si riferiscono a lavori di questo, considerano solo salti dal trampolino; inoltre il Problema 9.1 differisce dai precedenti problemi sullo sci trattati dall'autore (magari con collaboratori) per la presenza in esso dell'attrito, resistenza dell'aria e vincoli unilaterali. Per gli stessi motivi l'attuale sistema Σ non può riguardarsi come un caso speciale di qualche sistema olonomo a cui l'autore ha applicato la teoria dei controlli (eventualmente con collaboratori). In relazione a ciò, invece di risolvere il Problema 9.1 con questa teoria (principio di massimo di Pontriagin), conviene associargli il seguente:

PROBLEMA 6.1. Dato $T > 0$ e le condizioni iniziali at $t = 0$, come massimizzare la lunghezza $\xi(T) - \xi(0)$ del tratto di pista «percorso» da C nel tempo T ?

Si trovano ∞^∞ soluzioni di questo problema in $C^1 \cap PC^2$, ossia ben più regolari delle soluzioni (in L^1) assicurate dal più noto teorema di esistenza in teoria dei controlli (se applicabile). Infine si mostra che le soluzioni dei Problemi 6.1 e 9.1 sono le stesse per T e $\bar{\xi}$ legati da una certa relazione. I valori ottimali di $\xi(T)$ e $\bar{\tau}$ nei suddetti problemi possono esprimersi mediante i dati, indipendentemente dai moti ottimali di Σ . Si considerano varie proprietà di questi moti. Alcune di esse mostrano che in genere lo sciatore può influire ben poco sui valori di $\xi(T)$ e $\bar{\tau}$.

La Parte 3 riguarda i moti di Σ senza salti, ossia i più comuni. Tra l'altro, riferendosi a questi, si limita superiormente la suddetta piccola influenza dello sciatore. Inoltre per ogni $t \in [0, T[$ si danno condizioni necessarie e sufficienti per l'estendibilità di un moto senza salti in $[0, t]$ ad un tale moto in $[0, T]$. Si mostra, tra l'altro, che ciò può essere utile per realizzare lo sciatore U come un robot, al fine di confrontare moti reali di Σ_R , largamente arbitrari ma senza salti, con i corrispondenti moti di Σ considerati nella teoria sviluppata nella Parte 2, ove si assume $m\mathcal{R} \equiv 0$; l'artificio suaccennato permette di riferire il detto confronto a casi reali interessanti in cui $m\mathcal{R}$ sia approssimativamente costante e magari grande.

1. INTRODUCTION

The main aim of this work, divided in three parts, is to study a holonomic system $\Sigma = A \cup U$, of mass m and center of mass C , (consisting of or) schematizing a pair of skis A and its user U . We regard Σ as descending along a straight-line l of maximum steep in a plane ski-run, which forms the angle $\theta \in (0, \pi/2)$ with horizontal planes. The static and kinetic coefficients f_s and f_d of dry friction, between an element of the ski-run and one of the skis, are supposed to be independent of these elements; and the air resistance, of resultant $m\mathcal{R}$, is mainly assumed to have a simple form – see (2.8) – or even to be negligible. This however is useful, partly through a device – see A4.5 and above Remark 4.3 –, in certain practically interesting situations for the real skis-skier system Σ_R schematized by Σ ; and the major part of this work is devoted to the analysis of these cases.

(A) Air resistance and also dry friction are considered in [15, 18] respectively for ski-problems different from those in this work ⁽¹⁾ Differences of the present work from others concerning skis or wide classes of mechanical systems somehow including a skis-skier system similar to the above $\Sigma = A \cup U$, are mentioned in (C) to (D) below.

Various movements of the skier w.r.t. his skis are admitted, which may also cause jumps; sometimes the skier is considered to make steps with his skis, maintaining every ski both parallel with l and effectively sliding when it is touching the ski-run.

Calling T 's downward unit vector and \mathbf{n} the upward one normal to the ski-run, we use C 's coordinates ξ and η in a frame $OT\mathbf{n}$, where O belongs to l and is regarded as a starting point. Then we consider some reasonable initial conditions – see (3.6-7) – for Σ , and in particular for the motion $(\xi(\cdot), \eta(\cdot))$ of C in the time interval $[0, T]$. Thus Σ can be regarded as an holonomic system having an unspecified number n (≥ 4) of degrees of freedom and subjected to constraints that are partly one-sided

⁽¹⁾ In [15] one wants to optimize the angle of attack of the body for a ski jumper along the free-flight phase. In [18] the inrun phase is also considered; along it the possible movements of the skier and hence the control functions that are interesting in connection with friction, are different from those in the problems dealt with here; these problems are related to alpine ski.

and (by the afore-mentioned movements) also time-dependent. In fact, given (t, ξ, η) , briefly speaking the skier can determine Σ 's intrinsic configuration and hence the values of various variables which thus can be regarded as controls. However Σ 's dynamic equations can be written in a suitable form – see *e.g.* (3.1) – involving directly only one of these variables.

For Σ we consider especially symmetric motions w.r.t. the plane (l, C) , but we also deal with more general motions, along which *e.g.* steps can be made. Calling F_1 [F_2] U 's left [right] foot, the relevant scalar variables, besides ξ and η , are s_i , u_i , and w_i where $F_i C = (s_i + w_i)\mathbf{T} + u_i \mathbf{n}$ ($i = 1, 2$). Along, *e.g.*, symmetric motions $s_i = s$, $u_i = u$, and $w_i = w$ ($i = 1, 2$).

The variables u_i and w_i ($i = 1, 2$) are used as controls (implementable by U 's muscles) and are subjected to suitable constraints – see (2.10-14). However, in any case, after suitable definitions of s , u , and w – see (2.5) –, the variables directly involved by Σ 's dynamic equations are only ξ , η , and u ; for $i \in \{1, 2\}$ u_i and w_i affect u 's constraints, while s_i is involved in some qualitative requirement – *e.g.* the sliding condition A2.4 – for Σ 's admissible motions – see Definition 3.1; and besides satisfying the reasonable initial conditions hinted at above, these motions are in $C^1 \cap PC^2$. In spite of this regularity, the use of them is sufficient for our aims, and in particular to solve the optimization problems treated in Part 2.

Four versions of the air resistance law for Σ , A2.5,1 to A2.5,4, of decreasing generality degree are considered; the first two only for discussions, except that the version A5.1 of A2.5,2 is used to treat steps with sliding skis in Section 5.

In Part 1, after the afore-mentioned preliminaries, the first main result of this work can be stated on the basis of the linear law A2.5,3 including (2.8): the validity of a control-free first integral, (4.1), which implies that along every admissible motion of Σ , C 's motion $(\xi(\cdot), \eta(\cdot))$ is affected by the skier's behaviour only in that, for all $t \in [0, T]$, $\dot{\xi}(t)$ is determined by $\xi(t)$, $\eta(t)$, and $\dot{\eta}(t)$ – see Remark 4.1(b).

Furthermore in the case of negligible air resistance ($\mathcal{R} \equiv 0$), asserted by A2.5,4, the afore-mentioned first-integral can be further integrated, obtaining a control-free and derivative-free relation, (4.6). This relation renders it intuitively clear that the skier's behaviour has a small influence on the motion $\xi(\cdot)$ of C 's projection on the ski-run – see Remark 4.1(c)-(d) – in spite of the fact that by raising and lowering C the skier affects $\dot{\xi}$ because of dry friction.

E.g., between (2.8) and (2.9) and in the last parts of Sections 1, 2, and 4 one clarifies semi-intuitively at which extent in certain interesting cases some results rigorously obtained for some simple versions of the holonomic system Σ hold for the real skis-skier system Σ_R ; this extent appears good except in some cases considered at the end of Section 2. In particular by A4.5 one substantially notes that the addition of a constant value to the linear air resistance law (2.8) (in A2.5,3) is equivalent to a certain reduction of l 's steepness angle θ . This simple theorem suggests a useful device for obtaining some reliable informations on Σ_R from theorems based on A2.5,4 ($\mathcal{R} \equiv 0$) in certain practically interesting cases – see above Remark 4.3 and assertion (α) in this.

Part 1 ends with an application of its main result to a question concerning steps with sliding skies – see Section 5: briefly, for $\mathcal{R} \equiv 0$ on the one hand by some suitable such steps the total negative work W_{st} of dry friction in $[0, T]$ can be rendered arbitrarily small, and practically much smaller than along a rigid motion of Σ (under similar conditions); on the other hand these steps can be shown to be completely useless in order to, *e.g.*, increase the length $\xi(T) - \xi(0)$ of the ski-run covered by Σ in $[0, T]$. This fact is also discussed in connection with a law of air resistance, A5.1, more general than A2.5,2 and especially than those used to state the afore-mentioned main result of Part 1.

In Part 2, assuming the air resistance negligible ($\mathcal{R} \equiv 0$), two optimization problems on Σ are considered and solved on the basis of the afore-mentioned control-free and derivative-free relation (4.6). The second, whose solution is the main result of Part 2, briefly reads as follows.

PROBLEM 9.1. Given $\bar{\xi} > 0$, how can the skier behave in order to minimize the time $\bar{t} (> 0)$ for which $\xi(\bar{t}) = \bar{\xi}$ along an admissible motion \mathcal{M} of Σ in $[0, T]$ under given initial conditions (being $\mathcal{R} \equiv 0$, while T is sufficiently large)?

(B) This is similar to usual problems of minimum time in control theory; but it has some peculiar features, including the presence of some one-sided mechanical constraints.

(C) In particular the mechanical system Σ referred to in Problem 9.1 differs much from those considered in A. Bressan's paper on skis [6] and from the systems similar to a swing referred to in [10, 11] or Piccoli's paper [17]. In fact in the latter four papers (i) $\Sigma = A \cup U$ is assumed to be a holonomic system with two-sided constraints and with only two Lagrangian coordinates, s and u , the second of which is used as a control, (ii) applied external forces reduce to weights, and (iii) dry friction and air resistance are absent.

(D) Let us add to (A)-(C) that, since in the present work jumps and hence one-sided constraints are essentially considered (through *e.g.* steps), here Σ cannot be regarded as a special choice of any Lagrangian system with $1 + n$ scalar coordinates, s and $u = (u_1, \dots, u_n)$, treated in various other papers of A. Bressan possibly with *alii* – see [3-5, 8-9, 12] ⁽²⁾

As a counterpart of the above considerations, while in the above papers Pontryagin's maximum principle is used to solve some optimization problems, so far it appears convenient not to solve Problem 9.1 by using that principle or usual control theory. Instead here, first, another optimization problem is preliminarily considered: Problem 6.1, which briefly reads as follows.

(2) The skis-skier system dealt with in [6] is a special choice of the system considered in [5]. Incidentally, by the suitable choice of coordinates made in [6], the theory proposed in [5] (and initiated by the papers [3, 4] which have in part the character of an abstract), strictly speaking, is unnecessary to read [6]. However the afore-mentioned choice was suggested to the author just by [5].

PROBLEM 6.1. Given the instant $T > 0$, how can U behave in order to maximize $\xi(T)$ along an admissible motion in $[0, T]$, under given initial conditions, in the case $\mathcal{R} \equiv 0$?

Furthermore this problem is solved directly; and it is shown – see Theorem A7.1 – to have ∞^∞ solutions (in $C^1 \cap PC^2$, hence) more regular than the solutions assured (in L^1) by the main existence theorem of control theory ⁽³⁾

Lastly one shows that the above two problems have the same solutions when $\bar{\xi}$ and T are mutually related in a certain way – see Theorem A9.1.

Let us also mention the following facts concerning Part 2.

(E) In order to frame the existence theorems for Problem 6.1 and Problem 9.1, to every admissible motion \mathcal{M} of Σ in $[0, T]$ and every $\tau \in [0, T]$ one associates the motion $(\bar{\xi}_\tau(\cdot), \bar{\eta}_\tau(\cdot))$, of a freely falling mass point M_τ , that is tangent to C 's motion, at the instant τ – see (6.2); and for $\mathcal{R} \equiv 0$ various properties of these motions are shown in connection with either \mathcal{M} or a quadratic form, $\sigma(\cdot)$, which is basilar in the control- and derivative-free relation (4.6) – see *e.g.* (6.3) and A6.2.

(F) By the afore-mentioned relation, (4.6), the optimal values of $\xi(T)$ and \bar{z} , to be implemented according to Problems 6.1 and 9.1 respectively, can be expressed in simple ways not involving controls – see (7.2) and (7.2'), or A9.1 and (9.2-4) respectively.

(G) One shows – see A7.1(a) and thesis (b_3) in A7.1(b) – that in connection with \mathcal{M} – see (E) – there is a last instant $\tau \in]0, T]$ at which the skier U can turn \mathcal{M} into a solution to Problem 6.1.

Part 3 deals mainly with (admissible) motions that are jump-free (*i.e.* with $u(t) = \eta(t)$) in a last part $[t_1, T]$ of $[0, T]$ ($t_1 \in]0, T[$). Since these motions are the most common in practice ⁽⁴⁾ after some preliminaries set in Section 10, one states, *e.g.*,

(i) some conditions necessarily holding for them at every instant $t \in [t_1, T[$ – see A11.1(a)-(b),

(ii) a condition on the data $\eta(0)$ and $\dot{\eta}(0)$ necessary and sufficient for the skier to be able to implement an (admissible) jump-free motion in $[0, T]$ – see A11.1(c),

(iii) an upper bound, (11.5), valid along any jump-free motion, for the last instant τ considered in Part 2 and mentioned in (C) (which specifies how small $T - \tau$ is in practical cases); and briefly speaking

⁽³⁾ In spite of being in $C^1 \cap PC^2$, some motions of Σ in $[0, T]$ may have a «final» impulse $\mathbf{I} \neq 0$ – see (8.14) –, so that every extension of it beyond T is at most in $C^0 \cap PC^1 \cap PC^2$. This holds for all solutions to Problem 6.1 in an exceptional case; however in practical cases there are ∞^∞ solutions to Problem 6.1 having some extensions in $C^1 \cap PC^2$ again; this occurs for the solutions meeting a certain condition, precisely (7.3), which incidentally is considered for other purposes – see Remark 8.2(d)-(f) and above A7.1.

⁽⁴⁾ In fact, the book [14] for ski teachers advises skiers to jump as seldom as possible (on p. 210). Furthermore jumps are considered there only on non-plane parts of ski-runs (on pp. 209-211).

(iv) some conditions on $t \in]0, T[$, $\eta(t)$, and $\dot{\eta}(t)$ that enable the skier to extend any jump-free motion in $[0, t]$ to such a motion in $[0, T]$, or to a jump-free solution of Problem 6.1 – see A11.1(e).

The result (iv) is useful to implement Σ as a robot for this purpose: to detect, along jump-free motions, the errors of the theory presented in Part 2 due to, *e.g.*, its complete disregard of air resistance – see Remark 11.2(c)-(d).

After some comments on Theorem A11.1 – see Remarks 11.1 and 11.2 related to Problem 6.1 – and the proof of this theorem in Section 12, some of those comments are briefly extended to Problem 9.1 in Section 13.

* * *

In the present work some problems concerning the real system Σ_R are treated by using some versions of Σ subjected to A2.5,3 or A2.5,4. This scheme is obviously approximate with regard to both constraints and applied forces. Hence one tries to show how much in various real cases our results based on these versions are useful for those problems; in more details one aims at clarifying:

(i) *to which extent our results on some rigorously specified Σ 's versions keep holding for Σ_R , in some common situations, and especially*

(ii) *to which stronger extent some of the above results keep holding for Σ_R in certain situations having a practical interest.*

To reach the above aims and to compare this paper with other similar works, it is convenient to mention some well known applications of mathematical physics to the real world such as

(H) *the applications of mechanical similitude to biology in [1, II, 2] and*

(I) *ballistics*. For this let us specify that in the sequel we regard that

(I₁) *early ballistics* schematizes a gun projectile, p , as a heavy mass point p_M subjected to negligible air resistance ($\mathcal{R} \equiv 0$); and its main results are still taught in some university courses, which *e.g.* helps clarifying the respective influences of various situations on p 's motion;

(I₂) *main outer ballistics* schematizes p much better, by still using p_M but assuming that $m\mathcal{R} = -f(v)$ *versus* \mathbf{v} , where \mathbf{v} is p_M 's velocity and $f(0) = 0 < f'(v)$ for $v > 0$; and

(I₃) *refined outer ballistics* treats p as a special rigid body, or it considers either centrifugal forces or the spatial dependence of the gravity acceleration \mathbf{g} ⁽⁵⁾

⁽⁵⁾ (a) The fields of main and refined outer ballistics are formed, in usual terminology, by the main problem and secondary problems respectively of outer ballistics - see [16, II, 14, pp. 107-134].

(b) In [16, II, 14, pp. 107-108] one notes that if, for $\mathcal{R} \equiv 0$, p_M 's initial velocity v_0 is 625 m/s, then the maximum range R equals 40 km and the corresponding values for the projection angle α and the maximum

(iii) *The treatments presented in the present work are qualitatively more similar to those in ballistics than to the treatments in (H); however, incidentally, a property of this work partly shared by (H) is mentioned in Remark 2.4.*

Furthermore, our results on Σ with $\mathcal{R} \equiv 0$, in particular the control- and derivative-free second integral (4.6) and the results obtained in Part 2 and Part 3, are a priori only (at least) as useful for reliable informations on Σ_R as is early ballistics in connection with projectiles shot by (very early) guns (skiers are slower than projectiles – see footnote 5b). However certain special situations have a relevant practical interest, on the basis of these facts.

(K) *For Σ_R only jump-free motions are regarded as interesting – see footnote 4.*

(J) *In many ski-races of giant slalom the skis' speed is nearly constant, except in a brief initial part, provided the ski-run has a rather uniform steepness.*

Furthermore for Σ_R in the same situations, in my opinion the above information are «good», I mean roughly as reliable as main outer ballistics – see (I_2) –, at least provided one uses the device based on A4.5, mentioned above Remark 4.3, and precisely expressed by assertion (β) in this.

(L) *Conventions about notations.* (a) If a formula (r.s) has an upper (or first) line and a lower (or second line), then we denote these lines by $(r.s)^+$ and $(r.s)^-$ respectively.

(b) About labelling relations in formulas we note that, e.g., $(3.3)_3^+$ and $(3.3)_2^-$ are the relations $\ddot{u} \geq -g \cos \theta$ and $\ddot{\eta} = -g \cos \theta$ respectively. Furthermore, e.g. $(8.6)_2$ and $(8.6)_4$ are the inequality relation and the membership relation in (8.6). Obviously quantifiers, such as « $\forall t \in [0, T]$ » in (4.6), or expressions like «a.e. in \mathcal{N}_V » in (3.3) do not affect the labelling.

PART 1. A SKIS-SKIER HOLONOMIC SYSTEM Σ AND SOME CONTROL-FREE FIRST INTEGRALS FOR IT

2. SOME BASIC ASSUMPTIONS, DYNAMIC EQUATIONS, AND CONTROL PROPERTIES.

PRELIMINARIES TOWARDS MAINLY PRACTICAL REAL APPLICATIONS

We consider a Cartesian frame $O\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3$ for which the gravity acceleration is $\mathbf{g} = -g\mathbf{c}_2$; and we represent the straight line l , which contains the trajectory of the pair of skis A – see Section 1 –, by

$$(2.1) \quad P = P(s) = O + \mathbf{T}s, \quad \text{where } \mathbf{T} = \mathbf{c}_1 \cos \theta - \mathbf{c}_2 \sin \theta, \quad \theta \in (0, \pi/2).$$

A common situation, holding with tolerable approximation in many cases treated in this work, is that

height h are 45° and 10 km respectively. Instead, under the same initial conditions, the analogues for p_M of R , α , and h are (roughly) 3 km, 32° , and 0,5 km. Incidentally thus the influence of air resistance on p 's motion is clarified.

A2.1 (*symmetry*). The vertical plane $(O, \mathbf{T}, \mathbf{n})$ through l always is of material symmetry for Σ ⁽⁶⁾ \square

Jumps with skies are allowed provided

A2.2 (*skis' parallelism*). The skis always remain parallel with l . \square

Let A_1 and A_2 be U 's left and right ski respectively; and let A_i 's center of mass be the position F_i of U 's foot using A_i ($i = 1, 2$). Then in the case A2.1-2, F_1 and F_2 coincide with one point, F ; and we assume that $P = P(s)$ is F 's orthogonal projection on l . In the same case the equalities

$$(2.2) \quad \mathbf{n} = \mathbf{c}_1 \sin \theta + \mathbf{c}_2 \cos \theta, \quad OC = \xi \mathbf{T} + \eta \mathbf{n}, \quad PF = v \mathbf{n}, \quad FC = w \mathbf{T} + u \mathbf{n}$$

can define the quantities \mathbf{n} , η , u , v , and w (in part implicitly) in terms of Σ 's center of mass C , P , and F ⁽⁷⁾ Then (being $PC = PF + FC$) by (2.1)₁₋₂, i.e., the first two relations in (1.2) (written at the left of «where»), we have that

$$(2.3) \quad C = P + w \mathbf{T} + (u + v) \mathbf{n}, \quad \xi = s + w, \quad \eta = u + v.$$

To deal with Σ 's general motions in $[0, T]$ ($T > 0$), we note that, disregarding A2.1, (2.2)₃₋₄, and (2.3) but remembering (2.1) and (2.2)₁₋₂, we can define s_i , u_i , v_i , and w_i by

$$(2.4) \quad OF_i = s_i \mathbf{T} + v_i \mathbf{n}, \quad F_i C = w_i \mathbf{T} + u_i \mathbf{n} \quad (i = 1, 2);$$

hence by (2.2)₂ (being $OC = OF_i + F_i C$)

$$(2.3') \quad \xi = s_i + w_i, \quad \eta = u_i + v_i, \quad v_i \geq 0 \quad (i = 1, 2);$$

and we can determine s , w , u , and v by

$$(2.5) \quad 2s = s_1 + s_2, \quad 2w = w_1 + w_2, \quad u = \max\{u_1, u_2\}, \quad v = \min\{v_1, v_2\} \geq 0.$$

Then F , regarded as defined by (2.1) and (2.2)₃, may be strictly below the straight line $F_1 F_2$; however from (2.3') and (2.5) we deduce (2.3)₂₋₃ ⁽⁸⁾ which imply (2.3)₁ by (2.1) and (2.2)₂. Lastly (2.2)₄ follows from (2.3)₁ and (2.2)₃. Thus

A2.3. All equalities (2.2-3) hold for general motions of Σ , possibly incompatible with A2.1. \square

In any case, for $i = 1, 2$, Φ_i shall denote the resultant of the ski-run's reactions on the ski A_i when $v_i = 0$, and the zero vector $\mathbf{0}$ otherwise ($v_i > 0$); furthermore we set $\Phi = \Phi_1 + \Phi_2$, so that Φ is the analogous resultant for Σ [is $\mathbf{0}$] when $v = 0$ [$v > 0$].

⁽⁶⁾ In this work we regard Ar.s as the s -th assertion in section r; and it may be a theorem or a condition assumed in the sequel always or only sometimes.

⁽⁷⁾ One can regard (2.2) as a redefinition, because the same quantities \mathbf{n} to w were introduced in Section 1 in a slightly different way.

⁽⁸⁾ Indeed (2.3')₁ and (2.5)₁₋₂ imply (2.3)₂. Furthermore, in e.g. the case $u_1 \geq u_2$, (2.3')₂ and (2.5)₃₋₄ imply that $v_1 \leq v_2$, $u = u_1$, and $v = v_1$. Then (2.3')₂ for $i = 1$ yields (2.3)₃. In case $u_1 \leq u_2$, (2.3)₃ holds by an analogous reasoning.

Unless otherwise noted, we shall regard as valid that, as was hinted at in Section 1,

A2.4 (*sliding condition*). $\dot{s}_i > 0$ a.e. in $\mathcal{N}_{\Phi_i}^c \doteq \{t \in [0, T]; \Phi_i(t) \neq 0\}$ ($i = 1, 2$), which (with obvious notations) implies that $\dot{s} > 0$ a.e. in $\mathcal{N}_{\Phi}^c (= \mathcal{N}_{\Phi_1}^c + \mathcal{N}_{\Phi_2}^c)$ when A2.1 holds ⁽⁹⁾ \square

This condition is considered because, on the one hand, in alpine skiing it is generally impossible to stop Σ_R ; and on the other hand, when A2.4 fails to hold, our dynamic equations for Σ are essentially modified ⁽¹⁰⁾ so that our main results (4.1) and (4.6) are no longer holding – see Remark 3.3(c).

From Section 1 let us remember that *in this work the kinetic coefficient f_d of dry friction is always assumed to be constant (as well as the static one, f_s)* and that $m\mathcal{R}$ denotes the resultant of the air resistance acting on Σ . Then along any motion of Σ , symmetric or not and possibly with jumps,

$$(2.6) \quad \Phi = \Phi_T \mathbf{T} + \Phi_n \mathbf{n} \quad \text{with} \quad \Phi_n \geq 0, \quad \Phi_T = -f_d \Phi_n \quad (\text{a.e.}).$$

Consequently *the first balance equation for Σ reads*

$$(2.7) \quad m\ddot{C} = m\mathbf{g} + (\mathbf{n} - f_d \mathbf{T})\Phi_n + m\mathcal{R}, \quad \text{where} \quad \Phi_n = 0 \quad \text{if} \quad v > 0.$$

Denoting U 's configuration w.r.t. the frame $(C, \mathbf{T}, \mathbf{n})$ by \mathcal{C}_U , for $h = 1$ to 4 we consider the following law for Σ , whose generality degree decreases with h .

A2.5, h . For some continuous function $f(\dot{\xi}, \mathcal{C}_U)$,

(i) $\mathcal{R} = \mathcal{R}_T \mathbf{T}$ where

(ii) $\mathcal{R}_T = f(\dot{\xi}, \mathcal{C}_U)$ and the h -th of conditions (A_1) to (A_4) below holds.

(A_1) For every choice of \mathcal{C}_U , setting $f(\cdot) \doteq f(\cdot, \mathcal{C}_U)$ we have that

(iii) $f(\cdot) \in C^2(\mathcal{R}^+, \mathcal{R}^+)$, $f(\mathcal{R}^+) = \mathcal{R}^+$, $f'(\dot{\xi}) > 0$ ($\forall \dot{\xi} > 0$).

(A_2) The function $f(\cdot) = f(\cdot, \mathcal{C}_U)$ satisfying (iii) is independent of \mathcal{C}_U .

(A_3) For some constants k and \bar{k} (i) holds with

$$(2.8) \quad \mathcal{R}_T = -k\dot{\xi} + \bar{k} \quad (\forall \dot{\xi} > 0); \quad k \geq 0, \quad \bar{k} \geq 0; \quad \bar{k} = 0 \quad \text{if} \quad k = 0.$$

(A_4) $\mathcal{R}_T \equiv 0$ ($\equiv \mathcal{R}$). \square

In the assumption below a convention is included.

A2.5. For some $h \in \{1, \dots, 4\}$ Σ is subjected to the law A2.5, h ; and in this case it can also be denoted by Σ_h .

* * *

⁽⁹⁾ This implication may fail to hold along general motions. In fact (e.g. making backwards steps with sliding skies) in some short time interval we can have $0 < \dot{s}_1 < -\dot{s}_2$, $v_1 = 0 < v_2$, $\Phi_1 > 0$; then A2.4 is complied with but $\dot{s} < 0 = v$.

⁽¹⁰⁾ Indeed, if $\dot{\xi} = 0$, then the upper part of (3.1)₁ has to be replaced by $m(\ddot{\xi} - g \sin \theta - \mathcal{R}_T) = \Phi_T$ where $-f_d \Phi_n \leq \Phi_T \leq f_d \Phi_n$, ($\dot{\xi} = 0$).

Of course, for $h = 1$ to 4 , assuming Σ subjected to the law A2.5, h renders this holonomic system an approximate model of the real skis-skier system Σ_R , having various degrees of tolerableness in various situations; and in each of these the degree decreases with h , while the simplicity increases.

For instance the situations (S_1) to (S_2) below will always be presupposed in this work.

(S_1) *The lift $m\mathcal{R} \cdot \mathbf{n}$ is negligible for Σ_R , which occurs along many motions and especially for $\dot{\xi}$ not too high.* \square

In this situation the assumption

(iv) $\mathcal{R} \cdot \mathbf{n} = 0$ on Σ is roughly as tolerable as main outer ballistics, in that this theory treats projectiles as mass points instead of suitably shaped rigid bodies – see (I_2) and (I_3) in Section 1.

(S_2) *Winds and the (occasionally and variably present) asymmetries of Σ_R w.r.t. the vertical plane through the trajectory l can be neglected, roughly at the same tolerableness degree; and this appears to occur commonly, as far as I know, likely because the effects of those asymmetries are practically destroyed by the ski-run's reactions.* \square

Thus, for Σ_R in the situations (S_1) to (S_2) , \mathcal{R} 's parallelism to the plane (C, T, \mathbf{n}) has the above tolerableness degree. Then, by (iv),

(v) the same holds for condition (i) in A2.5, h ($h = 1$ to 4) and hence for A2.5,1.

Now we first note that the observed fact (J) – see near Section 1's end – refers to a (racing) situation in which for Σ_R we have that

(S_3) at practically every time t , \mathcal{C}_U renders Σ_R 's aerodynamic properties very near their best level. \square

Second, (J)'s analogue (J') for a touring situation also holds, and the corresponding analogue (S'_3) of (S_3) arises from (S_3) by changing «best level» with «best level obtained by U 's touring configurations». Therefore, by (v) and the continuity of $f(\cdot) \doteq f(\cdot, \mathcal{C}_U)$ w.r.t. \mathcal{C}_U ,

(vi) in the situations (S_1) , (S_2) , and either (S_3) or (S'_3) , the law A2.5,2 – besides A2.5,1 – is well tolerable; and since U 's touring configurations are more comfortable than racing ones,

(vii) the touring value of $|\mathcal{R}_T| = f(\dot{\xi})$ is lower than the racing one $\forall \dot{\xi} > 0$.

In connection with A2.5,2 let us note that when the function $f(\cdot)$ mentioned in (A_2) is known, for $\dot{\xi}$ in some neighborhood N_V of an arbitrarily fixed value V of $\dot{\xi}$, Taylor's formula affords the approximate expression (2.8) for \mathcal{R}_T , with $k = f'(V)$ and $\bar{k} = f'(V)V - f(V)$. In particular

(viii) if V is small, e.g. ≤ 10 km/h [large, e.g. between 20 and 200 km/h], then, for $\dot{\xi}$ near V , $f(\dot{\xi}) \cong c\dot{\xi}$ [$f(\dot{\xi}) \cong c\dot{\xi}^2$] where c (> 0) is constant. Hence the law (2.8) becomes $\mathcal{R}_T = -c\dot{\xi}$ [$\mathcal{R}_T = -cV(2\dot{\xi} - V)$ (< 0 for $\dot{\xi} > V/2$)].

Lastly it is substantially well known that

(ix) if $\mathcal{R} = \mathcal{R}_T \mathbf{T}$ and either (A_1) or (A_2) holds, then there is a unique value V_∞ of $\dot{\xi}$ (> 0) to which, for U in a (prefixed) constant configuration \mathcal{C}_U and for $u \equiv \eta$, $\dot{\xi}(t)$ tends as t tends to $+\infty$; hence V_∞ is the (positive) stationary value $V^* = V_{\mathcal{C}_U}^*$ of $\dot{\xi}$ relative to the configuration \mathcal{C}_U ; and in the case (A_2) $V^*(= V_\infty)$ is independent of \mathcal{C}_U ⁽¹¹⁾

Furthermore, if V is very close to V^* ($> V$) and, e.g., $\dot{\xi}(0) = V$, then by some skier's behaviour, rather periodic (or with \dot{u} bounded) and occurring naturally in various practical cases, $\dot{\xi}(t) \in N_V \forall t \in [0, T]$ for some small choice of N_V (of diameter not much larger than $2|V - V^*|$).

Incidentally the above considerations on the one hand explain in part the fact (J) – see at the end of Section 1; and on the other hand they specify it by adding that

(x) the skier's speed always belongs to some small neighborhood N_{V^*} of the above stationary value V^* for Σ (subjected to A2.5,2).

The fact (J) [(J')] includes the travel of a ski-racer [ski-tourist] between two (non-initial) consecutive gates; and it practically shows that this travel is very similar to Σ_R 's motion in the first [second] of the following situations.

(S₄) [(S'₄)] Σ'_R is travelling in racing- [tourist-] conditions along a (not too long) initial part Δl of the ski-run l with $\dot{\xi}_0 = \dot{\xi}(0)$ very near V^* . \square

Of course, by (v),

(xi) any touring value of V^* is lower than the racing one ⁽¹²⁾

Since the neighborhood N_{V^*} mentioned in (x) is small,

(xii) for Σ_R in either the situations (S₁) to (S₄) or their touring analogues (S₁), (S₂), (S'₃), and (S'₄) – so that jumps are absent by (K) in Section 1 –, the use of Σ_3 with e.g. the choice $k = f'(V^*)$ and $\bar{k} = kV - f(V)$ for the constants in (2.8), is well tolerable (roughly as much as main outer ballistics).

Thus an important special case of good tolerableness for the law A2.5,3 is pointed out. This fact and the device considered above Remark 4.3 will allow us to use many results of this work, based on the law A2.5,4 – e.g. the results in Part 2 and Part 3 –, to give good information on Σ_R in the same case.

⁽¹¹⁾ To hint at a proof of (ix), let us note that, by A3.2, the relations (3.1-3.3) below follow from only the first equality (i) in any among the laws A2.5,1 to A2.5,4; hence they follow from the assumption (α) $\mathcal{R} = -f(\dot{\xi})\mathbf{T}$. Then, by (3.3)₁, (β) $\ddot{\xi} = B - f(\dot{\xi}) - f_d \ddot{u}$ a.e. in N_v , i.e. for $v \doteq \eta - u = 0$. Assume that \mathcal{C}_U is constant and $\dot{\xi} (> 0)$ too, so that $\ddot{u} = \ddot{\xi} = 0$. Then, by (β), (γ) $f(\dot{\xi}) = B$. By (α) the positive solution V^* of this equation in $\dot{\xi}$ is unique and obviously dependent on \mathcal{C}_U . Now with usual methods one can prove that V^* is $\dot{\xi}$'s limit value V_∞ in $[0, \infty[$ for every constant \mathcal{C}_U and for $v \equiv 0$.

⁽¹²⁾ In fact the racing- [touring-] value of V^* is the maximum of $V_{\mathcal{C}_U}^*$ – see (ix) below (2.8) – for all configurations \mathcal{C}_U [all touring-configurations \mathcal{C}_U] of U .

REMARK 2.1. (a) Some recent experiments have improved Coulomb's dry friction laws: e.g., along Σ_R 's symmetric motions f_d depends on $\dot{\xi}$ and for $|\dot{\xi}|$ large we may have $f_d > f_s$ – see e.g. [13, p. 6057].

(b) However, when Σ_R is in the situations (S_1) to (S_4) , it can still be schematized by Σ (subjected to A2.5,3 or A2.5,4) taking the new facts into account, by simply identifying f_d 's constant value for Σ with the value that f_d takes for Σ_R when $|\dot{\xi}|$ equals V^* . \square

* * *

In connection with (2.8) we stress that in the symmetric [general] case (considered here) the variables u and w [u_i and w_i ($i = 1, 2$)] have a control character in that, roughly speaking, the skier U can implement the functions

$$(2.9) \quad u = u(t), \quad w = w(t) \quad [u_i = u_i(t), w_i = w_i(t) (i = 1, 2)] \quad \forall t \in [0, T]$$

by means of his muscles, arbitrarily within certain bounds.

Obviously *motions with $u_1 \equiv u_2$ i.e., with U 's feet always at the same height, properly include symmetric motions.*

It is reasonable to assume that

A2.6 (*partial constraints*). For some constants U_i , W_i , U'_i , and W'_i , ($i = 1, 2$) with

$$(2.10) \quad 0 < U_1 < U_2 < U'_2, \quad W'_1 < W_1 < 0 < W_2 < W'_2,$$

(i) at any instant, if $u_1 = u_2$ holds along Σ 's motion, then (u, w) – see (2.5) – satisfies the first two among the conditions

$$(2.11) \quad u \in [U_1, U_2], \quad w \in [W_1, W_2]; \quad u_i \in [U_1, U'_2], \quad w_i \in [W'_1, W'_2] \quad (i = 1, 2),$$

while the last two hold in any case; furthermore

(ii) the reduction of any segment written in (2.11) to a proper subset of it would render (i) false ⁽¹³⁾ \square

Furthermore the conditions (2.11) must be regarded as consequences of some stronger and more complex constraints acting on Σ . Before writing the assumption A2.7 on them, let us say that every motion \mathcal{M} of Σ in $[0, T]$ determines or *induces* the 4-tuple $\sigma_4 = (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$ and the 6-tuple $\sigma_6 = (\xi(\cdot), \eta(\cdot), u_1(\cdot), w_1(\cdot), u_2(\cdot), w_2(\cdot))$ of functions, which express C 's motion along \mathcal{M} and how the above variables u , w , u_i and w_i ($i = 1, 2$) vary along \mathcal{M} . The induced σ_4 and σ_6 satisfy the relations (2.5)₂₋₃ pointwise on $[0, T]$:

$$(2.5') \quad u(t) = \max\{u_1(t), u_2(t)\}, \quad 2w(t) = w_1(t) + w_2(t) \quad \forall t \in [0, T].$$

⁽¹³⁾ Let U be steady in his most upright position, and with $v_1 = v_2 = 0$ – see (2.4). Thus, by (2.3')₂ and (2.5)₃₋₄, $v = 0$ and $u = u_1 = u_2 = U_2$. Now let U raise his foot F_2 as much as possible, so that C also raises; then by (2.3')₂ (and the experience of skiers), $0 = v = v_1 < v_2$, $u = u_1 = U'_2 > U_1$, and $u_2 \in]U_1, U_2[$. Something similar can be said about W'_1 and W'_2 .

DEFINITION 2.1. We shall say that $\sigma_6 \doteq (\xi(\cdot), \dots, w_2(\cdot))$ induces $\sigma_4 \doteq (\xi(\cdot), \dots, w(\cdot))$ – see above – if (2.5') holds. \square

A2.7 (*general constraints*). There are two sets $K_2 \subset \mathcal{R}^2$ and $K_4 \subset \mathcal{R}^4$ that are compact, convex, the closure of their own interiors, and such that, first,

$$(2.12) \quad \{U_1\} \times [W'_1, W'_2] \cup [U_1, U'_2] \times \{0\} \subset K_2, \quad K_2 \times K_2 \subset K_4, \quad \partial K_2 \in C^2, \quad \partial K_4 \in C^2,$$

second, under the conditions (2.5)₂₋₃ the implication

$$(2.13) \quad (u_1, w_1, u_1, w_2) \in K_4 \implies (u, w) \in K_2$$

holds, third,

(i) the inclusions

$$(2.14) \quad (u, w) \in K_2, \quad (u_1, w_1, u_2, w_2) \in K_4$$

constitute the control constraints for Σ 's motions with $u_1 \equiv u_2$ and for general ones respectively (in the sense specified in Remark 2.2(b) below), and fourth,

(ii) K_4 is invariant under the transformation $(a, b, c, d) \rightarrow (c, d, a, b)$. \square

REMARK 2.2. (a) We can consider only choices of Σ 's motion \mathcal{M} that induce some r -tuple σ_r formed by functions in $C^1 \cap PC^2$ ($r = 2, 4$) in that, besides being rather realistic, this allows us to find some solutions to our optimization problems. In this work only some hints to less regular choices of σ_r (with functions in $C^0 \cap PC^1 \cap PC^2$) are substantially given (in Remark 8.2 (c)).

(b) By part (a), we can mean the last assertion in A2.7 in this (idealized) sense: if the functions $u(\cdot)$ and $w(\cdot)$ [$u_i(\cdot)$ and $w_i(\cdot)$ ($i = 1, 2$)] are in $C^1 \cap PC^2$, then

(i) the skier U can implement them along a motion with $u_1 \equiv u_2$ [a general motion] iff

(ii) those functions satisfy condition (2.14)₁ [(2.14)₂] pointwise, i.e. (2.9)₁₋₂ [(2.9)₃₋₄] imply (2.14)₁ [(2.14)₂] $\forall t \in [0, T]$. \square

* * *

Below some cases different from, e.g., the situations (S₁) to (S₄) are mentioned in which Σ schematizes Σ_R or other real systems with at least a fairly good approximation.

REMARK 2.3. The results obtained in this work on the holonomic system Σ ($= \Sigma_3$ or Σ_4) subjected to A2.5,3 or A2.5,4, such as the control-free first integrals (4.1) and (4.6) as well as theorems A7.1 and A9.1, can also give fairly good information on Σ_R in the situations (S₁) to (S₂) provided they are used in connection with short enough (jump-free) motions and after having reduced the intervals $[U_1, U_2]$ to $[W'_1, W'_2]$ in (2.11) and the compact sets K_2 and K_4 in (2.16) to suitably small parts of themselves respectively. \square

REMARK 2.4. As is in part remembered above, the relations (4.1) and (4.6) on Σ , simple and basilar for this work, are obtained under rather strong simplifications:

the air resistance laws A2.5,3 and A2.5,4 respectively; furthermore most part of the work is devoted to some consequences of those relations that are interesting and that in certain admittedly special but important cases also hold for Σ_R with a rather good approximation.

By the above aspects *the present work does not aim at a very good description of Σ_R and it is rather similar to the applications of mechanical similitude to biology written in [1, II, 2]*. In fact this similitude appears to occur only roughly in the cases considered there ⁽¹⁴⁾; and this occurrence is also disturbed by some biological facts: *e.g.*, muscles get tired, when stressed near their maximum possibilities, even without doing any physical work. \square

REMARK 2.5. (a) Briefly let Σ_{CT} be the real system formed by a closed and narrow toboggan piloted by a man, U , and descending along the ski-run l . For it

(i) the analogues of the situations (S_1) to (S_3) are rigorously (and necessarily) present. Hence some choice of Σ_2 can obviously schematize Σ_{CT} well.

Furthermore, in connection with Σ_{CT} 's (jump-free) motions along which $\dot{\xi}$ always is near V^* , the same practically holds even for some choice of Σ_4 , as the afore-mentioned device based on A4.5(a) shows. In particular the information on Σ_{CT} , related to Problem 6.1 or 9.1 and thus obtained from A7.1 or A9.1, also hold with a good approximation.

(b) By (i), *e.g.*, the above informations on Σ_{CT} are expected to hold with a slightly better approximation than their analogues for Σ_R ; hence the experimental use of Σ_{CT} would be more significant than the one of Σ_R to check the present theory on Σ . \square

3. ON Σ 'S ADMISSIBLE MOTIONS AND THEIR INDUCED 4-TUPLES AND 6-TUPLES

In the optimization problems considered in this paper, for simplicity reasons one assumes, first, A2.5,4 (and in particular that $\mathcal{R} \parallel \mathbf{T}$ even along non-symmetric motions); and second, that Σ 's dynamic motions satisfy the following simplifying condition in $v(= \eta - u)$ and $v_i(= \eta_i - u_i)$ for $i = 1, 2$. This additional assumption is not strictly needed. However it simplifies proofs and is largely satisfied in applications.

A3.1 (*realistic simplification*). The sets $\mathcal{N}_{v_i} = \{t \in [0, T]; v_i(t) = 0\}$ ($i = 1, 2$), and hence \mathcal{N}_v too, are finite unions of intervals. \square

Since $\mathbf{g} = -g\mathbf{c}_2$, by $(2.1)_3$ and $(2.2)_1$ $\mathbf{g} \cdot \mathbf{T} = g \sin \theta$ and $\mathbf{g} \cdot \mathbf{n} = -g \cos \theta$. Hence by $(2.2)_2$ the ODE (2.7) implies the following.

A3.2. If the common part $\mathcal{R} = \mathcal{R}_T \mathbf{T}$ of the laws A2.5,1 to A2.5,4 holds (so that

⁽¹⁴⁾ As is well known, given a statue, sometimes (briefly speaking) with the same materials one can construct a similar statue whose linear dimensions are λ times those of the former; however this occurs for λ not too large; and for λ very large no existing materials allow the construction of the latter statue.

either $\mathcal{R}_T = f(\dot{\xi}, \mathcal{C}_U)$ or $\mathcal{R}_T \equiv 0$), then for a.e. t in the domain $[0, T]$ of C 's motion

$$(3.1) \quad \begin{cases} m(\ddot{\xi} - g \sin \theta - \mathcal{R}_T) = -f_d \Phi_n, \\ m(\dot{\eta} + g \cos \theta) = \Phi_n \begin{cases} \geq 0 & \text{when } v = \eta - u = 0, \\ = 0 & \text{otherwise (when } v > 0) \end{cases} \end{cases} \quad \square$$

This easily yields that

$$(3.2) \quad \ddot{\xi} + f_d \dot{\eta} = B + \mathcal{R}_T \quad \text{a.e. in } [0, T], \quad \text{where } B \doteq g(\sin \theta - f_d \cos \theta).$$

Now, by A3.1 and (2.3)₃, the validity of (3.1) for some $\Phi_n = \Phi_n(t)$ is obviously equivalent to ⁽¹⁵⁾

$$(3.3) \quad \ddot{\xi} = \begin{cases} B + \mathcal{R}_T - f_d \ddot{u} \\ g \sin \theta + \mathcal{R}_T \end{cases}, \quad \dot{\eta} = \begin{cases} \ddot{u} \geq -g \cos \theta \\ -g \cos \theta \end{cases}, \quad \text{a.e. in } \begin{cases} \mathcal{N}_V \\ [0, T] \setminus \mathcal{N}_V \end{cases}.$$

As a differential constraint, (3.3) appears rather troublesome by its inclusion of \ddot{u} , even if u is the only scalar control occurring in it explicitly. Furthermore (3.3) holds only if the sliding condition A2.4 does. Hence by (2.3)₂₋₃ [(2.3')], in connection with any symmetric [general] motion \mathcal{M} of Σ we must consider both controls u and w [all controls u_i and w_i ($i = 1, 2$)]. Thus the whole 4-tuple [6-tuple] induced by \mathcal{M} is relevant – see Remark 2.2(b).

However, fortunately, the optimization problems on Σ considered in this work can be solved directly, *e.g.*, without using Pontriagin's maximum principle. Indeed in every case one can completely describe a set of optimal 4-tuple or 6-tuple formed by functions in $C^1 \cap PC^2$, that satisfy the sliding and (realistic) simplifying conditions A2.4 and A3.1. Also much less regular solutions obviously exist.

It will be useful to note that by (3.2)₂

$$(3.4) \quad B \geq 0 \quad \text{iff} \quad \theta \geq \theta_d \doteq \arctg f_d \in (0, \pi/2);$$

and that, since $f_s > f_d$,

$$(3.5) \quad B_{f_s} \doteq g(\sin \theta - f_s \cos \theta) < B.$$

We shall consider only the initial conditions

$$(3.6) \quad \xi(0) = 0, \quad \dot{\xi}(0^+) = \dot{\xi}_0, \quad \eta(0) = \eta_0, \quad \dot{\eta}(0^+) = \dot{\eta}_0, \quad u_1(0) = u_2(0) = \eta_0,$$

generally implemented in practical cases, assuming that

$$(3.7) \quad \dot{\xi}_0 \geq 0, \quad \eta_0 \in [U_1, U_2], \quad \dot{\eta}_0 \geq 0; \quad B \geq 0, \quad B_{f_s} > 0 \vee \dot{\xi}_0 > 0 \quad (\text{see (3.5)}).$$

Most part of this work is concerned with the case $B > 0$ and the second of the alternatives $B_{f_s} > 0$ and $\dot{\xi}_0 > 0$; however the case $B = 0$ is essential to consider a device – see above Remark 4.3 – used in the same part.

⁽¹⁵⁾ By (3.3)₂[–] – *i.e.* the lower part of (3.3)₂ by (L) at the end of Section 1 –, for $v > 0$ the relation (3.2)₁ is equivalent to (3.3)₁[–]; hence it is substantially free of f_d in spite of f_d 's occurrence in (3.2)₂.

Incidentally $(3.7)_2$ follows from $(3.6)_{5-6}$ by A2.6(i); and the structural conditions $(3.7)_{4-5}$ have been added here because they allow Σ to start effectively when $\dot{\xi}_0 = 0$ and, e.g., $u_1(\cdot) = u_2(\cdot) = \text{const.}$

The following multiple definition is basilar for our treatment of the optimization problems Problem 6.1 and 9.1, mentioned in Section 1 and regarded as endowed with the initial conditions in $(3.6-3.7)$.

DEFINITION 3.1. Let the 6-tuple $\sigma_6 \doteq (\xi(\cdot), \eta(\cdot), u_1(\cdot), w_1(\cdot), u_2(\cdot), w_2(\cdot))$ be formed by functions in $C^1 \cap PC^2([0, T])$. Then for $\Sigma = \Sigma_h$ ($h = 1, 2, 3$) – see A2.5 –

(a) we say that (for Problem 6.1) σ_6 is *admissible* [strongly *admissible* (see Remark 3.2(c))], briefly *ad.* [str. *ad.*], if

(i) $u_i(\cdot) \leq \eta(\cdot)$ ($i = 1, 2$) (hence $u(\cdot) \leq \eta(\cdot)$) [and in addition $u_1(\cdot) = u_2(\cdot) = u(\cdot)$, i.e. U 's feet always have the same distance from the ski-run – see $(2.4)_2$],

(ii) the 6-tuple σ_6 satisfies the control condition $(2.14)_2$ as well as the sliding condition A2.4, the simplifying one A3.1, and the initial conditions (3.6) where (3.7) holds, and

(iii) for some $\Phi_n = \Phi_n(t) \geq 0$, σ_6 solves the ODE (3.1);

(b) we say that σ_6 is *symmetric*, briefly *sym.*, if it is str. *ad.* and with $w_1(\cdot) = w_2(\cdot) = w(\cdot)$, so that U 's feet always have the same position – see $(2.4)_3$;

(c) we say that the 4-tuple $\sigma_4 \doteq (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$ is *ad.*, *str. ad.*, or *sym.*, if it is induced by – i.e. related through $(2.5)_{2-3}$ to – some *ad.*, *str. ad.*, or *sym.* choice of σ_6 respectively.

(d) in case σ_6 is *ad.* [sym.] and it induces σ_4 ,

(i) we write $c_4 \doteq ((u_1(\cdot), w_1(\cdot), u_2(\cdot), w_2(\cdot)) \in \text{Ad}C_4 [c_4 \in \text{Sym}C_4]$ and $c_2 \doteq (u(\cdot), w(\cdot)) \in \text{Ad}C_2 [c_2 \in \text{Sym}C_2]$,

(ii) we say that $c_2 [c_4]$ is an *ad.* or *sym. control couple* [fourtuple] respectively,

(iii) we write $\xi(\cdot) = \xi_2(\cdot, c_2)$ and $\xi(\cdot) = \xi_4(\cdot, c_4)$, and

(iv) we do the analogue with $\eta(\cdot)$ (see Remark 3.3(d) below).

(e) A motion of Σ is said to be *ad.*, *str. ad.*, or *weakly sym.*, if it induces an *ad.*, *str. ad.*, or *sym.* 6-tuple (and hence such a 4-tuple) respectively. \square

REMARK 3.1. Every *ad.* [symmetric] 6-tuple will (obviously) be regarded to be induced by some motions [symmetric motions] of Σ . \square

This allows us to assert, e.g., Remark 3.2(a) below.

REMARK 3.2. (a) Some *ad.* 4-tuples $\sigma_4 \doteq (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$ fail to be symmetric and even to be induced by some str. *ad.* 6-tuple $\sigma_6 \doteq (\xi(\cdot), \dots, w_2(\cdot))$ – i.e. with $u_1(\cdot) = u_2(\cdot)$. In fact by $(2.10)_{1-3}$ and the maximality condition (ii) in A2.6, the inclusion (i) $u(\tau) \in]U_2, U'_2[$ holds for some $\tau \in]0, T]$ along some nonsymmetric motion \mathcal{M}_{ns} of Σ . Then A2.6(i) and Definition 3.1(c)-(d) yield the thesis.

(b) Along some choice of \mathcal{M}_{ns} , besides the inclusion (i), the condition (ii) both $U_2 < \eta = u < U'_2$ and $\dot{u} = 0$ can hold in some neighborhood N_τ of τ . Hence

(iii) $\ddot{\eta}(\tau) + g \cos \theta > 0$ a.e. in N_τ by $(3.3)_{2-3}^+$. Instead, along every *str. ad.* motion \mathcal{M}_s satisfying (i), the strong inequality $\eta > u$ must hold in some smaller choice of N_τ , so that by $(3.3)_2$, (iv) $\ddot{\eta}(\tau) + g \cos \theta = 0$ a.e. there.

(c) Teachers of alpine ski advise skiers not to jump on plane parts of ski-run – see footnote 4 –, not to raise one foot, and substantially *to use str. ad. motions*. However for a discussion on steps with sliding skies, in Section 5 we need *ad.* motions that fail to be *str.* admissible. \square

REMARK 3.3. (a) A natural objection against *ad.* motions is that, given (i) $u(\cdot) \in C^1 \cap PC^2([0, T], [U_1, U_2])$ arbitrarily, generally no corresponding *ad.* solution to ODE (3.3) exists because hits arise. However, strictly sapeaking, only so regular motions occur in practical applications; and Problem 6.1 as well as Problem 9.1 – see Section 1 – have ∞^∞ solutions among them.

(b) Let us add that in practical applications (to Σ_R), even the condition $\eta(\cdot) \in C^1$ is too weak. An upper bound on $\ddot{\eta}(\cdot)$ is practically necessary in order to avoid deformations of the ski-run. In fact these cause an additional lost of Σ_R 's kinetic energy; they practically increase f_d ⁽¹⁶⁾

(c) Some choices of $u(\cdot)$ satisfying (i) in part (a) are incompatible with the sliding condition A2.4. It can be proved that some among them render $\xi(T)$ – to be maximized in Problem 6.1 – much larger than when A2.4 holds.

(d) A natural objection against the notations $\xi_2(\cdot, c_2)$ and $\xi_4(\cdot, c_4)$ introduced in Definition 3.1(d) is that the uniqueness of what they denote is not sure ⁽¹⁷⁾ However they can also be meant to express multifunctions – as it occurs in connection with *e.g.* (7.2). When in the sequel some uniqueness property is needed, this can easily be proved referring to *ad.* motions – see *e.g.* Remark 8.1 and footnote 26 placed on it. \square

A3.3. Assume that (i) $u(\cdot) \leq \eta(\cdot)$, (ii) the 4-tuple $\sigma_4 \doteq (\xi(\cdot), \eta(\cdot), u(\cdot), 0)$, formed by functions in $C^1 \cap PC^2([0, T])$, solves the ODE (3.1) for some $\Phi_n = \Phi_n(t) \geq 0$, (iii) it also satisfies the initial conditions (3.6) as well as the simplifying condition A3.1, (iv) $(\dot{\xi} =) \dot{s} > 0$ a.e. in \mathcal{N}_Φ^c holds along σ_4 , and (v) $u([0, T]) \subseteq [U_1, U_2]$. Then σ_4 is symmetric – see Definition 3.1(c). \square

In fact let $t \in [0, T]$. Then by (ii) and (v), $(u(t), 0) \in [U_1, U_2] \times \{0\}$ on $[0, T]$. Furthermore, by (2.12)₁, $[U_1, U_2'] \times \{0\}$. Hence $(u(t), 0, u(t), 0) \in K_4$ on $[0, T]$ by (2.10)₂₋₃ and (2.12)₂. Furthermore for $\sigma_6 \doteq (\xi(\cdot), \eta(\cdot), u(\cdot), 0, u(\cdot), 0)$ (vi) $s \equiv s_1 \equiv s_2$ holds by (2.3')₁ and (2.5)₁. Hence by (i) to (iv) and Definition 3.1(a)-(b), σ_6 is obviously symmetric, so that the same holds for σ_4 by Definition 3.1(c). q.e.d.

⁽¹⁶⁾ This effect of ski-run's deformation is practically well known by ski-racers in that along curves they must angulate their skis as little as possible to avoid just unnecessary such deformations, which practically increase f_d .

⁽¹⁷⁾ An example of such non-uniqueness is shown in [2] for a bouncing ball with positive restitution coefficient. Fortunately hits of skis against a ski-run can reasonably be regarded as completely inelastic.

4. A CONTROL-FREE FIRST INTEGRAL OF Σ 'S DYNAMIC EQUATIONS, FOR \mathcal{R}_T
 LINEAR IN $\dot{\xi}$; ITS INTEGRATION IN THE CASE $\mathcal{R}_T \equiv 0$. A WAY OF USING THIS
 WHEN $\mathcal{R}_T \neq 0$ FOR Σ_R

A4.1. Regarding the law A2.5,3 of air-resistance as valid, along any admissible motion \mathcal{M} of Σ – see Definition 3.1(e) – the control-free first integral

$$(4.1) \quad \dot{\xi} + f_d \dot{\eta} + k\xi = \overline{B}t + \dot{\sigma}(0) \quad \forall t \in [0, T] \quad (\overline{B} \doteq B + \overline{k}, \sigma(t) \doteq \xi(t) + f_d \eta(t))$$

– see (3.2)₂ – holds for C 's motion $(\xi(\cdot), \eta(\cdot))$ in $[0, T]$; and it is equivalent to the relation

$$(4.2) \quad \dot{\sigma}(t) + k\sigma(t) = \overline{B}t + kf_d \eta(t) + \dot{\sigma}(0) \quad \forall t \in [0, T],$$

where, besides (3.6-7), we have that

$$(4.3) \quad \sigma(0) = f_d \eta_0, \quad \dot{\sigma}(0) = \dot{\xi}_0 + f_d \dot{\eta}_0 \geq 0, \quad \overline{B} \geq 0, \quad \dot{\sigma}(0) > 0 \vee \overline{B} > 0. \quad \square$$

Indeed by definitions (4.1)₂₋₃, the conditions (3.6)₁₋₄, (3.7), (3.5), and (2.8)₃ imply (4.3). Furthermore, by Definition 3.1(a), (c), (e), the ODE (3.1) and hence the ODE (3.2) hold a.e. in $[0, T]$ for the ad. 4-tuple $\sigma_4 \doteq (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$ induced by \mathcal{M} . In addition, by (4.1)₂₋₃ and the linearity assumption (2.8) asserted in A2.5,3, (3.2) reads $\ddot{\sigma} = \overline{B} - k\dot{\xi}$, which by (3.6)₁ easily yields the relation (4.1)₁. Lastly by (4.1)₃, (4.1)₁ is equivalent to (4.2). q.e.d.

A4.2. *In the case*

$$(4.4) \quad \mathcal{R}_T(t) = 0 \quad \forall t \in [0, T], \text{ equivalent to A2.5,3 with } k = 0 = \overline{k},$$

along any *ad.* motion \mathcal{M} of Σ in $[0, T]$ the *control-free relation*

$$(4.5) \quad \begin{aligned} \dot{\xi}(t) + f_d \dot{\eta}(t) &= \dot{\sigma}(t) = \dot{\sigma}(0) + Bt > 0 \\ \forall t \in [0, T] \quad (B \geq 0, \dot{\sigma}(0) > 0 \vee B > 0) &- \text{ see (3.2)}_2 \end{aligned}$$

holds for C 's motion $(\xi(\cdot), \eta(\cdot))$; hence we have the *control-free (second) integral*

$$(4.6) \quad \xi(t) + f_d \eta(t) \equiv \sigma(t) = \sigma(0) + \dot{\sigma}(0)t + Bt^2/2 \quad \forall t \in [0, T]. \quad \square$$

Indeed, by A4.1, (4.2) holds for $(\xi(\cdot), \eta(\cdot))$. Hence (4.1)₂₋₃, (4.4)₂₋₃, and (3.7)₄₋₅ imply (4.5)_{1,2,4}. Now the inequality (4.5)₃ follows from (4.3)₂₋₃ and (4.5)₄. q.e.d.

DEFINITION 4.1. For Σ subjected to A2.5, b ($1 \leq b \leq 4$), we say that the ad. motion \mathcal{M} of Σ in $[0, T]$, as well as its induced 4-tuple and 6-tuple, are *jump-free [str. (i.e. strongly) jump-free]* in $[t', t'']$ ($\subset [0, T]$), if along \mathcal{M} $v = \eta - u = 0$ [$v_i = \eta - u_i = 0$ ($i = 1, 2$)] there; and «in $[t', t'']$ » can be omitted for $[t', t''] = [0, T]$. □

REMARK 4.1. If Σ is subjected to A2.5,3, \mathcal{M} is an ad. motion of Σ on $[0, T]$, and $t \in [0, T]$, then the followig holds.

(a) If $t \in \overset{\circ}{N}_v$, hence \mathcal{M} is jump-free in some neighborhood of t , then $\eta(t) = u(t)$ and $\dot{\eta}(t) = \dot{u}(t)$.

(b) The speed $\dot{\xi}(t)$ of C parallel to the ski-run is determined – see (4.1) – by (the value of t together with those of) $\xi(t)$ and $\dot{\eta}(t)$ ($= \dot{u}(t)$, if $t \in \overset{\circ}{N}_v$).

(c) [(d)] In the case $\mathcal{R}_T \equiv 0$ – by the validity of (4.6) [(4.5)] – C 's absciss $\xi(t)$ [C 's speed $\dot{\xi}(t)$] at the instant t is determined by C 's ordinate $\eta(t)$ [by $\dot{\eta}(t)$], and hence by $u(t)$ [by $\dot{u}(t)$] if $t \in \overset{\circ}{N}_v$ – see A3.1. \square

Thus, for $\mathcal{R}_T \equiv 0$ and $t \in \overset{\circ}{N}_v$, Σ 's intrinsic configuration [intrinsic velocity distribution] at the instant t determines $\xi(t)$ [$\dot{\xi}(t)$]. Therefore in practical cases the value of this quantity intuitively appears independent of U 's behaviour in a large time interval $[0, t_1] \subset [0, t[$. This is confirmed and specified later – see thesis (b_3) in A7.1(b).

REMARK 4.2. (a) In some cases it is convenient to use the subclass C_{K_2} formed by the possibly nonsymmetric motions that share the induced 4-tuple with some symmetric one ⁽¹⁸⁾; and this subclass is proper by (2.10) and (ii) in A2.6. Incidentally, by A3.3 one easily proves the property of being proper for the class C_A formed by Σ 's ad. motions along which $\dot{s} > 0$ a.e. in \mathcal{N}_Φ^c and U always has his most areodynamic position compatible with the present values of u_1 , u_2 , and $\dot{\xi}$ – the law A2.5,1 (or A5.1) being assumed.

(b) Str. jump-free (ad.) motions are in C_{K_2} – see part (a) – by the implication (2.13).

(c) *Some str. jump-free 4-tuple is induced by some motion (or 6-tuple) that fails to be so; in fact, for $\dot{\eta}_0 = 0$ and $\eta_0 \in]U_1, U_2[$, $\sigma_6 = (\xi(\cdot), \eta_0, \eta_0, 0, U_1, 0)$ (hence $u_1 = \eta_0 > U_1 = u_2$) induces $\sigma_4 = (\xi(\cdot), \eta_0, \eta_0, 0)$. Instead the analogue for 6-tuples is obviously false.* \square

A4.3. For every ad. 4-tuple $\sigma_4 = (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$, of «domain» $[0, T]$ and jump-free in $[t_1, T]$ where $t_1 \in [0, T[$ (as is the one considered in A11.1(a)),

(a) if σ_4 is str. ad., then it is str. jump-free in $[t_1, T]$; and

(b) if σ_4 is str. jump-free in $[0, T]$, then it is str. admissible. \square

Indeed, by Definition 4.1, (i) $u(t) = \eta(t) \forall t \in [t_1, T]$. Furthermore we, first, assume that σ_4 is str. admissible. Then, by Definition 3.1(a)-(b), σ_4 is induced by some ad. 6-tuple (i) $\sigma_6 = (\xi(\cdot), \eta(\cdot), u_1(\cdot), \dots, w_2(\cdot))$ with $u_1(\cdot) = u_2(\cdot) (= u(\cdot))$. Then, by Remark 3.1(a), some motion \mathcal{M} of Σ induces σ_6 and hence σ_4 . Therefore, by (i) and Definition 4.1, \mathcal{M} , σ_6 , and σ_4 are str. jump-free in $[t_1, T]$. Thus (a) holds.

Now let σ_4 be str. jump-free in $[0, T]$. Then by Definition 4.1 it is induced by a motion with $u_1 \equiv u_2$, so that by Definition 3.1(c) σ_4 is str. admissible. Hence (b) holds. q.e.d.

Let us note this corollary of A4.3.

A4.4. A jump-free 4-tuple is str. ad. iff it is str. jump-free. \square

⁽¹⁸⁾ These motions are characterized by (2.14)₁, which explains the use of « C_{K_2} ».

* * *

Now we aim at showing a useful way of using the result (4.6), obtained for $\mathcal{R}_T \equiv 0$, in case air resistance is present for Σ_R . To this end we consider Σ_R in the situations (S_1) to (S_2) – see Section 1 – and either (S_3) to (S_4) or (S'_3) to (S'_4) – see between (2.8) and (2.9). Hence it can be schematized rather well by some version of Σ_2 – see (xii) above Remark 2.1, which is being specified here. Let V^* be Σ_2 's stationary speed (at $\eta = u = \text{const}$). Since by definition V^* satisfies condition (3.2)₁ in $\dot{\xi}$ for $\dot{\xi} = 0 = \dot{\eta} = \ddot{u}$, we have that (i) $f(V^*) = B$. Furthermore let (ii) both $k = f'(V^*)$ and $\bar{k} = kV^* - f(V^*)$ hold for Σ_3 – see A2.5. Hence, by (i), (iii) $B = kV^* - \bar{k}$. Then, by (3.2)₁ for $\dot{\xi} = 0 = \dot{\eta} = \ddot{u}$, (iv) V^* also is Σ_3 's stationary speed at $\eta = u = \text{const}$.

By (i) and A2.5,2 [by (ii), (iii), and (2.8)] (v) for Σ_2 [Σ_3] the ODE (3.2)₁ is expressed by the first [third] of the equalities

$$(4.7) \quad \ddot{\sigma} = f(V^*) - f(\dot{\xi}) = k(V^* - \dot{\xi}) + o_1(V^* - \dot{\xi}), \quad \ddot{\sigma} = k(V^* - \dot{\xi}) (\sigma \dot{=} \xi + f_d \eta);$$

and (4.7)₂, where $o_1(\lambda)/\lambda \mapsto 0$ as $\lambda \mapsto 0$, follows from (ii)₁.

In addition (vi) let $\Sigma_{4,d}$ be obtained from Σ_4 (for which $\mathcal{R}_T \equiv 0$) by changing l 's steepness θ into the angle $\theta_d \doteq \arctg f_d \in (0, \pi/2)$ of kinetic friction. Then, by (3.2)₂, (vii) B 's analogue for $\Sigma_{4,d}$ is $B_d = 0$, so that

A4.5. the analogue for $\Sigma_{4,d}$ of ODE (3.2)₁ reads $\ddot{\sigma} = 0$ – see (4.7)₄.

A4.6. (a) Assume that, besides (i) to (ii), (viii) \mathcal{M}_2 is a jump-free (ad.) motion of $(\Sigma =) \Sigma_2$ in $[0, T]$ – see A2.5 – for which $\dot{\xi}_0 \cong V^* > 0$, (ix) $\sigma_{2,6} \doteq (\xi_2(\cdot), \eta(\cdot), u_1(\cdot), \dots, w_2(\cdot))$ is its induced 6-tuple, (x) $\sigma_{4,6} \doteq (\xi_4(\cdot), \eta(\cdot), u_1(\cdot), \dots, w_2(\cdot))$ where $\xi_4(t) = \dot{\xi}_0 t + f_d[u(0) - u(t)]$, and (xi) $f_d \dot{u}_i(t) + \dot{w}_i(t) < \dot{\xi}_0$ ($0 \leq t \leq T$) ($i = 1, 2$). Then (xii) $\sigma_{4,4} \doteq (\xi_4(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$, where (2.5)_{2,3} hold, is an ad. 4-tuple for the system $\Sigma_{4,d}$ defined by (vi); and

$$(4.8) \quad |\ddot{\xi}_2(t) - \ddot{\xi}_4(t)| \leq \varepsilon \doteq \max \{ |f(V^*) - f(\dot{\xi}_2(\tau))|; 0 \leq \tau \leq T \} \quad (0 \leq t \leq T),$$

so that

$$(4.9) \quad \begin{aligned} |\dot{\xi}_2(t) - \dot{\xi}_4(t)| &= \left| \int_0^t [f(V^*) - f(\dot{\xi}_2(\tau))] d\tau \right| \leq \varepsilon t \quad (0 \leq t \leq T), \\ |\xi_2(t) - \xi_4(t)| &= \left| \int_0^t [f(V^*) - f(\dot{\xi}_2(\tau))](t - \tau) d\tau \right| \leq \varepsilon t^2/2 \quad (0 \leq t \leq T). \end{aligned}$$

(b) The jump-free condition in (viii) is essential for part (a), if $\theta \neq \theta_d$.

(c) Besides (i), (ii), and (vii) to (ix), we assume that \mathcal{M}_3 is a jump-free (ad.) motion of Σ_3 in $[0, T]$ satisfying the same initial conditions as \mathcal{M}_2 and that $\sigma_{3,6} \doteq (\xi_3(\cdot), \eta(\cdot), u_1(\cdot), \dots, w_2(\cdot))$ is its induced 6-tuple. Then, using the function $o_1(\cdot)$

involved by (4.7), we have that for $t \in [0, T]$

$$(4.10) \quad |\dot{\xi}_2(t) - \dot{\xi}_3(t)| = \left| \int_0^1 o_1[V^* - \dot{\xi}_2(\tau)] e^{k(t-\tau)} d\tau \right| \leq |o_1(\mu)| e^{kt} \quad \text{and} \\ |\xi_2(t) - x_3(t)| \leq |o_1(\mu)| e^{kt}/k, \text{ where } \mu \geq |V^* - f(\dot{\xi}_2(\tau))| \quad \forall \tau \in [0, t]. \quad \square$$

Indeed, by (viii) and Definition 4.1, (xiii) $\eta(\cdot) = u(\cdot)$, while by Definition 3.1(e) and (ix), the 6-tuple $\sigma_{2,6}$ in $[0, T]$ is admissible. Then by Definition 3.1, first, $\sigma_{2,6}$ is formed by functions in $C_1 \cap PC^2([0, T])$; hence by (x) (xiv) the same holds for $\sigma_{4,6}$. Second, by (ix) $\sigma_{2,6}$ satisfies conditions (i) to (iii) in Definition 3.1(a); hence, by (x), it is easy to check that (xv) $\sigma_{4,6}$ too satisfies conditions (i) to (iii) in Definition 3.1(a) with the exception of the sliding condition A2.4 included in (ii).

However by (2.3')₁, (x), and (xi), for $\sigma_{4,6}$ we have that $\dot{s}_i = \dot{\xi}_4(t) - \dot{w}_i(t) = \dot{\xi}_0(t) - f_d \dot{u}_i(t) - \dot{w}_i(t) > 0$ ($0 \leq t \leq T$) ($i = 1, 2$). Hence A2.4 too holds for $\sigma_{4,6}$.

On the one hand by (x)₂, (xvi) $\ddot{\xi}_4(t) = -f_d \ddot{u}(t)$, so that by (xiii) $\sigma_{4,6}$ satisfies (3.3)₁⁺ with $B + \mathcal{R}_T = 0$, which is its version for $\Sigma_{4,d}$ (equivalent to $\ddot{\sigma} = 0$ by (vii)).

On the other hand (iii) in Definition 3.1(a) holds for $\sigma_{2,6}$; hence by (xiii) and Definitions (2.5)_{2,3}, (xvii) $\sigma_{2,6}$ solves (3.3)⁺, and incidentally also (3.2)₁ for $\sigma = \xi(t) + f_d \eta(t)$, which will be used later.

Since $B \geq 0$ by (3.7)₄, (3.4) yields that $\theta \geq \theta_d (= \arctg f_d)$; hence (xviii) $\cos \theta \leq \cos \theta_d$. Furthermore, since (3.3)_{2,3}⁺ hold for $\sigma_{4,6}$, (xix) $\ddot{\eta} = \ddot{u} \geq -g \cos \theta_d$, i.e. the versions of (3.3)_{2,3}⁺ for $\Sigma_{4,d}$ hold for $\sigma_{4,6}$. Then by (x) and (xiii) the same can be said of (3.3)⁺, and even of (3.3) which is equivalent to (3.1). Thus the versions for $\Sigma_{4,d}$ of conditions (i) to (iii) in Definition 3.1(a) hold for $\sigma_{4,6}$. Hence thesis (xii) in Part (a) holds.

Remembering (xvii)₂, by (v) (and (xiii)) $\sigma_{2,6}$ solves (4.7)₁ for $\sigma = \xi_2(t) + f_d u(t)$. Hence, by the equality in (xvi), (xx) $\ddot{\xi}_2 - \ddot{\xi}_4 = f(V^*) - f(\ddot{\xi}_2)$, which yields thesis (4.8). Thus part (a) is substantially proved.

To prove Part (b), let \mathcal{M}_2 fail to be jump-free. Then, by (iii) in Definition 3.1(a), the relations $u(t) < \eta(t)$ and (3.3)₂⁻, i.e. $\ddot{\eta} = -g \cos \theta$, hold for $\sigma_{2,6}$ on some nonempty subset A of $\{0, T\}$. Then, by (xviii) and (x), the relations $u(t) < \eta(t)$ and $\ddot{\eta} > -g \cos \theta_d$ hold for $\sigma_{4,6}$ on A if $\theta \neq \theta_d$. Thus $\sigma_{4,6}$ fails to satisfy the version of (3.3) for $\Sigma_{4,d}$. Hence, by Definition 3.1(a), $\sigma_{4,6}$ cannot be ad. for $\Sigma_{4,d}$.

To prove part (c) we note that its hypotheses imply that (xxi) $\xi_2(0) = \xi_3(0) = 0 = \dot{\xi}_2(0) - \dot{\xi}_3(0)$ and that the jump-free 6-tuples $\sigma_{2,6}$ and $\sigma_{3,6}$ solve the ODEs (4.7)_{1,3} in σ respectively. Then by (4.7)_{2,4}

$$(4.11) \quad \ddot{\xi}(t) = -k\dot{\xi}(t) + o_1[V^* - \dot{\xi}_2(t)], \quad \text{where } \xi(\tau) = \xi_2(\tau) - \xi_3(\tau) \quad \forall \tau \in [0, T].$$

By the admissibilities of \mathcal{M}_2 and \mathcal{M}_3 , $|\dot{\xi}|$ is absolutely continuous and $t \vdash o_1[V^* - \dot{\xi}_2(t)]$ is integrable on $[0, T]$. Hence

$$d|\dot{\xi}(t)|/dt \leq |\ddot{\xi}(t)| \leq k|\dot{\xi}(t)| + |o_1[V^* - \dot{\xi}_2(t)]| \text{ for a.e. } t \in [0, T],$$

so that Gronwall's lemma yields $(4.10)_1$ for $t \in [0, T]$. Furthermore $(4.10)_4$ yields $(4.10)_{2,3}$. Thus part (c) holds. q.e.d.

Theorem A4.5(a) suggests *a device to obtain good information on Σ_R in, e.g., the racing situations (S_1) to (S_4) , by using theorems on $\Sigma = \Sigma_4$; it simply consists of using $\Sigma_{4,d}$, instead of Σ_4 , as a model of Σ_R – see (α) in Remark 4.3 below.*

REMARK 4.3. We have considered Σ_R in the racing situations (S_1) to (S_4) or their touring analogues; and its schematizations Σ_2 to Σ_3 for which (i) to (vii) hold. In this case, where $|\dot{\xi} - V^*|$ and (hence) $|f(\dot{\xi}) - V^*|$ always are very small by (S_4) , both Σ_2 and Σ_3 were said to be good models for Σ_R in Section 2 – see (vi) and (xii) between (2.8) and (2.9). Here, first, we remark that by A4.5(a)

(α) in the above case $\Sigma_{4,d}$ is a model for Σ_R as good as Σ_2 , up to zeroth order infinitesimals in that (4.9) holds; and (β) this can be satisfactory especially when ε is very small – see (4.8)₂ – and either T is not very large, or (4.9)_{2,4} hold even with « \leq » replaced by «much less than».

Second, we note – especially in connection with Section 5 – that

(γ) in the same case Σ_3 is model for Σ_R as good as Σ_2 , up to first order infinitesimals, in that (4.10) holds; of course the inequalities in (4.10) are better than their corresponding equalities in (4.9) only for $t (> 0)$ sufficiently small; this is specified below in an interesting case.

(δ) Consider the maximum $\mu (> 0)$ such that, roughly speaking, the inequality

$(xxii) |\Delta| \leq \mu$ with $\Delta \doteq \dot{\xi} - V^*$ holds along every Σ_R 's jump-free motion in $[0, T]$, for which e.g. the situations (S_1) to (S_4) occur; and assume that $(xxiii) f(\dot{\xi}) = c\dot{\xi}^2$ hold along them, so that the same happens for every Σ_2 's choice schematizing Σ_R well. Then – see (4.8)₂

$$(4.12) \quad \varepsilon = \max\{|c\dot{\xi}^2 - cV^{*2}|; |\Delta| \leq \mu\} = 2cV^*\mu \quad (\Delta \doteq \dot{\xi} - V^*)$$

while along the corresponding motions of the analogous choice of Σ_3

$$(4.13) \quad 0 \leq -o_1(\Delta) = c\Delta^2 \leq c\mu^2.$$

Indeed $(4.12)_{1,2}$ hold by $(xxiii)$ and $(4.8)_2$; and (4.12) holds along the aforementioned jump-free motions of $\Sigma_{4,d}$ by (v) (involving $(4.7)_1$) and A4.5. Furthermore, by $(xxii)_2$ and $(4.7)_2$, $o_1(-\Delta) = k\Delta + f(V^*) - f(\dot{\xi})$, where $k = f'(V^*)$ by $(ii)_1$. Then, by $(xxiii)$, $o_1(-\Delta) = 2cV^*\Delta + cV^{*2} - c(V^* + \Delta)^2 = -c\Delta^2$, which by $(xxii)_1$ yields (4.13). q.e.d.

(ε) We conclude that for all afore-mentioned Σ_R 's jump-free motions (in e.g. the situations (S_1) to (S_4)) – which are well represented by $\sigma_{2,6} \doteq (\xi_2(\cdot), \eta(\cdot), u_1(\cdot), \dots, w_2(\cdot))$ –, $(xxiv)$ the inequality $(4.9)_{1,2}$ [(4.9)_{3,4}] is «better» than the inequality $(4.10)_{1,2}$ [(4.10)_{3,4}] for

(xxv) $0 \leq t < t_1$ [$0 \leq t < t_2$], while (xxvi) the converse occurs for (xxvii) $t_1 < t[t_2 < t]$, where t_1 and t_2 are determined by the conditions

$$(4.14) \quad 2V^*t_1 = \mu e^{kt_1}, \quad 2V^*(t_2)^2 = \mu e^{kt_2}.$$

Indeed, e.g. (xxiv) holds iff $\varepsilon t < |o_1(\mu)|e^{kt}$ [$\varepsilon t^2 < 2|o_1(\mu)|e^{kt}$], which by (4.12) [(4.13)] is equivalent to (xxv) iff (4.14)₁ [(4.14)₂] holds. An analogous reasoning proves the converse of (xxiv) under condition (xxvii). q.e.d. \square

REMARK 4.4. A proof of this is expected: many results obtained in Part 2 or Part 3 for $\Sigma = \Sigma_4$ or $\Sigma = \Sigma_{4,d}$, and holding for Σ_R in, e.g., the case (S₁) to (S₄) with a good approximation, cannot hold rigorously for $\Sigma = \Sigma_3$. \square

5. ON STEPS WITH SLIDING SKIES. A SIMPLE CASE WITH $\mathfrak{R} \equiv 0$ AND A GENERALIZATION

Some among the ad. 6-tuples that fail to be sym. – see Definition 3.1(a)-(b) – are induced by nonsymmetric motions \mathcal{M}_{st} along which some steps are made with sliding skies. For the sake of simplicity we consider any $\sigma_6 = (\xi(\cdot), \eta(\cdot), u_1(t), \dots, w_2(t))$ among these ad. 6-tuples, of «domain» $[0, T]$ and such that

(C₁) it is jump-free – see Definition 4.1 – (and with $\dot{\xi}(0) > 0$),

(C₂) along it, at a.e. instant $t \in [0, T]$, only one foot of U is touching the ski-run l (or its corresponding l 's reaction vanishes), so that σ_6 and its induced 4-tuple $\sigma_4 = (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$ – see (2.4), (2.3')₂, and (2.5), – are jump-free, i.e. with $\eta(\cdot) = u(\cdot)$, and

(C₃) the trajectory of C is parallel to the ski-run, i.e. $\eta \equiv \eta_0$.

Then, briefly, it is clear that we can choose \mathcal{M}_{st} in such a way that along it

- (i) the total length λ covered by sliding skies is as small as desired, or
- (ii) \mathcal{M}_{st} is practically implementable and λ is appreciably smaller than $\xi(T)$.

Consequently the (negative) work $W_{\text{st}}^* = -mf_d g \lambda \cos \theta$ of dry friction along \mathcal{M}_{st} can be rendered (with $|W_{\text{st}}^*|$) as small as desired and, in practical cases, much smaller than its analogue $W_R^* = -mf_d g \xi(T) \cos \theta$ for either any rigid motion \mathcal{M}_R of Σ or (more generally) any sym. motion \mathcal{M}_s satisfying (C₁) and (C₃). This fact may induce the following

CONJECTURE 5.1. By means of suitable steps with sliding skies, under given initial conditions, the length $\xi(T)$ covered by C along \mathcal{M}_{st} can be rendered larger than its analogue for \mathcal{M}_R or \mathcal{M}_s . \square

Note that under condition (C₃) the action of dry friction, $-f_d \Phi_n = -mf_d g \cos \theta$, can be regarded as a constant applied force. In spite of this, Conjecture 5.1, based on the assumptions (C₁) to (C₃), obviously fails to follow rigorously from the reasonings made above it. In fact, when U is making steps, Σ is not rigid and it can be regarded as a holonomic system with time-dependent constraints, so that the mechanical energy

may not be conserved for it (even when W_{St} is reduced to zero). However it is not unreasonable to think that something similar to Conjecture 5.1 may be implemented to some extent, in some cases.

Instead, *e.g.* when $\mathcal{R} \equiv 0$, even a partial validity of that conjecture can be denied (in connection with \mathcal{M}_{St}). In fact, then the relation (4.6) holds, which by (4.3) and (C_3) (*i.e.* $\eta \equiv \eta_0$) becomes

$$(5.1) \quad \xi(t) = \sigma(t) - f_d \eta_0 = \dot{\xi}_0 t + Bt^2/2 \quad (B \geq 0)(0 \leq t \leq T)$$

for both \mathcal{M}_{St} and \mathcal{M}_R (or \mathcal{M}_s). The following can be concluded

REMARK 5.1. (a) In connection with $\Sigma = \Sigma_4$ – see A2.5 – let conditions (C_1) to (C_3) hold for \mathcal{M}_{St} . Then, however small $|W_{St}|$ may be with respect to $|W_R|$, the part $W_R - W_{St}$ (< 0) of the work W_R (of dry friction along \mathcal{M}_R) eliminated by the skier along \mathcal{M}_{St} through some steps, is replaced along \mathcal{M}_{St} with an equal negative work done by the internal forces exerted by the skier on Σ , in order to make the steps.

(b) Along above motions \mathcal{M}_{St} and \mathcal{M}_R , Σ_4 schematizes Σ_R rather well provided (i) T is very short or (ii) \mathcal{R}_T is very small for Σ_R , which could happen (by an extremely low air density) in skiing on very high mountains or on the moon (*e.g.*, with some snow substitute). Even this seems to me significant about the uselessness of the above steps, which may even tire skiers.

(c) The practical case in which the above results on Σ_4 give the best information on Σ_R is, briefly, when (C_1) to (C_3) and, *e.g.*, (S_1) to (S_4) hold for Σ_R , so that both some choice of Σ_2 schematizes Σ_R well and $\dot{\xi}_0 \cong V^*$. Then – see A4.5, A4.6(a), and Remark 4.3 – it is convenient to use (5.1) with $\theta = \theta_d$ (so that $\Sigma = \Sigma_4 = \Sigma_{4,d}$) and $B = 0$. Then $\xi(t) = \dot{\xi}_0 t$ ($0 \leq t \leq T$) and T need not be very small; furthermore (α) and (β) in Remark 4.3 clarify in which sense the information on Σ_R thus obtained is «good» up to infinitesimals of zeroth order. \square

In order to generalize the considerations in Remark 5.1(a), (α) we now abandon the assumptions (C_3) (of parallelism), the one $\dot{\xi}_0 \cong V^*$ considered in Remark 5.1(c), and that $\mathcal{R} \equiv 0$. Furthermore *we assume that* (β) $\Sigma (= \Sigma_1)$ *is subjected to the following approximate law of air resistance, more general than A2.5,2 but included in A2.5,1.*

A5.1. There is some C^1 -function $f(\dot{\xi}, u)$ with $f(0, u) = 0$, $f(\mathcal{R}^+) = \mathcal{R}^+$, and $\partial f(\dot{\xi}, u)/\partial \dot{\xi} > 0$ for $\dot{\xi} \geq 0$ and $u \in [U_1, U_2]$, such that $-mf(\dot{\xi}, u)\mathbf{T}$ is the resultant $m\mathcal{R}$ of the forces of air resistance acting on Σ in connection with nonnegative admissible values of $\dot{\xi}$, u_i , and w_i ($i = 1, 2$) where $u = \max\{u_1, u_2\}$ (at least for $u_1(\cdot) \cong u_2(\cdot)$ and $|w_1(t) - w_2(t)|$ not too large, which is compatible with steps). \square

REMARK 5.2. By A3.2, assumption (β) above A5.1 yields that Σ 's dynamic ODE (3.1) is still holding; hence the same is true of its consequences (3.2-3). \square

REMARK 5.3. Of course, the new law A5.1, acting on Σ by (β) , is useful especially for information on Σ_R about some family $F_{u(\cdot)}$ that is related to a given choice of $u(\cdot)$

and that is formed by some dynamic motions of Σ_R along which we always have that $u_1(\cdot) \cong u_2(\cdot) \cong u(\cdot)$, $|w_1(t) - w_2(t)|$ is never too large, and the equality $f(\dot{\xi}, \mathcal{C}_U) = f(\dot{\xi}, u)$ always holds with a good approximation; and this is significative – see Remark 5.5. \square

Preliminarily let us briefly show that

A5.2. if assumption (β) above A5.1 holds, (i) $u(\cdot) \in C^1 \cap PC^2([0, T], [U_1, U_2])$ (with $\ddot{u} \geq -g \cos \theta$ a.e.), and (ii) $\mathcal{M}_{u(\cdot)}$ is a jump-free motion of $\Sigma = \Sigma_4$ inducing the 4-tuple $\sigma_4 = (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$, then C 's motion $(\xi(\cdot), \eta(\cdot))$ along $\mathcal{M}_u(\cdot)$ is determined by the condition $\eta(\cdot) = u(\cdot)$ and the Cauchy problem

$$(5.2) \quad \begin{aligned} \ddot{\xi} &= B - f(\dot{\xi}(t), u(t)) - f_d \ddot{u}, & \xi(0) &= 0, & \dot{\xi}(0) &= \dot{\xi}_0 \\ & & (B \geq 0, B_f > 0 \vee \dot{\xi}_0 > 0 - \text{see (3.5)}) & & \square \end{aligned}$$

Indeed Σ 's dynamic ODE (3.1), as well as (3.3), depends on $u_i(\cdot)$ and $w_i(\cdot)$ ($i = 1, 2$) only through $u(\cdot)$; furthermore $\eta(\cdot) = u(\cdot)$ by (ii), and $\mathcal{R}_T = -f(\dot{\xi}, u)$ by A5.1. Hence the consequence (3.2) of (3.1) becomes (5.2)₁ ⁽¹⁹⁾. q.e.d.

Now we consider any version $\mathcal{M}_{\text{St}, u(\cdot)}$ of the motion $\mathcal{M}_{u(\cdot)}$ introduced in A5.2, that includes (forward) steps and satisfies the conditions (C_1) to (C_2) , the simplifying week conditions

$$(5.3) \quad \dot{\xi}_0 > 0, \quad u(t) \in]U_1, U_2] \quad \forall t \in [0, T],$$

and the following specialization of (C_1) to (C_2) . Let $[0, T]$ be divided by means of the instants (i) $\tau_0 = 0 < \tau_1 < \tau_2 < \dots < \tau_{2n} = T$; and assume that (ii) for $h = 0$ to $n - 1$, in the time interval $[\tau_{2h}, \tau_{2h+1}]$ U 's left foot F_1 is sliding on l 's segment $[a_{2h}, b_{2h+1}]$ – i.e. F_1 's ξ -coordinate s_1 is describing it and $v_1 = 0$ –, while in $]\tau_{2h}, \tau_{2h+1}[$ U 's right foot F_2 fails to touch l (hence so does U 's right ski by the parallelism condition A2.2); incidentally thus, by (5.3)₂ and (2.5)₃, (iii) $U_2 > u_1(t) = u(t) > u_2(t) \geq U_1 \quad \forall t \in]\tau_{2h}, \tau_{2h+1}[$.

Likewise (iv) for $k = 1$ to n , in $[\tau_{2k-1}, \tau_{2k}]$ both F_2 is sliding on l 's segment $[a_{2k-1}, b_{2k}]$ and F_1 fails to touch l ; incidentally thus (iii') $U_2 > u_2(t) = u(t) > u_1(t) \geq U_1 \quad \forall t \in]\tau_{2k-1}, \tau_{2k}[$.

Furthermore let

$$(5.4) \quad 0 < a_0 < b_1 < a_1 < b_2 < \dots < a_{2n-2} < b_{2n-1} < a_{2n-1} < b_{2n} < \xi(T).$$

Hence

$$(5.5) \quad \lambda \doteq \text{meas } (\lambda) = \sum_{r=0}^{2n} (b_r - a_r) \in]0, \xi(T)[, \quad \text{where } \lambda = \bigcup_{r=0}^{2n}]a_r, b_r[.$$

⁽¹⁹⁾ Incidentally, since $\eta(\cdot) = u(\cdot)$, $\mathcal{N}_v = [0, T]$; hence, by $\mathcal{M}_{u(\cdot)}$'s admissibility, σ_4 is required to solve the upper part (3.3)⁺ of (3.3). Thus (3.3)_{2,3}⁺ imply the inequality written in the hypothesis (i). Being super-abundant, it has been put between parentheses.

In addition, along $\mathcal{M}_{\text{St},u}(\cdot)$, we have that $v = \eta - u = 0$; hence the work of dry friction, *i.e.* of the force $-f_d \Phi_n T$, has by (3.1)₂ the expression

$$(5.6) \quad W_{\text{St},u(\cdot)} = \sum_{r=1}^{2n} \int_{\tau_{r-1}}^{\tau_r} f_d m[\ddot{u}(t) + g \cos \theta] \dot{s}_i(t) dt \quad \left(i = i_r \doteq \frac{1}{2} \text{ for } r \begin{array}{l} \text{even} \\ \text{odd} \end{array} \right).$$

By the admissibility of $\mathcal{M}_{\text{St},u(\cdot)}$,

(v) $\ddot{u} \in PC^0([0, T])$ and $s_i(\cdot) \in C^1$, ($i = i_r$; $r = 1, \dots, 2n$);

furthermore, by the sliding condition A3.1,

(vi) $\dot{s}_i(t) > 0$ for $i = i_r$ and t in $] \tau_{r-1}, \tau_r[$, *i.e.* when the foot F_i is sliding on $]a_{r-1}, b_r[$ ($r = 1, \dots, 2n$).

Hence a strictly increasing inverse $t = t_i(s)$ of $s = s_i(t)$ exists on the segment

$$(5.7) \quad]a_{r-1}, b_r[= s_i(] \tau_{r-1}, \tau_r[) \quad (r = 1, \dots, 2n) - \text{see } (5.6)_{2-3}.$$

Setting

$$(5.8) \quad t = t(s) = t_i(s) \text{ for } i = i_r \quad \forall s \in]a_{r-1}, b_r[\quad (r = 1, \dots, 2n) - \text{see } (5.6)_{2-3},$$

$t(\cdot)$ is uniformly continuous on λ ; furthermore (5.5)₄ and (5.6) clearly imply that

$$(5.9) \quad W_{\text{St},u(\cdot)} = - \int_{\lambda} f_d m[\ddot{u}[t(s)] + g \cos \theta] ds.$$

A5.3. Given $\varepsilon > 0$, there is a version of the above motion $\mathcal{M}_{\text{St},u(\cdot)}$ involving steps with sliding skies, along which the work (5.9) of dry friction is smaller than ε . \square

Indeed \ddot{u} is bounded by the first inclusion in (v); hence the above integrand, positive by (3.3)₂₋₃⁺, is $< \beta \quad \forall s \in \lambda$ for some constant β . Then, by (5.5)₁, (vii) $|W_{\text{St},u(\cdot)}| < \lambda\beta$.

Given any $\varepsilon > 0$, by (vi) and (5.7) we can obviously choose a new (ad. jump-free) version of $\mathcal{M}_{\text{St},u(\cdot)}$ for which

(viii) $0 < \dot{s}_i(t) < \varepsilon/T\beta$ and hence $b_r - a_r < (\tau_{r-1} - \tau_r)\varepsilon/T\beta \quad \forall t \in]\tau_{r-1}, \tau_r[$ ($i = i_r$; $r = 1, \dots, 2n$) $\forall t \in]\tau_{r-1}, \tau_r[$ ⁽²⁰⁾.

Then (i) below (5.3) and (5.5)₁₋₂ yield that $\lambda < \varepsilon/\beta$, so that $|W_{\text{St},u(\cdot)}| < \varepsilon$ by (vii).
q.e.d.

By A5.3, Remark 5.1(a) can easily be extended to the motions of the type $\mathcal{M}_{\text{St},u(\cdot)}$ and in particular to some family $F_{u(\cdot)}$, of the type considered in Remark 5.3, formed with these motions:

REMARK 5.4. If $\Sigma (= \Sigma_1)$ is subjected to the law A5.1, any (dynamic) motion $\mathcal{M}_{\text{St},u(\cdot)}$ of the above type is referred to, and $u(\cdot)$ is kept fixed, then

(i) there is a new version of that motion along which the (negative) work $W_{\text{St},u(\cdot)}$ of dry friction has an absolute value smaller than an arbitrarily preassigned quantity; and in spite of this, the distance $\xi(T)$ covered by Σ 's center of mass equals the one along

⁽²⁰⁾ To check all admissibility conditions for $\mathcal{M}_{\text{St},u(\cdot)}$'s new version is easy but tedious. For this job it is convenient to use an arbitrarily large value of n and the simplifying conditions (5.3) (besides the constraint properties (2.12)).

any motion $\mathcal{M}_{u(\cdot)}$ of Σ , possibly without steps and related to the same choice of $u(\cdot)$; and

(ii) the analogue for $\mathcal{M}_{u(\cdot)}$ of the (negative) work $W_{\text{St}, u(\cdot)}$ has a larger absolute value than $W_{\text{St}, u(\cdot)}$; but the difference is replaced by a negative work of internal forces analogous to that mentioned in Remark 5.1(a) where the case $\mathcal{R} \equiv 0$ is considered. Furthermore, if $u(\cdot)$ can be practically implemented, then the above assertions hold for some new version of $\mathcal{M}_{\text{St}, u(\cdot)}$ along which $|W_{\text{St}, u(\cdot)}|$ is appreciably smaller than its analogue for the above motion $\mathcal{M}_{u(\cdot)}$ without steps. \square

REMARK 5.5. (a) Consider any (practically) maximal family $F_{u(\cdot)}$, of the type mentioned in Remark 5.3. Then, as far as its motions are concerned, theorems A5.2 and A5.3 hold for Σ_R with a good approximation; hence the same can be said of Remark 5.4, which substantially implies that (i) *steps with sliding skies are useless*.

(b) Incidentally, for Σ_R in e.g. the situations (S_1) , (S_2) , and (S'_3) where (S'_3) is a tourist analogue of the situation (S_3) – see between (2.8) and (2.9) –, all Σ_R 's motions of the type $\mathcal{M}_{u(\cdot)}$, with a same $u(\cdot)$, form a maximal family $F_u(\cdot)$ of the above type.

(c) Part (a) practically implies that (ii) if for two motions of Σ_R having the type $\mathcal{M}_{u(\cdot)}$ with the same $u(\cdot)$ but not belonging to a same family $F_{u(\cdot)}$, the distance $\xi(T)$ covered by Σ_R 's center of mass has different values, then this depends on different values of air resistance experienced by Σ_R along them; and consequently that (iii) *the value of $\xi(T)$ along a motion of Σ_R having the type $\mathcal{M}_{u(\cdot)}$ with a given $u(\cdot)$ is maximum in case along it at every $t \in [0, T]$ \mathcal{C}_U has the best aerodynamic properties compatible with the values of $u(t)$ and $\dot{\xi}(t)$.*

Since (obviously) steps generally may only cause \mathcal{C}_U to loose the above properties, (iv) *it appears convenient for Σ_R not to make steps in order to maximize $\xi(T)$.* Furthermore (v) *in order to implement the absolute maximum of $\xi(T)$ it is convenient for Σ_R to stay, at every instant $t \in [0, T]$, in the configuration \mathcal{C}_U with the best (tourist) aerodynamic properties compatible with the value thus reached by $\dot{\xi}$ at t .* \square

PART 2. TWO OPTIMIZATION PROBLEMS FOR Σ SUBJECTED TO A NEGLIGIBLE AIR RESISTANCE ($\mathcal{R} \equiv 0$). THEIR SOLUTIONS

6. A MAXIMIZATION PROBLEM FOR $\xi(T)$ ON $\Sigma = \Sigma_4$. THE MOTION $((\tilde{\xi}_\tau(\cdot), \bar{\eta}_\tau(\cdot))$ ASSOCIATED TO C 'S MOTION $(\xi(\cdot), \eta(\cdot))$

The following optimization problem has several assumptions, for the sake of simplicity. It is natural to change them in several ways. The corresponding solutions are easily found by reasonings similar to those made for the original problem.

PROBLEM 6.1. Assume (4.4), i.e. $\mathcal{R}_T \equiv 0$, but that $f_d > 0$. Then how must U behave along a motion \mathcal{M} of Σ that induces the ad. 4-tuple $\sigma_4 = (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$ and the ad. 6-tuple $\sigma_6 = (\xi(\cdot), \eta(\cdot), u_1(\cdot), \dots, w_2(\cdot))$, in order to to maximize $\xi(T)$,

$$(6.1) \quad \xi(T) = \sigma(T) - f_d \eta(T) \mapsto \sup? \quad \square$$

REMARK 6.1. (a) If $f_d > 0$, then U 's goal in Problem 6.1 is attained iff $\eta(T) = u(T) + v(T)$ is minimized.

(b) If f_d were zero, then U 's behaviour, as well as the initial conditions (3.6)_{3,4} would become completely irrelevant by (4.3) and (4.6). \square

Part (a) practically says much about solutions, the more so as, necessarily $u(T) \geq U_1$ and $v(T) \geq 0$; furthermore, in practice, generally $v \equiv 0$ and the condition $\eta(T) = u(T) = U_1$ can be implemented in infinitely many ways independently of U 's behaviour in $[0, \tau]$ for some instant $\tau \in]0, T[$, close to T but not too much (see thesis (b₃) in A7.1(b), Remark 8.2(a), and especially the bound (11.5) for τ in A11.1(d), as well as Remark 11.1(c) in Part 3.)

REMARK 6.2. Regards e.g. Problem 6.1, U 's optimal behaviours can be searched for among ad. 4-tuples [6-tuples] because of Remark 3.1. \square

Here are some preliminaries. To any ad. motion \mathcal{M} or ad. r -tuple $\sigma_r = (\xi(\cdot), \eta(\cdot), \dots)$ «defined» on $[0, T]$ ($r \in \{2, 4\}$), and to any $\tau \in [0, T]$ we *associate the motion* $((\bar{\xi}_\tau(\cdot), \bar{\eta}_\tau(\cdot)))$ in $[0, +\infty[$ of a mass point M_τ moving in empty regions and having, at $t = \tau$, the same position and velocity assigned to C by \mathcal{M} or σ_r , respectively:

$$(6.2) \quad \begin{aligned} \bar{\xi}_\tau(t) &\doteq \xi(\tau) + \dot{\xi}(\tau)(t - \tau) + g \frac{\sin \theta}{2} (t - \tau)^2, \\ \bar{\eta}_\tau(t) &\doteq \eta(\tau) + \dot{\eta}(\tau)(t - \tau) - g \frac{\cos \theta}{2} (t - \tau)^2. \end{aligned}$$

A6.1. We assume that (i) $\mathcal{R}_T \equiv 0$, (ii) $\sigma_r \doteq (\xi(\cdot), \eta(\cdot), \dots)$ is an ad. 6-tuple of «domain» $[0, T]$ for $r \in \{2, 4\}$, and (iii) $\tau \in [0, T]$. Then theses (a) to (b) below hold.

(a) For $\tau = 0$ the above motion $((\bar{\xi}_\tau(\cdot), \bar{\eta}_\tau(\cdot)))$ associated to σ_r is independent of σ_r in that by (3.6-7)

$$(6.2') \quad \bar{\xi}_0(t) \doteq \dot{\xi}_0 t + g \frac{\sin \theta}{2} t^2, \quad \bar{\eta}_0(t) \doteq \eta_0 + \dot{\eta}_0 t - g \frac{\cos \theta}{2} t^2 \quad \forall t > 0.$$

(b) For arbitrary choices of σ_r and τ above, (iv) the couple $((\bar{\xi}_\tau(\cdot), \bar{\eta}_\tau(\cdot)))$ solves the ODE (3.2) in $(\xi(\cdot), \eta(\cdot))$; and (v) for any choice of $u(\cdot) \leq \bar{\eta}_\tau(\cdot)$, in $]t_1, t_2[\doteq \{t \in]0, T[; \bar{\eta}_\tau(t) > U_1\}$ the same couple solves the ODE (3.1) forcing Φ_n to vanish on $]t_1, t_2[$ ⁽²¹⁾; lastly (vi) we have that

$$(6.3) \quad \sigma(t) = \bar{\xi}_0(t) + f_d \bar{\eta}_0(t) = \xi(t) + f_d \eta(t) = \bar{\xi}_\tau(t) + f_d \bar{\eta}_\tau(t) \quad \forall t \in [0, T]. \quad \square$$

Indeed thesis (a) holds obviously; and (6.2) yields (b)'s assertion (iv).

For $(\xi(\cdot), \eta(\cdot)) = (\bar{\xi}_\tau(t), \bar{\eta}_\tau(t))$ we have that $\ddot{\eta} = -g \cos \theta$. Hence (3.1)⁻ implies that $\Phi_n = 0$ on $]t_1, t_2[$, so that (3.1)⁺ with $\mathcal{R}_T \equiv 0$ also holds there. Thus assertion (v) in (b) is proved.

⁽²¹⁾ Part (v) of (b) will be used to prove theorem A7.1(a) on the auxiliary Problem 6.1, and precisely Step 1 in its proof – see Section 7.

Now note that by (6.2) the s -th time derivative of the equality (6.3)₃ holds at $t = \tau$ for $s \in \{0, 1\}$. Furthermore $(\xi(\cdot), \eta(\cdot))$ solves the ODE (3.2) for $\mathcal{R}_T \equiv 0$ a.e. on $[0, T]$; and the same holds for $(\bar{\xi}_\tau(\cdot), \bar{\eta}_\tau(\cdot))$ by thesis (iv). Hence B equals, on $[0, T]$, the second time-derivatives of both $t \mapsto \xi_\tau(t) + f_d \eta(t)$ and $t \mapsto \bar{\xi}_\tau(\cdot) + f_d \bar{\eta}_\tau(\cdot)$. Then (6.3)₃ holds.

For $\tau = 0$ (6.3)₃ becomes (6.3)₂. Now (4.6)₁ yields the expression (6.3)₁ of $\sigma(\cdot)$, which, like (4.6)₂, is directly based on the initial conditions (3.6) ⁽²²⁾

A6.2. (a) For $\tau \in [0, T]$ let $\bar{\eta}_\tau(\cdot)$ be associated to the ad. 4-tuple $\sigma_4 = (\xi(\cdot), \eta(\cdot), \dots)$. Then

$$(6.4) \quad \dot{\eta}(t) \geq -g \cos \theta = \ddot{\bar{\eta}}_\tau(t) \text{ a.e. on } [0, T], \quad \eta(t) \geq \bar{\eta}_0(t) \quad \forall t \in [0, T],$$

$$(6.5) \quad \dot{\eta}(t) \begin{cases} \leq \dot{\bar{\eta}}_\tau(t) & \forall t \in \left\{ [0, \tau] \right\} \\ \geq \dot{\bar{\eta}}_\tau(t) & \forall t \in \left\{ [\tau, T] \right\} \end{cases}, \quad \eta(t) \geq \bar{\eta}_\tau(t) \quad \forall t \in [0, T].$$

and (i) the function $\tau \mapsto (\bar{\xi}_\tau(T), \bar{\eta}_\tau(T))$ is in $C^1 \cap PC^2$.

(b) Add the assumptions $0 \leq \tau < \tau' \leq T$. Then

$$(6.6) \quad \dot{\bar{\eta}}_\tau(t) \leq \dot{\eta}(t) \leq \dot{\bar{\eta}}_{\tau'}(t) \quad \forall t \in [\tau, \tau'],$$

$$(6.7) \quad \dot{\bar{\eta}}_\tau(\tau') \leq \dot{\bar{\eta}}_{\tau'}(\tau'), \quad \bar{\eta}_\tau(\tau') \leq \bar{\eta}_{\tau'}(\tau'), \quad \bar{\eta}_\tau(t) \leq \bar{\eta}_{\tau'}(t) \quad \forall t \in [\tau', T],$$

and conditions (ii) to (v) below are pairwise equivalent.

(ii) $\bar{\eta}_\tau(T) = \bar{\eta}_{\tau'}(T)$, (iii) both $\bar{\eta}_\tau(\tau') = \bar{\eta}_{\tau'}(\tau')$ and $\dot{\bar{\eta}}_\tau(\tau') = \dot{\bar{\eta}}_{\tau'}(\tau')$,

(iv) $\bar{\eta}_\tau(\cdot) = \bar{\eta}_{\tau'}(\cdot)$, (v) $\eta(t) = \bar{\eta}_\tau(t) \quad \forall t \in [\tau, \tau']$. \square

PROOF. By Definition 3.1(a), the functions $\xi(\cdot)$, $\eta(\cdot)$, and $u(\cdot)$ are in $C^1 \cap PC^2$ and they satisfy the condition (3.3)_{2,3} [(3.3)₂]⁺ – i.e. the upper [lower] part of (3.3)_{2,3} [(3.3)₂]. Hence by (6.2)₂ it is easy to check (6.4)_{1,2} and (6.5), while (6.4)₃ is (6.5)₂ for $\tau = 0$. Furthermore, since $\xi(\cdot)$ and $\eta(\cdot)$ are in $C^1 \cap PC^2$, (6.2) with $t = T$ easily yields assertion (i). Thus part (a) holds.

To prove part (b), note that (6.6)_{1,2} follow from (6.5)₁[−] and the analogue for τ' of the condition (6.5)₁⁺ on τ respectively. Furthermore, for $t = \tau'$ they yield the inequality (6.7)₁.

To prove (6.7)₂ we note that (6.2)₂ and (6.5)₁[−] imply the first of the relations (vi) $\bar{\eta}_\tau(\tau') \leq \eta(\tau') = \bar{\eta}_{\tau'}(\tau')$; the second follows from the analogue for τ' of the condition (6.2)₂ on τ . By (v), (6.7)₂ holds; lastly (6.7)_{1,2} imply (6.7)₃ because of condition (6.4)₂ on τ and its analogue for τ' .

Now assume (ii) and (iii)'s falsity. Then, by (6.7)_{1,2}, either $\dot{\bar{\eta}}_\tau(\tau') < \dot{\bar{\eta}}_{\tau'}(\tau')$ or $\bar{\eta}_\tau(\tau') < \bar{\eta}_{\tau'}(\tau')$, so that definition (6.2)₂ (valid for all $\tau \in [0, T]$) and $\bar{\eta}_\tau(\cdot)$'s Taylor's expansion, of initial point τ' , yield that $\bar{\eta}_\tau(T) < \bar{\eta}_{\tau'}(T)$, absurd by (ii). Hence (iii) holds, which implies condition (iv). This yields (ii), so that (ii) to (iv) are pairwise equivalent.

⁽²²⁾ The expression (6.3)₁ follows from (4.6)₂ directly by (6.2')₁, (4.3)_{1,2}, and (3.2)₂.

Furthermore condition (iv) obviously follows from (v) by definition (6.2)₂. To prove the converse we now assume (iv). Then, by (6.2)₂ and (6.6)₁, (v)'s falsity implies that $\dot{\bar{\eta}}_\tau(t) < \dot{\eta}(t)$ for some $t \in]\tau, \tau'[$. By (iv) this yields the strict inequality $\dot{\eta}(t) > \dot{\bar{\eta}}_{\tau'}(t)$ which contrasts to (6.6)₂. Now we can assert the mutual equivalence of (iv) and (v), which by a preceding result implies part (b). q.e.d.

7. A THEOREM ON SOLUTIONS TO PROBLEM 6.1. PROOF OF ITS FIRST PART

The next theorem, A7.1, is divided in the parts (a) and (b), which are proved in Sections 7 and 8 respectively; they treat the ad. and the sym. solutions to Problem 6.1 in two respective cases, (α) and (β), the latter of which generally occurs in practice. Theorem 7.1 together with Step 3 in its proof (Section 8) shows, among other things, how to construct the afore-mentioned solutions.

Note that thesis (b₃) in A7.1(b) below is not used to prove the main theorem of Part 2 (A9.1) on Problem 9.1 of optimal time. Furthermore the optional properties (7.3), assured by thesis (b₁) of A2.1(b) to ∞^∞ solutions in case (β), are considered simply because they imply the existence of some regular extensions (in $C^1 \cap PC^2$) for the same solutions – see Remark 8.2(d)-(f).

A7.1(a) In the case (α) $\bar{\eta}_0(T) \geq U_1$ the symmetric [ad.] solutions to Problem 6.1 — see Remark 6.2 — are (Σ's motions inducing any among) the infinitely many symmetric (or ad.) 4-tuples and 6-tuples — see Definition 3.1(a)-(c), (e) — of the respective forms ⁽²³⁾

$$(7.1) \quad \sigma_4^* = (\bar{\xi}_0(\cdot), \bar{\eta}_0(\cdot), u^*(\cdot), w^*(\cdot)), \quad \sigma_6^* = (\bar{\xi}_0(\cdot), \bar{\eta}_0(\cdot), u_1^*(\cdot), \dots, u_2^*(\cdot));$$

and along them $\Phi_n \equiv 0$, hence $\ddot{u} = -g \cos \theta$ (for $u = u^*(t)$) when $v = 0$ a.e. [$\ddot{u}_i = -g \cos \theta$ when $v_i = 0$ ($i = 1, 2$) a.e.]. Furthermore, in case (α) — see Definition 3.1(d) and the Remark 3.3(d) —, for $r \in \{2, 4\}$ we have that

$$(7.2) \quad \begin{aligned} \max\{\xi_r(T, c_r); c_r \in \text{Ad}C_r\} &= \{\xi_4(T, c_4); c_4 \in \text{Sym}C_4\} = \\ &= \sigma(T) - f_d \max\{U_1, \bar{\eta}_0(T)\} = \bar{\xi}_0(T) = \dot{\xi}_0 T + \frac{g}{2} T^2 \sin \theta - \text{see (4.6)}_2. \end{aligned}$$

(b) In the remaining case, (β) $\bar{\eta}_0(T) < U_1$, theses (b₁) to (b₃) below hold.

(b₁) Problem 6.1 is solved by some symmetric 4-tuples $\sigma_4 = (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$: those with $(u(T) =) \eta(T) = U_1$; and σ_4 can be chosen, in ∞^∞ ways, satisfying all the conditions

$$(7.3) \quad u(T) = U_1 = \eta(T), \quad \dot{u}(T) = 0 = \dot{\eta}(T), \quad w(T) = 0 = \dot{u}(T).$$

(b₂) The equality (7.2)₁ — relating general solutions to Problem 6.1 with the symmetric ones — also holds in case (β), as well as (7.2)₂; instead equalities (7.2)₃₋₄ have

⁽²³⁾ For the sake of simplicity, e.g. $\bar{\xi}_0(\cdot)$ is used in (7.1) by abuse of language, instead of its restriction to $[0, T]$.

to be replaced – see (3.2)₂ – by, e.g.,

$$(7.2') \quad \sigma(T) - f_d \max\{U_1, \bar{\eta}_0(T)\} = \sigma(T) - f_d U_1 = f_d(\eta_0 - U_1) + (\xi_0 + f_d \dot{\eta}_0)T + BT^2/2.$$

(b₃) Consider any 4-tuple $\sigma_4 \doteq (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$ [6-tuple $\sigma_6 \doteq (\xi(\cdot), \eta(\cdot), \dots, w_2(\cdot))$]. Then either $\eta(T) = U_1$ so that the condition $(\gamma) \bar{\eta}_\tau(\tau) = U_1$ holds for $\tau = T$ – see (6.2)₂ –, and σ_4 [σ_6] solves Problem 6.1; or there is a last $\tau \in]0, T[$ satisfying (γ) . In the latter subcase Problem 6.1 is solved by any ad. 4-tuple σ_4^* [6-tuple σ_6^*] of the form (7.4) [(7.5)] below (see Remark 8.2(a) and condition (11.5) on τ) ⁽²⁴⁾

$$(7.4) \quad (\xi^*(t), \dots, w^*(t)) = \begin{cases} (\xi(t), \dots, w(t)) & \forall t \in [0, \tau], \\ (\bar{\xi}_\tau(t), \bar{\eta}_\tau(t), \bar{u}(t), \bar{w}(t)) & \forall t \in [\tau, T], \end{cases}$$

$$(7.5) \quad (\xi^*(t), \dots, w_2^*(t)) = \begin{cases} (\xi(t), \dots, w_2(t)) & \forall t \in [0, \tau], \\ (\bar{\xi}_\tau(t), \bar{\eta}_\tau(t), \bar{u}_1(t), \dots, \bar{w}_2(t)) & \forall t \in [\tau, T]. \quad \square \end{cases}$$

PROOF. Let case (α) hold. Then either (i) $U_1 \leq \bar{\eta}_0(T) < U_2$ or, by (3.7) and (6.2')₂, for some $t_1 \in]0, T[$ and some $t_2 \geq T$, (ii) $\{t > 0; \bar{\eta}_0(t) > U_2\} =]t_1, t_2[$. We note that in both of these mutually excluding subcases, when e.g. σ_4^* in (7.1) is assumed symmetric, then by Definition 3.1(a), (c), and the simplifying condition A3.1, $u^*(\cdot)$ must satisfy the conditions

$$(7.6) \quad v(t) = \bar{\eta}_0(t) - u^*(t) \geq 0 \quad \forall t \in [0, T], \quad u^*(\cdot) \in C^2([0, T], [U_1, U_2])$$

and (iii) $v^{-1}(0)$ is a finite union of possibly improper intervals. Note that *such* $u^*(\cdot)$ *obviously exists* and, if preferred, *in the subcase (i) $u^*(\cdot)$ can also satisfy the condition (iv) $u^*(t) = \bar{\eta}_0(t) \quad \forall t \in [t', T]$ for some $t' \in]0, T[$.*

Let us also note, with a view to treating case (β) , that (v) for every $t' \in]0, T[$ the choices of the set $v^{-1}(0) \cap [0, t']$ compatible with the above conditions are ∞^∞ (even in the subcase $\bar{\eta}_0(T) = U_1$), which obviously implies the same for the choices of the C^1 -function $u^*(\cdot)$ satisfying (7.6), (iii), and the non-compulsory additional condition (iv) below (7.6).

Now (vi) *we call \mathcal{C}' the class formed by the 4-tuples $\sigma'_4 \doteq (\bar{\xi}_0(\cdot), \bar{\eta}_0(\cdot), u^*(\cdot), 0)$ satisfying the conditions (7.6), (iii) below (7.6), and in addition (iv) below (7.6) when subcase (i) below (7.5) holds.*

To prove part (a) and (b)'s thesis (b₁), we briefly check the following

STEP 1. (a) *In the case (α) the above class \mathcal{C}' consists of ∞^∞ symmetric 4-tuples σ'_4 – see Definition 3.1(b)-(c) – having the form (7.1)₁; and (b) so many are their restrictions to every proper subinterval $[0, t']$ of $[0, T]$.*

⁽²⁴⁾ Of course by (7.4-5) $\bar{u}(\cdot)$, $\bar{w}(\cdot)$, $\bar{u}_i(\cdot)$, and $\bar{w}_i(\cdot)$ are in $C^1 \cap PC^2([\tau, T])$, $\bar{u}^{(s)}(\tau) = u^{(s)}(\tau)$, $\bar{w}^{(s)}(\tau) = w^{(s)}(\tau)$, $\bar{u}_i^{(s)}(\tau) = u_i^{(s)}(\tau)$, and $\bar{w}_i^{(s)}(\tau) = w_i^{(s)}(\tau)$ ($i = 1, 2; s = 0, 1$).

Indeed, first, the elements of \mathcal{C}' , obviously of the form (7.1)₁ with $w^*(\cdot) = 0$, are ∞^∞ by (v) below (7.6). Furthermore the same elements are in $C^1 \cap PC^2$ by (6.2'), (vi), and (7.6)₃; hence condition A3.3(ii)₁ holds.

Now assume that $\sigma'_4 \in \mathcal{C}'$. Then, by (7.6)_{2,3}, condition (i) in A3.3 holds for it, as well as (ii)₁. In addition, by A6.1(b) for $\tau = 0$ and $]t_1, t_2[$ replaced by $]0, T[$, in case (α) the couple $(\bar{\xi}_0(\cdot), \bar{\eta}_0(\cdot))$ solves the ODE (3.1) for $\mathcal{R}_T \equiv 0$ and $u = u^*(t)$ – see (7.6) – on $[0, T]$. Thus σ'_4 satisfies A3.3(ii)₂ too.

Since case (α) holds, $v(\cdot)$'s zero-set \mathcal{N}_v is a finite union of intervals by (iii) below (7.6), so that the simplifying condition A3.1 obviously holds for σ'_4 . Thus by (6.2')₁, σ'_4 satisfies A3.3(iii).

Since $w \equiv 0$ for σ'_4 , (2.3)₂ yields that $s = \xi = \bar{\xi}_0(t)$; hence by (6.2')₁ and (3.7)₁ $\dot{s} > 0$ on $]0, T[$. Thus the 4-tuple σ'_4 also satisfies A3.3(iv), so that it is sym. by A3.3. Thus Step 1(a) is proved. Step 1(b) easily follows from (v) below (7.6).

Now we consider any ad. r -tuple $\sigma_r = (\xi(\cdot), \eta(\cdot), \dots)$ ($r \in \{2, 4\}$). Then, by (6.4)₃, $\eta(T) \geq \bar{\eta}_0(T)$. Hence, by Remark 6.1(a),

STEP 2. *In case (α), (a) every ad. r -tuple of the form σ_r^* in (7.1) ($r \in \{2, 4\}$) solves Problem 6.1; and (b) in particular this holds for the ∞^∞ symmetric elements of the class \mathcal{C}' defined above Step 1.*

Conversely we now assume that $\sigma_r = (\xi(\cdot), \eta(\cdot), \dots)$ solves Problem 6.1. Hence (vii) $\bar{\eta}_0(T) = \eta(T) = \bar{\eta}_T(T)$ by Step 1 together with Remark 6.1(a), and by (6.2)₂ respectively. Then, by the mutual equivalence of the conditions (ii) and (v) in A.6.2(b) considered for $\tau = 0$ and $\tau' = T$, (vii) implies that $\eta(t) = \bar{\eta}_0(t) \ \forall t \in [0, T]$. Hence (6.3)_{1,2} yield that $\xi(t) = \bar{\xi}_0(t) \ \forall t \in [0, T]$. We can now conclude, remembering Step 2, that in case (α), for $r \in \{2, 4\}$, σ_r solves Problem 6.1 iff it has the form σ_r^* in (7.1).

Consequently, in case (α) both sides of (7.2)₁ equal $\bar{\xi}_0(T)$, and hence $\sigma(T) - f_d \bar{\eta}_0(T)$ by (6.3)₁. Therefore, in the same case (7.2)_{1,3} hold; and (7.2)₄ follows from (6.2')₁. Thus A7.1(a) holds.

8. PROOF OF THEOREM A7.1(b). SOME COMMENTS ON THE SOLUTIONS TO PROBLEM 6.1

To prove A7.1(b) let case (β) written in it hold. In order to define a solution $\sigma_4 = (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$ to Problem 6.1 we set (viii) $w(\cdot) = 0$ again.

First we consider the subcase (ix) $\eta_0 - U_1 = 0 = \dot{\eta}_0$; and in it we set

$$(8.1) \quad \begin{aligned} \eta(\cdot) &= u(\cdot) = U_1, \\ \xi(t) &= \dot{\xi}_0 t + Bt^2/2 \quad (\dot{\xi}_0 \geq 0, B \geq 0, B_{f_j} > 0 \vee \dot{\xi}_0 > 0) \quad (t \in [0, T]) \end{aligned}$$

– see (3.7)_{4,5}, (3.5). Then σ_4 solves the ODE (3.1) with $\mathcal{R}_T \equiv 0 \equiv v$ and $\Phi_n \equiv mg \cos \theta$; and it is obviously symmetric. Furthermore, remembering Remark 6.1(a), σ_4 is easily seen to solve Problem 6.1 and to satisfy (7.3). In addition, by a suitable change of (8.1)_{1,2} that keeps the validity of the conditions (7.3)_{1,3} on $u(\cdot)$, the corresponding sym.

solution σ_4 to Problem 6.1 appears to can be chosen in ∞^∞ ways and to still satisfy all conditions (7.3) by (viii) and (8.1)₁ ⁽²⁵⁾

Thus theses (b_1) and (b_2) obviously hold in (β) 's subcase (ix) .

Now let subcase (ix) fail to hold. Then, by (β) and (3.6-7), (x) exactly one instant $T_1 \in]0, T[$ satisfies the conditions

$$(8.2) \quad \bar{\eta}_0(T_1) = U_1, \quad V_1 \doteq -\dot{\bar{\eta}}_0(T_1) > 0.$$

Hence (xi) for some $\varepsilon \in]0, T_1[$ and every $\tau_1 \in]0, \varepsilon[$, the instant $T_0 \doteq T_1 - \tau_1$ renders

$$(8.3) \quad \bar{\eta}_0(T_0) \in]U_1, U_2[, \quad V_0 \doteq -\dot{\bar{\eta}}_0(T_0) > 0 \quad (0 < T_0 < T_1 < T)$$

true. Then for Problem 6.1 with T replaced by T_0 , say Problem _{T_0} , case (α) holds by (8.3)₁. Hence, by Step 2 (in Section 7), it is solved by every sym. 4-tuple $\sigma_{4, T_0} \doteq (\bar{\xi}_0(\cdot), \bar{\eta}_0(\cdot), u_0^*(\cdot), 0)$ belonging to the analogue \mathcal{C}'_{T_0} for T_0 of the class \mathcal{C}' – see above Step 1. Then $u_0^*(\cdot)$ and T_0 satisfy the conditions (7.6) in $u^*(\cdot)$ and T , as well as (iii) below (7.6); and in addition, since (8.3)₁ implies (i) _{T_0} $U_1 \leq \bar{\eta}_0(T_0) < U_2$, we have that (iv) _{T_0} $u^*(t) = \bar{\eta}_0(t) \quad \forall t \in [t', T_0]$ for some $t' \in]0, T_0]$ – see (iv) below (7.6) and the definition of \mathcal{C}' above Step 1. Then, by (8.3)₁ and (6.2')

$$(8.4) \quad u(T_0^-) = \bar{\eta}_0(T_0) \in]U_1, U_2[, \quad \dot{u}(T_0^-) = \dot{\bar{\eta}}_0(T_0), \quad \ddot{u}(T_0^-) = -g \cos \theta.$$

Furthermore, by Step 1(b), the afore-mentioned choices of σ_{4, T_0} are ∞^∞ .

We now want to extend the above symm. 4-tuple σ_{4, T_0} to a symmetric solution $\sigma_4 = (\xi(\cdot), \eta(\cdot), u(\cdot), 0)$ of Problem 6.1 that satisfies (7.3) in the remaining subcase (x) above (8.2). Then (xii) we identify σ_4 's restriction to $[0, T_0]$ with (the above) σ_{4, T_0} . Hence (iv) _{T_0} and (8.4) yield that (xiii) $v = \dot{v} = \ddot{v} = 0$ at $t = T_0^-$ for σ_4 , where $v(t) = \eta(t) \geq u_0^*(t) \quad \forall t \in [0, T_0]$.

STEP 3. We fix the quantity ε mentioned in (xi) above (8.3) and any τ_1 with

$$(8.5) \quad 0 < \tau_1 < \min\{\varepsilon, V_1/2g \cos \theta, T - T_1\} \quad (> 0).$$

⁽²⁵⁾ Briefly, this is clear by noting that we can both keep (8.1)₁ and replace the equalities (8.1)₂₋₃ with the conditions (7.3)_{1,3},

$$(A) \quad u(\cdot) \in C^1 \cap PC^2([0, T], [U_1, U_2]), \quad \ddot{u}(t) \geq -g \cos \theta \text{ a.e. in } [0, T],$$

$$(B) \quad \dot{u}(0) \geq 0, \quad \dot{u}(T) = 0, \quad u(T) = U_1,$$

and

$$(C) \quad \xi(t) = \dot{\xi}_0 t - \frac{1}{f_d} \int_0^t (t - \phi) \ddot{u}(\phi) d\phi + Br^2/2 (t \in [0, T]).$$

Then the conditions (A) to (C) and (8.1)₁ imply (7.3), are compatible with (3.6-7), and together with (7.3)_{1,3} are satisfied by ∞^∞ choices of $u(\cdot)$. By (8.1)₁, (A)₁, and (C), (3.2) (with $\mathcal{R} \equiv 0$) holds a.e. on $[0, T]$; then, by (A)₂ and (8.1)₁, the conditions (3.3)⁺ – hence (3.3) and (3.1) too – hold a.e. on $[0, T]$ for σ_4 .

Then

$$(8.6) \quad a \doteq \frac{V_1 + \tau_1 g \cos \theta)^2}{2V_1\tau_1 - \tau_1^2 g \cos \theta} > 0, \quad T_2 \doteq T_1 + \tau_1 \frac{V_1 - 2\tau_1 g \cos \theta}{V_1 + \tau_1 g \cos \theta} \in]T_1, T[;$$

and by the determination

$$(8.7) \quad u(t) = \begin{cases} \bar{\eta}_0(T_0) - V_0(t - T_0) + 2^{-1}a(t - T_0)^2 & \forall t \in]T_0, T_2], \\ U_1 & \forall t \in]T_2, T] \end{cases}$$

of $u(\cdot)$ on $]T_0, T[$ (whence $\ddot{u} = a \geq 0$ on $]T_0, T[$) we have that

$$(8.8) \quad u(T_2) = U_1, \quad \dot{u}(T_2) = 0; \quad \dot{u}(t) \leq 0 \quad \forall t \in [T_0, T], \quad \ddot{u} \geq -g \cos \theta \text{ a.e. in } [T_0, T];$$

furthermore

$$(8.9) \quad u(\cdot) \in C^1 \cap PC^2([0, T], [U_1, U_2]), \text{ being } u(t) = u_0^*(t) \quad \forall t \in [0, T_0].$$

Indeed, by (8.5) $0 < \tau_1 < V_1/2g \cos \theta$, hence $0 < 2\tau_1 g \cos \theta < V_1$. Then the inequalities (8.6)₂ and $T_2 > T_1$ – see definition (8.6)₃ – hold. Furthermore $T - T_1 > \tau_1$ by (8.5); hence (8.6)₄ too holds. Thus thesis (8.6) is proved.

In addition, by T_0 's definition above (8.3), (6.2')₂, and (8.2)

$$(8.10) \quad \bar{\eta}_0(T_0) = U_1 + V_1\tau_1 - g \frac{\cos \theta}{2}\tau_1^2, \quad V_0 = V_1 + \tau_1 g \cos \theta \quad (T_0 \doteq T_1 - \tau_1).$$

The definitions (8.10)₃ and (8.6)₃ respectively yield the first two of the equalities

$$(8.11) \quad T_2 - T_0 = T_2 - T_1 + \tau_1 = \frac{2V_1 - \tau_1 g \cos \theta}{V_1 + \tau_1 g \cos \theta} \tau_1 = \frac{2V_1\tau_1 - \tau_1^2 g \cos \theta}{V_0} = \frac{V_0}{a},$$

while (8.11)_{3,4} respectively follow from (8.10)₂ and from the definitions (8.6)₁ and (8.12)₃.

By (8.7), the conditions (8.8)_{1,2} hold iff (xiv) $u(T_2^-) = U_1$ and (xv) $\dot{u}(T_2^-) = 0$ respectively. By (8.7)⁺, $\dot{u}(T_2^-) = -V_0 + a(T_2 - T_0)$ so that (8.11) yields (xv) and hence (8.8)₂.

Furthermore, (8.7)⁺ and (8.10)₁ imply the first of the equalities

$$2u(T_2^-) - 2U_1 = 2V_1\tau_1 - \tau_1^2 g \cos \theta - 2V_0(T_2 - T_0) + a(T_2 - T_0) = 0.$$

The second holds because both $T_2 - T_0 = a^{-1}V_0$ by (8.11), and (8.11)₄ is valid. Thus (xiv) and hence thesis (8.8)₁ are proved.

By (8.7)⁺, (8.3)_{2,3}, and (8.8)₂, for $t \in]T_0, T_2[$ $\dot{u}(t) = -V_0 + a(t - T_0) \leq \dot{u}(T_2) = 0$; hence (8.7)⁻ yields (8.8)₃. Furthermore, since $a > 0$ by (8.6), (8.7) yields the last relation in thesis (8.8).

Lastly, by (xii) below (8.4), $u(\cdot)$'s restriction to $[0, T_0]$ is in $C^1 \cap PC^2([0, T_0], [U_1, U_2])$. Hence (8.3)_{1,2}, (8.4)_{1,3}, (8.7), and (8.8)_{1,3} imply thesis (8.9). Thus Step 3 holds.q.e.d.

In order to complete the construction of σ_4 's restriction to $]T_0, T[$, we set (xvi) $\eta(t) = u(t)$ (besides $w(t) = 0$) $\forall t \in]T_0, T[$. Hence, by (8.4)_{1,3} and (xii) to (xiii) below (8.4), both (xvii) $w(t) = 0 \leq v(t)$ $\forall t \in [0, T]$ and (xviii) $\eta(\cdot) \in C^1 \cap PC^2([0, T])$.

By $(xvi)_1$, $[T_0, T] \subset \mathcal{N}_V$ and $(3.3)_2^+$ holds on $[T_0, T]$; then $(8.8)_4$ implies the validity of $(3.3)_3^+$ there.

Now **(xix)** we extend $\xi(\cdot)$ to $]T_0, T]$ by requiring it to solve the ODE $(3.3)_1^+$ in $[T_0, T]$ for $\mathcal{R}_T = 0$ and $u = u(t)$ given by (8.7). Furthermore, remembering (xii) below (8.4), we also require $\xi(\cdot)$ to satisfy the initial conditions $\xi^{(s)}(T_0^+) = \bar{\xi}_0^{(s)}(T_0)$ (> 0 by the analogue for T_0 of $(8.1)_{3-5}$) ($s = 0, 1$). Then $\xi(\cdot) \in C^1 \cap PC^2([0, T])$, so that by $(8.9)_1$ the results $(xvii)_1$ and $(xviii)$ yield that

(xx) σ_4 is formed by functions in $C^1 \cap PC^2([0, T])$.

Since $a > 0$ by $(8.6)_{1-2}$, (8.7) and (xvi) yield that $\ddot{\eta} = \ddot{u} \geq 0$ on $[T_0, T] \subset \mathcal{N}_V$, so that $(3.3)_{2-3}^+$ also hold for σ_4 on $[T_0, T]$. Then, by (xii) below (8.4) and (xix) ,

(xxi) σ_4 satisfies the condition (3.3), equivalent to the ODE (3.1) for some Φ_n that is ≥ 0 on $[T_0, T]$.

By (xvi) and $(8.8)_3$, $\dot{\eta} = \dot{u} \leq 0$ on $[T_0, T]$. Furthermore, by $(4.5)_{1-3}$, $\dot{\xi}(t) + f_4 \dot{\eta}(t) = \dot{\sigma}(t) > 0 \quad \forall t \in]0, T]$. Then, by $(xvii)$, $\dot{s} = \dot{\xi} > 0$ holds on $[T_0, T]$ for σ_4 ; hence the sliding condition A2.4 holds for σ_4 's restriction to $[T_0, T]$; then, by (xii) below (8.4), A2.4 holds for σ_4 .

Now it is easy to check that, by (xii) below (8.4) and (8.7), the constraint $u([0, T]) \subset [U_1, U_2]$ is also satisfied by $\sigma_4 = (\xi(\cdot), \eta(\cdot), u(\cdot), 0)$; hence, by the preceding properties proved for it, it is easy to deduce by A3.3 that **(xxii)** σ_4 is symmetric – see Definition 3.1(c).

Furthermore, by $(8.6)_{3-4}$, $(8.7)^-$, and (xvi) , (7.3) holds; thus, by $(7.3)_2$ and Remark 6.1(a), σ_4 solves Problem 6.1 in the class of all ad. 4-tuples. Since by (xii) below (8.4) the sym. solution σ_4 can be chosen in ∞^∞ ways, thesis (b_1) of part (b) is proved.

REMARK 8.1. Since the function $\eta(\cdot)$ in the ad. 4-tuple σ_4 is not a control, unlike $w(\cdot)$ and $u(\cdot)$, we note that equality $(xvi)_1$, which expresses that σ_4 is jump-free in $[T_0, T]$ and provides σ_4 with the expected properties, *must hold for every ad. 4-tuple* $\sigma_4 = (\xi(\cdot), \eta(\cdot), u(\cdot), 0)$ that induces $\sigma_{4, T_0} = (\bar{\xi}_0(\cdot), \bar{\eta}_0(\cdot), u_0^*(\cdot), 0)$ on $[T_0, T]$ and satisfies (8.7) or at least its consequence $(8.8)_4$ ⁽²⁶⁾

To deduce thesis (b_2) of part (b) we note that, by thesis (b_1) , some symmetric 4-tuple solves Problem 6.1 and satisfies condition $(7.3)_1$. Hence $(7.2)_2$ also holds in case (β) . The same can be said of $(7.2)_1$ because sym. 4-tuples are admissible. Of course, in case (β) $\max\{U_1, \bar{\eta}_0(T_0)\} = U_1$; hence, by (4.6) and $(4.3)_{1-2}$, $(7.2)_{3-4}$ have to be replaced by, e.g., $(7.2')_{1-2}$. Thus (b_2) holds.

To deduce thesis (b_3) , for $r \in \{2, 4\}$ we consider the ad. r -tuple σ_r in (b_3) and

⁽²⁶⁾ Indeed, since σ_4 is ad., $\eta(\cdot)$ and $v(\cdot)$ are in $C^1 \cap PC^2([0, T])$. Now let $(xvi)_1$ not hold, for reduction ad absurdum. Then, for some $\bar{t} \in]0, T[$, $(\mathcal{A}) \quad v(\bar{t}) = \eta(\bar{t}) - u(\bar{t}) > 0$. Furthermore $v(T_0) = 0 = \dot{v}(T_0)$ by $(xiii)$ below (8.4). Hence $(\mathcal{B}) \quad t_2 = \sup\{t \in [T_0, \bar{t}]; v(t) = 0\} \in [T_0, \bar{t}[\subset [T_0, T]$. Then $(\mathcal{C}) \quad v(t_2) = 0 = \dot{v}(t_2)$. By (\mathcal{A}) and $(\mathcal{C})_1$, for some $t_3 \in]T_0, \bar{t}[$, $(\mathcal{D}) \quad 0 < \dot{v}(t_3) = \int_{t_2}^{t_3} \ddot{v} dt$. However, by (\mathcal{B}) , $]t_2, t_3[\subset [0, T] \setminus \mathcal{N}_V$, so that by $(3.3)_2^-$ and the consequence $(8.8)_4$ of (8.7) we have that $\ddot{v} = \ddot{\eta} - \ddot{u} = 0$ on $]t_2, t_3[$, in contrast to (\mathcal{D}) . Therefore $(xvi)_1$ is true.

condition (γ) there; furthermore we exclude the trivial case $\eta(T) = U_1$ where σ_r solves Problem 6.1 by Remark 6.1(a), and (γ) holds with $\tau = T$.

Then (xxiii) $\bar{\eta}_\tau(T) = \eta(T) > U_1 > \bar{\eta}_0(T)$ by (6.2)₂ with $\tau = T$, the constraint $\eta \geq U_1$, and (β) . Furthermore $\tau \mapsto \bar{\eta}_\tau(T)$ is continuous by (i) in A6.2(a). Hence the last $\tau \in]0, T[$ such that (γ) $\bar{\eta}(T) = U_1$ exists in case (β) . Let us now set

$$(8.12) \quad \mathcal{C}_2^{(s)} \doteq (u^{(s)}(\tau), w^{(s)}(\tau)), \quad \mathcal{C}_4^{(s)} \doteq (u_1^{(s)}(\tau), \dots, w_2^{(s)}(\tau)) \quad (s = 0, 1).$$

Since σ_r is ad., and hence formed by functions in $C^1 \cap PC^2$, the assumption A2.7 yields that (xxiv) either $\mathcal{C}_r^{(0)} \in K_r$ or $\mathcal{C}_r^{(1)}$ is tangent to ∂K_r . Then for $r = 2$ [$r = 4$] there is a function $t \mapsto \bar{\mathcal{C}}_r(t) \doteq (\bar{u}(t), \bar{w}(t))$ [$\doteq (\bar{u}_1(t), \dots, \bar{w}_2(t))$] in $C^2([\tau, T], K_r)$ such that

$$(8.13) \quad \bar{\mathcal{C}}_r(t) = \mathcal{C}_r^{(0)}, \quad \dot{\bar{\mathcal{C}}}_r = \mathcal{C}_r^{(1)}; \quad \bar{u}(t) < \bar{\eta}_\tau(t), \quad \bar{\mathcal{C}}_r(t) \in K_r, \quad v(t) > 0 \quad \forall t \in]t, T[,$$

– see (2.14) – where $v(t) = \bar{\eta}_\tau(t) - \bar{u}(t) = \min\{\bar{\eta}_\tau(t) - \bar{u}_1(t), \bar{\eta}_\tau(t) - \bar{u}_2(t)\} \quad \forall t \in]t, T[$ – see (2.3)₃, (2.3')₂, and (2.5)_{3,4}.

Now one can easily see that (xxv) the r -tuple σ_r^* defined by (7.4) or (7.5) in connection with the above choice of $\bar{\mathcal{C}}_r(\cdot)$ is admissible. In particular, by (8.13)₅ and σ_r 's admissibility, σ_r^* trivially satisfies the sliding condition A2.4 on $[\tau, T]$ and $[0, \tau]$ respectively.

Now we note that, since $\bar{\eta}_\tau(T) = U_1$ by (γ) , (7.4) or (7.5) yields that σ_r^* solves Problem 6.1 by Remark 6.1(a). Thus thesis (b_3) too holds. q.e.d.

REMARK 8.2. (a) The instant τ associated to any ad. 4-tuple σ_4 or 6-tuple σ_6 in A7.2(b) is the last instant at which the skier U , supposed to be implementing a motion of Σ that induces σ_4 and σ_6 , can start a correction of this motion, capable to turn it into a solution of Problem 6.1; this appears from τ 's definition above (7.4), the forms of the corresponding solutions σ_4^* and σ_6^* , and Remark 6.1(a). A lower bound for τ will be given in (11.5).

(b) The afore-mentioned correction – see (7.4-5) – consists of a small traction exerted by the skier on his skis, that balances their weight and thus renders $\Phi_n = 0$ on $[\tau, T]$. Incidentally, on the one hand the skier can also disjoin his skis from the ski-run; on the other hand the above traction can practically be replaced by a relaxation of the skier's legs, especially if this relaxation starts a bit earlier than τ .

(c) Along, e.g., the version (7.4) of the above correction, in the case $\tau < T$, at the end instant T the ski-run exerts on Σ an impulse, reasonably inelastic and hence expressed by

$$(8.14) \quad \mathbf{I} = m \dot{\bar{\eta}}_t(T) (f_d \mathbf{T} - \mathbf{n}) \quad (\dot{\bar{\eta}}_t(T) < 0).$$

(d) By part (c), e.g. every solution to Problem 6.1, of the form (7.4)₂, is obviously in $C^1 \cap PC^2([0, T])$ and fails to be extendible to any so regular (ad.) 6-tuple («defined» on

$[0, T']$ for some $T' > T$) because of (8.14); and this may appear somewhat nonrealistic. However

(e) the solutions to Problem 6.1 satisfying the conditions (7.3), whose existence in case (β) is stated in thesis (b_1) of A7.1(b), have some ad. 4-tuples extending them.

(f) In the subcase $\bar{\eta}_0(T) = U_1$ of the case (α) , generally absent in practice, all solutions to problem 6.1 – see (7.1) – have the «final» impulse $I \neq 0$ expressed by (8.14). However, briefly, the $\sup\{\xi(T); (\xi(\cdot), \dots, w_2(\cdot))\}$ is an extendible ad. 6-tuple²⁷ obviously equals the $\max\{\xi(T); (\xi(\cdot), \dots, w_2(\cdot))\}$ is ad.²⁸

9. A NATURAL PROBLEM OF MINIMUM TIME WITH NEGLIGIBLE AIR RESISTANCE

We use the preceding considerations on Problem 6.1 to solve:

PROBLEM 9.1. Assume that, unlike f_d , the air resistance is negligible ($\mathcal{R} \equiv 0$) and that $\bar{\xi} > 0$. Then how can U behave along a motion of Σ that induces the ad. 4-tuple $\sigma_4 \doteq (\xi(\cdot), \dots, w(\cdot))$ – see Definition 3.1(a) –, in order to minimize the time \bar{t} at which C 's ξ -coordinate $\xi = \xi(\cdot)$ reaches the value $\bar{\xi}$,

$$(9.1) \quad \bar{t} \doteq \min\{t > 0; \xi(t) = \bar{\xi}\} \mapsto \inf? \quad \square$$

The above ad. 4-tuple σ_4 , as well as any ad. 6-tuple σ_6 inducing it, is obviously regarded as *ad. for Problem 9.1* in the sense that (i) $\xi(\bar{t}) = \bar{\xi}$ must hold for some $\bar{t} \in [0, T]$. Lastly, by Definition 3.1(a), $\xi(\cdot)$ is C^0 so that the minimum in (9.1) exists. Furthermore any extension of a given σ_4 , *i.e.* any increase of T , does not affect \bar{t} 's value satisfying (i).

The next theorem shows that Problem 9.1 is always solved by *e.g.* infinitely many sym. 4-tuples; and that these are the solutions to Problem 6.1 for a certain choice of T . In order to determine this choice we use the instants $t_* = t_*(\bar{t})$ and $T_* \doteq T_*(\bar{\xi})$ in $[0, +\infty[$ for which – see (6.2')₁:

$$(9.2) \quad \bar{\xi} = \bar{\xi}_0(t_*), \quad \text{i.e.} \quad t_* = \left(-\dot{\xi}_0 + \sqrt{\dot{\xi}_0^2 + 2\bar{\xi}g \sin \theta} \right) / g \sin \theta$$

and – see (4.6) and (6.3) ⁽²⁷⁾

$$(9.3) \quad \bar{\xi} = \sigma(T_*) - f_d U_1, \quad \text{i.e.} \quad T_* = \left\{ -\dot{\sigma}(0) + \sqrt{\dot{\sigma}(0)^2 + 2B[\bar{\xi} - f_d(\eta_0 - U_1)]} \right\} / B.$$

A9.1. Assume that (i) $\mathcal{R} \equiv 0$ and (ii) $\bar{\xi} > 0$. Then (a) to (b) below hold.

(a) in the case (α) , *i.e.* $U_1 \leq \bar{\eta}_0(t_*)$, the ad. 4-tuples [6-tuples] that solve Problem 9.1 are those having a «domain» $[0, T]$ with $T \geq t_*$ and solving Problem 6.1 for $T = t_*$; and these are infinitely many: those of the form (7.1)₁ [(7.1)₂]. Thus \bar{t} – see

⁽²⁷⁾ By (4.6)₂ and (4.3)₁, (9.3)₁ with $T_* > 0$ is equivalent to $f_d \eta_0 + \dot{\sigma}(0) T_* + B T_*^2 / 2 = \bar{\xi} + f_d U_1$, and hence to (9.3)₂.

(9.1) – has the minimum t_* both within Σ 's general motions and within the symmetric ones. Furthermore

(b) in the remaing case $\neg(\alpha)$, we have that (iii) both $t_* < T_*$ and $\eta_0(T_*) < U_1$; in addition, (iv) for $r \in \{2, 4\}$, the ad. r -tuples $(\xi(\cdot), \eta(\cdot), \dots)$ that solve Problem 9.1 are those solving Problem 6.1 for $T = T_*$, hence those for which $(u(T_*) =) \eta(T_*) = U_1$; (v) infinitely many among these solutions satisfy all conditions (7.3) for $T = T_*$; lastly for $r \in \{2, 4\}$

$$(9.4) \quad \begin{aligned} T_* &= \min\{t \in [0, T]; \xi_r(t, c_r) = \bar{\xi}, c_r \in \text{Ad } C_r, T \geq T_*\} = \\ &= \min\{t \in [0, T]; \xi_r(t, c_r) = \bar{\xi}, c_r \in \text{Sym } C_r, T \geq T_*\} \\ &\quad - \text{ see Definition 3.1(d).} \quad \square \end{aligned}$$

Indeed, first, we consider the case (α) ; and for $r \in \{2, 4\}$ let $\sigma_r \doteq (\xi(\cdot), \eta(\cdot), c_r)$ be an ad. r -tuple for Problem 9.1. Then $\xi(\cdot)$ and $\bar{\xi}_0(\cdot)$ satisfy the same initial conditions (3.6)₁₋₂; furthermore the ODE (3.1)₁⁺ is solved by $\xi(\cdot)$ for $\Phi_n \geq 0 = \mathcal{R}_T$ and T large enough, so that (6.2')₁ and (3.7)₁ imply that $\ddot{\xi}(t) \leq g \sin \theta = \ddot{\bar{\xi}}_0(t)$ for a.e. $t \in [0, T]$ and that $\bar{\xi}_0(\cdot)$ is strictly increasing on $[0, +\infty[$. Hence (vi) $\xi(t) \leq \bar{\xi}_0(t) \leq \bar{\xi}_0(t_*) = \bar{\xi} \forall t \in [0, t_*]$ by (9.2)₁. Then $t_* \leq \inf \bar{t}$ – see (9.1)₁; and every solution to Problem 6.1 for $T = t_*$ solves Problem 9.1, because by A7.1(a) it has the form (7.1) and (9.2)₁ holds; furthermore (vii) $t_* = \min \bar{t}$.

Conversely let now σ_r solve Problem 9.1 in case (α) . Hence $\xi(t_*) = \bar{\xi}$ by (vii). Then, by (9.2) and (vi), σ_r also solves Problem 6.1 for $T = t_*$; hence by A7.1(a) it has the form (7.1) for $T = t_*$. Thus part (a) holds.

To prove part (b) we consider the case $\neg(\alpha)$ $U_1 > \bar{\eta}_0(t_*)$. Then, by (6.3)₁, (9.2)₁, and (9.3)₁,

$$(9.5) \quad \sigma(t_*) = \bar{\xi}_0(t_*) + f_d \bar{\eta}_0(t_*) < \bar{\xi} + f_d U_1 = \sigma(T_*).$$

Furthermore $\dot{\sigma}(t) > 0 \forall t > 0$ by (4.5)₂₋₃ (with $T > 0$ arbitrary). Hence $t_* < T_*$ so that $U_1 > \bar{\eta}_0(T_*)$ by (3.7)₂, $\neg(\alpha)$, and (6.2')₂. Thus thesis (iii) in part (b) obviously holds.

Now, for $r \in \{2, 4\}$, let the ad. r -tuple $\sigma_r^* \doteq (\xi^*(\cdot), \eta^*(\cdot), \dots)$ solve Problem 6.1 for $T = T_*$. Then thesis (b₂) in A7.1(b) – precisely (7.2)₁₋₂ together with (7.2')₂ – and (9.3)₁ respectively yield the equalities (viii) $\xi^*(T_*) = \sigma(T_*) - f_d U_1 = \bar{\xi}$.

For reduction *ad absurdum* we now suppose that

HYP. (A) *some instant (ix) $T < T_*$ and some r -tuple $\sigma_r = (\xi(\cdot), \eta(\cdot), \dots)$, ad. for Problem 9.1, render (x) $\xi(T) = \bar{\xi}$ true.*

Then by (6.3)₁₋₂ the first two of the relations

$$(9.6) \quad \bar{\xi} = \xi(T) = \sigma(T) - f_d \eta(T) \leq \sigma(T) - f_d U_1 < \sigma(T_*) - f_d U_1 = \bar{\xi}$$

hold; the third follows from the constraint $\eta \geq U_1$, while (ix) and (4.5)₂₋₃ yield (9.6)₄. Lastly (9.6)₅ is (viii)₂. Thus Hyp.(A) is absurd.

We conclude that $\inf \bar{t} \geq T_*$ – see (9.1); and by (viii), $\min \bar{t} = T_*$. Thus the solutions to Problem 9.1 are those to Problem 6.1 for $T = T_*$. Now, by thesis (b_1) in A7.1(b), the whole theses (iv) and (v) in part (b) hold.

Furthermore, by thesis (iv) in part (b), we easily see that (9.4) in A9.1(b) follows from thesis (b_2) in A7.1(b) and (9.3). q.e.d.

REMARK 9.1. By thesis (iv) in A9.1(b), thesis in A7.1(b) is also useful to construct solutions to Problem 9.1 in the case $\neg(\alpha)$.

Further comments on solutions to Problem 9.1 can be found in Section 13, where other properties of those to Problem 6.1 are available.

PART 3. SOME PROPERTIES OF THE SOLUTIONS TO THE PRECEDING OPTIMIZATION PROBLEMS CONSIDERED IN PART 2

10. SOME PRELIMINARIES FOR Σ 'S JUMP-FREE DYNAMIC MOTIONS

In order to deal with Σ 's (strongly) jump-free dynamic motions – see Definition 4.1 and Remark 4.2(b) – we consider the function

$$(10.1) \quad \bar{\eta}_\tau(t, \eta, \dot{\eta}) \doteq \eta + \dot{\eta} \cdot (t - \tau) - g \frac{\cos \theta}{2} (t - \tau)^2$$

where τ, η , and $\dot{\eta}$ are real variables, and the definitions

$$(10.2) \quad M_\eta \doteq \sqrt{2(U_2 - \eta)g \cos \theta}, \Delta_\eta \doteq \sqrt{2(U_2 - \eta)/g \cos \theta}, \zeta \doteq \tau' - \tau.$$

Furthermore we prove the following theorem.

A10.1. (a) for all $\tau, \eta \in \mathcal{R}$ and for $\mathcal{R}^* \doteq \mathcal{R}/\{0\}$, the equation in $\dot{\eta}$

$$(10.3) \quad \bar{\eta}_\tau(\tau + \zeta, \eta, \dot{\eta}) = U_2 \text{ with } (\eta, \zeta) \in [U_1, U_2] \times \mathcal{R}^* \cup \{U_2\} \times \mathcal{R}$$

has the unique (τ -independent) solution expressed by the upper part of

$$(10.4) \quad \dot{\eta} = f_{(\eta)}(\zeta) \doteq g \frac{\cos \theta}{2} \zeta + \begin{cases} (U_2 - \eta)\zeta^{-1} \\ 0 \end{cases} \text{ for } (\eta, \zeta) \in \begin{cases} [U_1, U_2] \times \mathcal{R}^* \\ \{U_2\} \times \mathcal{R} \end{cases},$$

in case $\zeta \neq 0$; for $\zeta = 0$ it is solved by all $\dot{\eta} \in \mathcal{R}$; and in both cases $f_{(\eta)}(\zeta)$ expresses its solution having the minimum absolute value ⁽²⁸⁾

(b) Given $\tau \in \mathcal{R}$ and $\eta \in [U_1, U_2]$, (i) the solutions $(\zeta, \dot{\eta})$ to (10.3)₁ that have the least value of $|\dot{\eta}|$ among these solutions are (Δ_η, M_η) and $(-\Delta_\eta, -M_\eta)$. Furthermore

$$(10.5) \quad \max \bar{\eta}_\tau([\tau, \pm\infty[, \eta, \pm M_\eta) = U_2 = \bar{\eta}_\tau(\tau + \zeta, \eta, \pm M_\eta), \text{ iff } \zeta = \pm \Delta_\eta,$$

⁽²⁸⁾ We regard $f_{(\eta)}(\zeta)$ as being defined only under condition (10.3)₂; if this condition fails to hold, then the equation (10.3)₁ in $\dot{\eta}$ has no solutions.

(ii) $\Delta_\eta = 0 = M_\eta = f_{(\eta)}(\Delta_\eta)$, if $\eta = U_2$; and

$$(10.6) \quad \min_{f_{(\eta)}} \left(\begin{array}{c} \mathcal{R}^+ \setminus \{0\} \\ \mathcal{R}^+ \end{array} \right) = M_\eta = f_{(\eta)}(\zeta), \text{ iff } \zeta = \Delta_\eta \text{ for } \begin{cases} \eta < U_2 \\ \eta = U_2. \end{cases}$$

(c) Fix $\zeta > 0$ and η satisfying (10.3)₂; and assume that

$$(10.7) \quad \dot{\eta} \leq f_{(\eta)}(\zeta) \text{ if } 0 < \zeta \leq \Delta_\eta; \quad \dot{\eta} \leq M_\eta \text{ if } \zeta \geq \Delta_\eta.$$

Then

$$\bar{\eta}_\tau(\phi, \eta, \dot{\eta}) \leq U_2 \quad \forall \phi \in [\tau, \tau'], \text{ where } \tau' = \tau + \zeta. \quad \square$$

Indeed by definitions (10.1) and (10.2)₃, part (a) holds obviously.

To prove part (b) we assume that $\eta \in [U_1, U_2]$ and $(\zeta, \dot{\eta})$ solves (10.3)₁. Furthermore let $|\dot{\eta}|$ have the least value among such solutions, so that $(\zeta, \dot{\eta})$ solves (10.4)₁ too. Then either $\eta = U_2$ so that, by (10.4)⁻ and (10.2), $\zeta = \Delta_\eta = 0 = \dot{\eta} = M_\eta$; hence thesis (i), (10.5), (10.6)⁻, and (ii) below (10.5) hold trivially (for $\eta = U_2$).

Otherwise (iii) $U_1 \leq \eta < U_2$. Then, by (10.4)⁺, ζ satisfies the conditions

$$(10.9) \quad 0 = \partial f_{(\eta)}(\zeta) / \partial \zeta \equiv 2^{-1} g \cos \theta - (U_2 - \eta) \zeta^{-2}, \text{ equivalent to } \zeta = \pm \Delta_\eta,$$

while (10.2)_{1,2} and (10.4)⁺ yield the first two of the relations

$$(10.10) \quad \dot{\eta} = f(\pm \Delta_\eta) = \pm M_\eta, \quad \partial^2 f_{(\eta)}(\zeta) / \partial \zeta^2 = 2(U_2 - \eta) \zeta^{-3} \geq 0 \text{ for } \zeta \geq 0;$$

furthermore (10.10)_{3,4} follow from (10.9)₂. By the equivalence asserted in (10.9), in the case (iii) (10.10) yields (10.6)₁⁺ and (b)'s thesis (i). This thesis and (10.10)₂ obviously imply the equivalence between (10.6)₂ and (10.6)₃, i.e. – since by (10.4) $f_{(\eta)}(\cdot)$ is odd – the fact that, in case (iii), the equation $\pm M_\eta = f_{(\eta)}(\zeta)$ in ζ has the unique solution $\zeta = \pm \Delta_\eta$. Therefore, by the equivalence of the equations (10.4)₁⁺ and (10.3)₁ in $(\zeta, \dot{\eta})$ (for $\eta < U_2$ and hence $\zeta \neq 0$), in the case (iii) (10.5)₂[±] is equivalent to (10.5)₃[±]; and in particular we have that (iv) $\bar{\eta}_\tau(\tau \pm \Delta_\eta, \eta, \pm M_\eta) = U_2$.

Now let (10.5)₁[±] fail to hold by reduction *ad absurdum*. Then by (iv) the L.H.S. of (10.5)₁[±] is $> U_2$; hence, for some $t \in [\tau, \pm\infty[$, $\bar{\eta}_\tau(t, \eta, \pm M_\eta) > U_2$. Then by (iii) above (10.9) and by (10.1), for some $\dot{\eta} \in [0, \pm M_\eta[$, $\bar{\eta}_\tau(t, \eta, \dot{\eta}) = U_2$ in contrast to thesis (i). Therefore (10.5)₁[±] holds. Thus part (b) is proved.

To prove part (c) we assume (10.3)₂ and (10.7), we fix $\phi \in [\tau, \tau']$, and first we consider the case $\zeta = \tau' - \tau \geq \Delta_\eta$, so that $\dot{\eta} \leq M_\eta$ by the implication (10.7)₂. Hence by (10.6)₁

$$(10.11) \quad \dot{\eta} \leq \dot{\eta}_\phi = f_{(\eta)}(\phi - \tau).$$

In the remaining case $\Delta_\eta > \zeta = \tau' - \tau \geq \phi - \tau > 0$ – see (10.2)₃. Hence $\dot{\eta} \leq f_{(\eta)}(\zeta) \leq f_{(\eta)}(\phi - \tau)$ by the implication (10.7)₁ and by (10.6)_{1,2} with $\zeta = \Delta_\eta$. Thus (10.11) is still holding.

In every case, by definition (10.11)₂, the implication (10.3)₁ \implies (10.4)₁⁺ yields that $\bar{\eta}_\tau(\phi, \eta, \dot{\eta}_\phi) = U_2$. Then (10.11)₁ and (10.1) with $t = \phi > \tau$ imply the inequality

(10.8)₁. By ϕ 's arbitrariness and $\bar{\eta}_\tau(t, \eta, \dot{\eta})$'s continuity, (10.8) holds. Thus part (c) is proved. \square

Below, e.g. in Theorems A10.2 and A11.1, a dynamic motion $(\xi(\cdot), \eta(\cdot))$ of Σ in $C^1 \cap PC^2$ is often referred to; then $\bar{\eta}_t(\cdot)$ is regarded as associated to it, and the equalities (or notations)

$$(10.12) \quad \Delta \doteq t' - t, \quad \eta' = \eta(t'), \quad \dot{\eta}' = \dot{\eta}(t'), \quad \eta = \eta(t), \quad \dot{\eta} = \dot{\eta}(t) \quad \forall t, t' \in [0, T]$$

are used; thus

$$(10.13) \quad \bar{\eta}_t(\cdot) = \bar{\eta}_t(\cdot, \eta, \dot{\eta}) \quad \forall t \in [0, T].$$

Now some preliminaries for theorem A11.1 are stated, regarding $t, t' \in [0, T]$ and $\eta \in [U_1, U_2]$ as arbitrarily fixed.

A10.2. Along any (ad.) dynamic motion $(\xi(\cdot), \eta(\cdot))$ of C

$$(10.14) \quad \eta' - \eta - \dot{\eta} \cdot \Delta = \int_t^{t'} \ddot{\eta}(\phi) \cdot (t' - \phi) d\phi \geq -g \frac{\cos \theta}{2} \Delta^2, \quad \text{for } t, t' \in [0, T]$$

– see (10.12)₁₋₂ and (10.13). Furthermore, along every str. ad. motion of Σ

$$(10.15) \quad \begin{cases} +\dot{\eta} \\ -\dot{\eta} \end{cases} \leq f_{(\eta)}(|t' - t|), \quad \text{when } 0 \leq \begin{cases} t < t' \\ t' < t \end{cases} \leq T \quad (\eta, \eta' \in [U_1, U_2]). \quad \square$$

Indeed (10.12-13) yield (10.14)₁ ⁽²⁹⁾ while by (3.3)₂₋₃ $\ddot{\eta} \geq -g \cos \theta$, whence (10.14)₂ ⁽³⁰⁾ Thus thesis (10.14) holds.

To prove thesis (10.15) we assume (10.15)_{2,4}[±], whence $\Delta \geq \Delta$. Then inequality (10.14)₁₋₂ implies (10.16)₁[±] below,

$$(10.6) \quad \dot{\eta} \begin{cases} \leq \\ \geq \end{cases} \frac{\eta' - \eta}{\Delta} + g \frac{\cos \theta}{2} \Delta \begin{cases} \leq \\ \geq \end{cases} \pm f_{(\eta)}(|t' - t|)$$

and incidentally *it is equivalent to it*. Furthermore, along any str. ad. motion of Σ the inclusion (10.15)₅ must hold. Then, by definition (10.4)₂, the inequality (10.14)₁₋₂ yields, for $\Delta \geq 0$, (10.16)₂[±] ⁽³¹⁾ hence (10.16)[±] holds. This formula obviously implies thesis (10.15)[±]. q.e.d.

⁽²⁹⁾ Indeed, calling $\lambda(t')$ the R.H.S. of (10.14)₁, $(\alpha) \quad d\lambda(t')/dt' = \dot{\eta}(t') - \dot{\eta}(t) \quad \forall t' \in [t, T]$ and $\lambda(t) = 0$. Hence, by integration of (α) on $[t, t']$, $\lambda(t') = \eta(t') - \eta(t) - \dot{\eta}(t) \cdot (t' - t)$. Hence (10.14)₁ holds.

⁽³⁰⁾ For both $\Delta > 0$ and $\Delta < 0$, along the integral in (10.14)₂, regarded as curvilinear, we have that $0 < (t' - \phi)d\phi = -\psi d\psi$, where $\psi = t' - \phi$. Hence, since $\ddot{\eta} \geq -g \cos \theta$,

$$\int_t^{t'} \ddot{\eta} \cdot (t' - \phi) d\phi \geq -g \cos \theta \int_t^{t'} (t' - \phi) d\phi = g \cos \theta \int_\Delta^0 \psi d\psi = -g \frac{\cos \theta}{2} \Delta^2.$$

⁽³¹⁾ By (10.4)₂ the first of the relations

$$\pm f_{(\eta)}(\Delta) = \frac{U_2 - \eta}{\Delta} + g \frac{\cos \theta}{2} \Delta \begin{cases} \geq \\ \leq \end{cases} \frac{\eta' - \eta}{\Delta} + g \frac{\cos \theta}{2} \Delta, \quad \text{where } \Delta \begin{cases} \geq \\ \leq \end{cases} 0$$

holds. The second follows from the consequences $\eta < U_2$ and $\eta' < U_2$ of (10.15)₅. For $\Delta = t' - t$ these relations imply (10.16)₂[±].

11. ON SOLUTIONS TO PROBLEMS 6.1 AND 9.1. A BOUND FOR THEIR
ARBITRARINESS. ON Σ 'S JUMP-FREE MOTIONS, IN PART REFERRING TO ROBOTS

Here we state theorem A11.1 – where (10.12-13) are presupposed – and some remarks on it, concentrating on (partially) str. jump-free motions – see Definition 4.1 – by Remark 3.2(c). The proof of the theorem is given in Section 12.

A11.1. (a) We assume that (i) $\sigma_4 \doteq (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$ is a str. ad. 4-tuple «defined» on $[0, T]$, (ii) $\bar{\eta}_t(\cdot)$ is associated to σ_4 whenever $t \in [0, T]$ – see (6.2) – and (iii) σ_4 is jump-free in $[t_1, T]$ where $t_1 \in [0, T]$ – so that σ_4 is str. jump-free there by A4.3(a). Then, for all $t, t' (= t + \Delta) \in [t_1, T]$, both

$$(11.1) \quad \begin{cases} \dot{\eta} \\ -\dot{\eta} \end{cases} \leq f_{(\eta)}(\Delta) \text{ if } \begin{cases} T - \Delta_\eta \leq t < t' \leq T \\ t_1 \leq t' < t \leq \Delta_\eta \end{cases}; \quad \begin{cases} \dot{\eta} \\ -\dot{\eta} \end{cases} \leq M_\eta \text{ if } t \in \begin{cases} [t_1, T - \Delta_\eta] \\ [t_1 + \Delta_\eta, T]; \end{cases}$$

and (iv) $\bar{\eta}_t(\phi) \leq U_2 \quad \forall \phi \in [t, T]$.

(b) Assume (i) to (ii). Then, *given every* $t \in [0, T]$, *exactly one* $t' = t + \Delta \geq t$ (possibly $> T$) *satisfies the condition* (v) *both* $\bar{\eta}_t(t') = U_1$ *and, whenever* $t = 0 = \eta_0 \leq \leq \dot{\eta}_0 \geq 0, t' > 0$ *too* ⁽³²⁾ *the corresponding value* $\Delta_{(t)}$ *of* $\Delta \doteq t' - t$ *renders the first of the equalities*

$$(11.2) \quad \Delta_{(t)} = \frac{\dot{\eta} + \sqrt{\dot{\eta}^2 + 2(\eta - U_1)g \cos \theta}}{g \cos \theta} \begin{cases} = 0 & \text{if } \eta = U_1, \\ \leq \Delta_\eta + \Delta_{U_1} < 2\Delta_{U_1} & \text{if } \eta \in]U_1, U_2] \end{cases}$$

true ⁽³³⁾ and when (iii) above (11.1) holds, then all remaining (conditional) relations (11.2)₂⁺ and (11.2)₂₋₃⁻, as well as (iv) in part (a) are also valid.

(c) The (partly) initial conditions (3.6-7) are compatible with some str. jump-free choice of the above 4-tuple σ_4 – see Definition 4.1 (and A4.4) – iff

$$(11.3) \quad \dot{\eta}_0 \leq f_{(\eta_0)}(T) \text{ if } \Delta_{\eta_0} \geq T; \quad \dot{\eta}_0 \leq M_{\eta_0} \text{ if } \Delta_{\eta_0} \leq T \quad - \text{ see (10.2).}$$

Furthermore (11.3) implies that, for some $u(\cdot)$, both

$$(11.4) \quad u(0) = \eta_0, \dot{u}(0) = \dot{\eta}_0, \ddot{u} \geq -g \cos \theta \text{ on } [0, T], u(\cdot) \in C^1 \cap PC^2([0, T], [U_1, U_2])$$

and the corresponding str. ad. 4-tuple $\bar{\sigma}_4 \doteq (\xi_2(\cdot, c_2), \eta_2(\cdot, c_2), c_2)$ where $c_2 = (u(\cdot), w(\cdot)) = (u(\cdot), 0)$ – see Definition 3.1(d) – is str. jump free. Lastly any choice of $\bar{\sigma}_4$ for which $w(\cdot) = 0$ and (11.3-4) hold is str. jump free.

(d) Assume (i) to (ii) and that, for $r \in \{2, 4\}$, (vi) $\sigma_r \doteq (\xi(\cdot), \eta(\cdot), \dots)$ is str. ad. and also jump-free in $[t_1, T]$ for some $t_1 \in]0, T - 2\Delta_{U_1}[$, which generally occurs in practice. Then the last instant τ mentioned in thesis (b₃) of A7.1(b), at which σ_r can be turned into a solution σ_r^* to Problem 6.1 (see Remark 8.2(a)) satisfies the condition

$$(11.5) \quad t > T - 2\Delta_{U_1} \quad - \text{ see (10.2)}_2.$$

⁽³²⁾ Note that, in case $\eta = \eta(t) > U_1$, by (6.2)₂ $t' = t + \Delta (\geq t)$ is determined by the first equality in condition (v), which implies $t' > 0$.

⁽³³⁾ By (6.2)₂ and (v), both $\bar{\eta}_t(t') = U_1$ and $\dot{\bar{\eta}}_t(t') \leq 0$ for $t' = t + \Delta_{(t)}$, in every case.

(e) Assume (i), that (vii) σ_4 is (str.) jump-free in $[0, t]$ where $t \in]0, T[$, that (viii) $w(t) = 0 = \dot{w}(t)$, and that

$$(11.3') \quad \dot{\eta}(t) \leq f_{(\eta(t))}(T - t) \text{ if } \Delta_{\eta(t)} \leq T - t; \quad \dot{\eta}(t) \leq M_{\eta(t)} \text{ if } \Delta_{\eta(t)} < (T - t).$$

Then (ix) σ_4 's restriction σ_4^t to $[0, t]$ can be extended to a (str.) jump-free 4-tuple σ_4^T «defined» on $[0, T]$; and (x) if $t \in [0, t_1]$ with $t_1 = T - 2\Delta_{U_1}$ – see (10.2)₂ –, then the above σ_4^T can be identified with a solution to Problem 6.1 satisfying all conditions (7.3). \square

REMARK 11.1. (a) The implications (11.1) in A11.1(a) constitute a condition on $\dot{\eta}(t)$ ($= \dot{u}(t)$), and hence on both C 's speed and the skier's behaviour, that is necessary for Σ 's motion to be str. jump-free in $[t_1, T]$.

(a') A11.1(a) keeps holding if «str.» is cancelled everywhere, provided U_2 be replaced by U_2' – see (2.10) – in that, instead of the quantities M_η and Δ_η defined by (10.2)₁₋₂ in terms of U_2 , their analogues M'_η and Δ'_η for U_2' can be respectively used in (11.1). The analogous remark holds for *e.g.* the «only if-part» of A11.1(c).

(b) In A11.1(c) the condition (11.3) on initial data is necessary and sufficient for the existence of some str. ad. motion of Σ that is jump-free, and also symmetric if preferred – see Remark 3.1.

(c) In A11.1(d)-(e), among other things, one specifies the large arbitrariness that the skier has under usual practical conditions, *e.g.* when he wants to implement a str. jump-free solution to Problem 6.1: if $\dot{\eta}_0 \leq M_{\eta_0}$ and $2\Delta_{U_1} \leq T$, then at least in the time interval $[0, t_1]$ with $t_1 = T - 2\Delta_{U_1}$ – see (10.2)₂ – he can keep an arbitrary ad. str. jump-free behaviour for which $w_i(t_1) = 0 = \dot{w}_i(t_1)$ ($i = 1, 2$).

REMARK 11.2. (a) In A11.1(e) some conditions are also given, that at any instant $t \in [0, T[$ allow us to extend a str. jump-free motion of Σ in $[0, t]$ with $w(\cdot) = 0$, to such a motion in $[0, T]$ (see Remark 4.2(b)-(c)).

(b) The above condition $w(\cdot) = 0$ can obviously be weakened to: $w(\tau) \in \mathcal{N} \ \forall \ \tau \in [0, t]$, where \mathcal{N} is a suitable given neighborhood of zero (in \mathcal{R}). It suffices to replace the interval $[U_1, U_2]$ implicitly involved by condition (11.3'), with a suitable interval $[U_1^{\mathcal{N}}, U_2^{\mathcal{N}}] \subset [U_1, U_2]$. The same remark holds for every case considered in this paper, where $w(\cdot) = 0$ is assumed.

(c) Suppose that (xi) Σ has been implemented as a robot, *e.g.* in order to compare the results of the preceeding theory on Σ with experiments in which air resistance is not very small, (xii) one can influence $\ddot{u}(\phi)$ at every instant $\phi \in]0, T[$, furthermore (xiii) condition (11.3) holds and one is interested only in *e.g.* str. jump-free motions of Σ (with $w(\cdot) = 0$ – see part (b)). Then, for some small $\varepsilon \geq 0$, one can try and provide Σ with some gadgets that (xiv) allow us to influence $\ddot{u}(\phi)$ except when both $\dot{u}(\phi) > 0$ and in addition

$$(11.3'') \quad \begin{aligned} & f_{(\eta(\phi))}(T - \phi) - \dot{\eta}(\phi) \geq \varepsilon \text{ and } \Delta_{\eta(\phi)} \geq T - \phi, \text{ or} \\ & M_{\eta(\phi)} - \dot{\eta}(\phi) \geq \varepsilon \text{ and } \Delta_{\eta(\phi)} \leq T - \phi, \end{aligned}$$

while (xv) in the situation (11.3'') with $\dot{\eta}(\phi) > 0$, those gadgets practically render $\ddot{u} = -g \cos \theta$ holding till the above situation disappears. Thus in case, *e.g.*, $w(\cdot)$ is kept zero, by A11.1(e) those gadgets, on the one hand, practically assure that Σ 's motion is str. jump-free; and on the other hand, they leave us a large possibility of influencing $\ddot{u}(\cdot)$. Of course, if one wants to keep C 's motion in C^1 , when $\dot{u}(\phi) < 0$ and $0 < u(\phi) - U_1 \leq \varepsilon$ some other gadgets have to work in order to avoid any impulse – see (8.14) – *i.e.* the situation $u(t) = 0$ and $\dot{u}(t) < 0$ ⁽³⁴⁾

(d) To the assumptions (xi) to (xv) in (c), which render Σ 's motion str. jump-free, let us add the following three: $w(\cdot)$ is kept zero – see part (b); (xvi) $t_1 \doteq T - 2\Delta_{U_1} > 0$ so that, by A11.1(b), case (β) in A7.1(b) holds ⁽³⁵⁾ and (xvii) the robot Σ also has some gadgets that at t_1 automatically start a motion that extends σ_4 's restriction σ_4^t to $[0, t]$ into a str. jump-free 4-tuple σ_4^T that is «defined» on $[0, T]$, solves Problem 6.1, and also satisfies all conditions (7.3) if preferred; this motion can be similar to the one considered in A11.1(e) under the assumption $0 \leq t \leq t_1 = T - 2\Delta_{U_1}$ below (11.3'). Thus, for $t_1 > 0$ (and for $w(\cdot)$ kept zero), the above gadgets both assure us that Σ 's motion practically is a str. ad. jump-free solution to Problem 6.1 and they leave us a large possibility of influencing $\ddot{u}(\cdot)$.

12. PROOF OF THEOREM A11.1

To prove A11.1(a) we assume (i) to (iii) in it. Hence (iii), Definition 4.1, and A2.6 yield that (xi) $\eta, \eta' \in [U_1, U_2]$ when $t, t' \in [t_1, T]$ ⁽³⁶⁾ Then, by A10.2, the nonstrict inequality in (10.15)₁[±] holds when (xi)₂ is specified by (10.15)₃[±], *i.e.* $t \leq t'$. For $|\Delta| = |t' - t| \leq \Delta_\eta$ – see (10.12)₁ – this yields the first implication in (11.1)₁[±]. Furthermore, since $M_\eta = f_{(\eta)}(\Delta_\eta)$, by (10.6)₁₋₂ with $\zeta = \Delta_\eta$, this yields the second too; infact, if $t \in [t_1, T - \Delta_\eta]$ or $t \in [t_1 + \Delta_\eta, T]$, then we can choose $t' = t \pm \Delta_\eta$ so that $t' \in [t_1, T]$ and $|t' - t| = \Delta_\eta$. Thus (11.1) holds.

In order to deduce thesis (iv) in A11.1(a), we shall use A10.1(c). Therefore (xii) we choose the nonempty oriented interval $] \tau, \tau' [=]t, t' [=]t, T[\subset [t_1, T]$, so that (xiii) both $0 < \zeta = \Delta = t' - t = T - t \leq T - t_1$ by (10.2)₃, and $\eta, \eta' \in [U_1, U_2]$ by (xi). Now, by (xiii)₂, (10.3)₂ holds obviously.

To deduce the remaining hypotheses in A10.1(c), *i.e.* (10.7), we first assume that $0 < \zeta \leq \Delta_\eta$. Then, by (xiii)₁₋₃, $0 < T - \Delta_\eta \leq T - \zeta = t < t' = T$; and the implication

⁽³⁴⁾ Since $\dot{\eta} \geq -g \cos \theta$ by (3.3)₂, for $\eta \in]U_1, U_2[$ and $\dot{\eta} < 0$, along str. jump-free motions $|\dot{\eta}| \leq \sqrt{2g(U_2 - \eta) \cos \theta} = \sqrt{2}M_\eta$ – see (10.2)₁.

⁽³⁵⁾ Indeed, by A11.1(b), $(\alpha) \bar{\eta}_t(t') = U_1$ where $t' = t + \Delta_{(t)}$; and, since Σ 's motion is str. jump-free, condition (iii) above (11.1) holds so that all relations in (11.2) are valid; hence, by (11.2)₁[−], $\Delta_{(t)} < 2\Delta_{U_1}$. Then, for $t = t_1$, $(\alpha)_2$ and assumption (xvi) yield that $t' < t_1 + 2\Delta_{U_1} \in]0, T]$. Hence (6.7)₃ in A6.2(b) – with (τ, τ', t) replaced by $(0, t, T)$ – and $(\alpha)_1$ yield that $\bar{\eta}_0(T) \leq \bar{\eta}_t(T) < U_1$. Therefore case (β) in A7.1(b) holds.

⁽³⁶⁾ The labels like «(xi)» follow those in A11.1, disregarding the labels «(xi)» to «(xvii)» used in Remark 11.2.

in $(11.1)_1^+$ yields that $\dot{\eta} \leq f_{(\eta)}(\Delta)$; thus the implication $(10.7)_1$ holds.

Now we assume that $\zeta \geq \Delta_\eta$. Hence, by (xiii), $t_1 \leq t = T - \zeta \leq T - \Delta_\eta$ so that $t \in [t_1, T - \Delta_\eta]$; and the implication $(11.1)_2^+$ yields that $\dot{\eta} \leq M_\eta$; thus the implication $(11.7)_2^+$ also holds. Then A10.1(c) yields (10.8), which by $(10.12)_{4,5}$, (10.13), and (xii) is just thesis (iv) in A11.1(a).

To prove A11.1(b), first let (i) to (ii) in part (a) hold. Then by $(6.2)_2$ the first equality in (v) holds for exactly one $t' = t + \Delta > 0$ if $0 < \eta (= \eta(t) = \bar{\eta}_t(t))$. Otherwise, since $\eta(\cdot) \geq 0$ on $[0, T]$, for $t > 0$ the equality $\eta = U_1$ implies $\dot{\eta} = 0$, so that $t' = t$. Instead for $t = 0$ $(3.7)_{2,3}$ are compatible with the conditions $\eta_0 = 0 \leq \dot{\eta}_0$; then $\dot{\eta}_0(t') = 0$ for two nonnegative values of t' , only one of which is > 0 ; and just this is required by (v) for $t = 0$.

Thus, in every case, (v) holds for exactly one value of $t' = t + \Delta$; and it is now easy to check that Δ 's corresponding value $\Delta_{(t)}$ satisfies condition $(11.2)_1$ because – see $(6.2)_2$ with $]\tau, t[$ replaced by $]t, t'[$ – it simply expresses explicitly the time $\Delta (\geq 0)$ that implicitly satisfies condition (v).

At this point let σ_4 be str. jump-free in $[t_1, T]$, so that (iii) above (11.1) holds, and assume $t_1 \leq t$. Then, by the constraint $(2.11)_1$, $\dot{\eta} = 0$ for $\eta = U_1$, which yields $(11.2)_2^+$.

To deduce $(11.2)_2^-$, we first note that $\sqrt{\dot{\eta}^2 + 2(\eta - U_1)g \cos \theta} \leq |\dot{\eta}| + \sqrt{2(\eta - U_1)g \cos \theta}$. Hence, in the case (xiv) $\dot{\eta} \leq 0$ – which certainly holds as an equality for $\eta = U_2$ and $t_1 < t < T$ by (iii), Definition 4.1, and $(2.11)_1$ – the definition $(10.2)_2$ and $(11.2)_1$ yield the relations (xv) $t' - t = \Delta \leq \Delta_{U_1}$ and hence $(11.2)_2^-$.

In the opposite case, i.e. $\dot{\eta} > 0$ (whence $U_1 < \eta < U_2$), $(6.2)_2$ (with τ and t interchanged) together with condition (v) above (11.2) implies that (xvi) $\dot{\eta}_t(\tau) = 0$ for some $\tau \in]t, t'[$. Hence, by replacing $]t, t'[$ with $]\tau, t'[$ in the above deduction of (xv) from (xiv), we obtain a deduction of (xvii) $t' - \tau \leq \Delta_{U_1}$ from (xvi).

On the other hand, by $(6.2)_2$ $\dot{\eta}_t(\tau) = \dot{\eta} - g(\tau - t) \cos \theta$; hence (xviii) $\tau - t = \dot{\eta}/g \cos \theta > 0$ by (xvi). Furthermore, by A11.1(a), (iii) above (11.1) yields the implication $(11.1)_2^+$. Hence either $\Delta \leq \Delta_\eta$ and $(11.2)_2^-$ holds obviously; or $\Delta = t' - t > \Delta_\eta$ and, by the implication $(11.1)_2^+$, $\dot{\eta} \leq M_\eta$. Hence in the latter alternative, (xviii) and $(10.2)_{1,2}$ yield (xix) $\tau - t \leq M_\eta/g \cos \theta = \Delta_\eta$. By (xvii) and (xix), $(11.2)_2^-$ holds again. Lastly $(11.2)_3^-$ for $\eta \in]U_1, U_2]$ follows from $(10.2)_2$.

Since (i) to (iii) above (11.1) imply (iv) by A11.1(a), A11.1(b) is proved.

To prove A11.1(c), we first assume the existence of some str. jump-free 4-tuple $\sigma_4 = (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$ – see A4.3 – so that, by Definition 4.1, (i) to (iii) in A11.1(a) hold for $t_1 = 0$ and for the obvious choice of $\bar{\eta}_t(\cdot)$. Then by Corollary A4.4 and Definition 3.1(a), $(3.6-7)$ hold for σ_4 ; and A11.1(a) implies (11.1). Now set

(xx) $t = 0$, $\eta = \eta_0$, $\dot{\eta} = \dot{\eta}_0$, and $(\Delta =) t' = T$;

thus, if $\Delta_{\eta_0} \geq T$ so that $T - \Delta_{\eta_0} \leq t < t' \leq T$, then implication $(11.1)_2^+$ yields that $\dot{\eta}_0 \leq f_{(\eta_0)}(T)$; hence implication $(11.3)_1$ holds. Furthermore, if $\Delta_{\eta_0} \leq T$ so that $t(=0) \in [0, T - \Delta_{\eta_0}]$, then the implication $(11.1)_2^+$ with $t_1 = 0$ yields that $\dot{\eta}_0 \leq M_{\eta_0}$; thus the implication $(11.3)_2$ also holds. We conclude that the condition (11.3) is

necessary for the compatibility mentioned above it.

Now we conversely assume condition (11.3) and we note that, for (xxi) $(\tau, \tau', \eta, \dot{\eta}) = (0, T, \eta_0, \dot{\eta}_0)$, we have that (xxii) $\zeta \doteq \tau' - \tau = T > 0$ – see (10.2)₃ – and $\eta \in [U_1, U_2]$ by (3.7)₂, while (11.3) yields (10.7) where the equalities (xxi) to (xxii) hold; hence A10.1(c) yields (10.8), so that by (10.13), (xxi), and (xxii)

$$(12.1) \quad \bar{\eta}_0(t) = \bar{\eta}_0(t, \eta_0, \dot{\eta}_0) \leq U_2 \quad \forall t \in [0, T].$$

Furthermore, by (6.2')₂ and (3.7)_{2,3}, for $t_M \doteq \dot{\eta}_0/g \cos \theta$

$$(12.2) \quad \eta_M \doteq \bar{\eta}_0(t_M) = \max \bar{\eta}_0([0, +\infty[) = \eta_0 + \dot{\eta}_0(2g \cos \theta)^{-1}(U_1 \leq \eta_0 \leq U_2, \dot{\eta}_0 \geq 0).$$

By (12.1-2), $\bar{\eta}_0(t) \in [U_1, U_2] \quad \forall t \in [0, T]$ and $\dot{\bar{\eta}}_0(t_M) = 0$. Hence, setting

$$(12.3) \quad u(t) = \bar{\eta}_0(t) \quad \forall t \in [0, T] \cap [0, t_M]; \quad u(t) = \bar{\eta}_0(t_M) \quad \text{if } t_M < t \leq T,$$

we easily see that by (6.2')₂ $u(\cdot)$ satisfies (11.4).

Furthermore we set $w(\cdot) = 0$, $\eta(\cdot) = u(\cdot)$, and – see (3.2)₂

$$(12.4) \quad \xi(t) = \begin{cases} \dot{\xi}_0 t + g(\sin \theta) t^2/2 & \forall t \in [0, T] \cap [0, t_M], \\ \xi(t_M) + \dot{\xi}(t_M)(t - t_M) + B(t - t_M)^2/2 & \text{if } t_M < t \leq T. \end{cases}$$

Thus the 4-tuple $\sigma_4 \doteq (\xi(\cdot), \eta(\cdot), u(\cdot), w(\cdot))$ is clearly sym. by A3.3, and hence str. ad. – see Definition 3.1(a)-(c); it is str. jump-free by A4.4; and it obviously has the form $\bar{\sigma}_4$ mentioned in A11.1(c). Furthermore this result shows that condition (11.3) is sufficient, besides necessary, for the possibility of a str. jump-free choice of $\bar{\sigma}_4$, substantially asserted in A11.1(c).

Lastly we identify σ_4 with any choice of the above (str.) ad. 4-tuple $\bar{\sigma}_4$ mentioned below (11.4); thus, for σ_4 , (11.1) holds and $w(\cdot) = 0$, while $\xi(\cdot)$ and $\eta(\cdot)$ are in $C^1 \cap PC^2$. Then by (3.6)_{3,4}, Remark 8.1, and ⁽²⁶⁾ (placed at the end of it), $v(t) = \eta(t) - u(t) = 0 \quad \forall t \in [0, T]$. Now σ_4 is clearly jump-free. Thus A11.1(c) is completely proved.

To check A11.1(d) we assume its hypotheses, which include (i) to (iii) above (11.1); and we set (xxiii) $t = T - 2\Delta_{U_1}$ ($> t_1 > 0$ by assumption (vi)). Then, by A11.1(b), thesis (v) and (11.2) hold; hence (xxiv) $\bar{\eta}_t(t') = U_1$ for $t' = \Delta_{(t)}$. Furthermore, since $t \in]0, T[$, either $\eta - U_1 = 0 = \dot{\eta}$ and $t' = t$ by (11.2)₂, or $t' \in]t, T[$ by (11.2)⁻. Thus in every case (xxv) $0 < t_1 < t \leq t' < T$. Then by (6.2)₂, $\bar{\eta}_t(T) \leq U_1$ – see ⁽³³⁾ (placed below (11.2)). Furthermore $\bar{\eta}_T(T) = \eta(T) \geq U_1$. Hence the continuity of $\phi \mapsto \bar{\eta}_\phi(T)$ asserted by thesis (i) in A6.2(a) yields that $t \leq t' \leq \tau$ where τ is the last instant for which (xxvi) $\bar{\eta}_\tau(T) = U_1$.

To prove (11.5), it now suffices to exclude the case $\tau = t$ which by (xxv) yields that (xxvii) $\tau = t' = t + \Delta_{(t)} = t < T$. Therefore we assume it. Then $\Delta_{(t)} = 0$, so that $\eta - U_1 = 0 = \dot{\eta}$ by (11.2)₁. By (xxiv), (xxvii), and (6.2)₂ this yields that $\bar{\eta}_\tau(T) < U_1$ in contrast to (xxvi). We conclude that $\tau > t$, so that (11.5) holds.

Lastly, by Remark 8.2(a), the above last instant τ satisfying (xxvi) is the last instant at which σ_4 can be turned into a solution σ_4^* to Problem 6.1. Thus A11.1(d) is completely proved.

Now let the assumptions in A11.1(e) hold. They include (11.3'), *i.e.* (11.3) with $(0, T, \eta_0, \dot{\eta}_0)$ replaced by $(t, T-t, \eta(t), \dot{\eta}(t))$. Furthermore by them, through (i) and (vii), σ_4 is str. jump-free in $[0, t]$; hence (xxviii) both $\eta(t) \in [U_1, U_2]$ and $\dot{\xi}(t) \geq 0$. Hence the analogues of the conditions (3.7)₁₋₃ on the instant 0 hold for the (fixed) instant t .

Case (xxix) $\dot{\eta}(t) \geq 0$. In it, by A11.1(c), there is a str. jump-free 4-tuple $\bar{\sigma}_4 \doteq (\bar{\xi}(\cdot), \bar{\eta}(\cdot), \bar{u}(\cdot), \bar{w}(\cdot))$ «defined» on $[t, T]$ and with $\bar{w}(\cdot) = 0$ – see (viii) –, $\xi^{(s)}(t) = \xi^{(s)}(\bar{t})$, and $\eta^{(s)}(t) = \eta^{(s)}(\bar{t})$ ($s = 0, 1$). Furthermore, by A11.1(e), (xxx) the 4-tuple σ_4^T obtained by «joining» σ_4 's restriction σ_4^t to $[0, t]$ with $\bar{\sigma}_4$ is str. admissible⁽³⁷⁾. Hence σ_4^T is str. jump-free by (vii) in A11.1(e) and A4.4.

Now let the case (xxix) not hold. Hence by (vii) in A11.1(e), $\dot{u}(t) = \dot{\eta}(t) < 0$ and $u(t) = \eta(t) > U_1$. Then for some a and t'

$$(12.5) \quad a > -\dot{u}(t)/(T-t), \quad a > \dot{u}(t)^2/[2u(t) - 2U_1], \quad \dot{u}(t) + a \cdot (t' - t) = 0,$$

so that $a > 0$ and $t' \in]t, T[$. We can set

$$(12.6) \quad \bar{\eta}(\phi) = \bar{u}(\phi) = \begin{cases} u(t) + \dot{u}(t) \cdot (\phi - t) + a \cdot (\phi - t)^2/2 & \forall t \in [t, t'] \\ \bar{u}(t') & \forall t \in [t', T]; \end{cases}$$

hence $\dot{\bar{u}}(t^-) = \dot{u}(t)$, $\dot{\bar{u}}(t) = 0$, $\dot{\bar{u}}(\phi) < 0$, and $\bar{u}(\phi) > \bar{u}(t') > U_1$ on $]t, t'[($ ⁽³⁸⁾. Furthermore, remembering (viii) in A11.1(e) and (6.3) we can set

$$(12.7) \quad \bar{w}(\phi) = 0, \quad \bar{\xi}(\phi) = \sigma(\phi) - \bar{\eta}(\phi) \quad \forall \phi \in]t, T[.$$

It is easy to check that by joining σ_4^t with $\bar{\sigma}_4$ – see⁽³⁷⁾ –, a str. jump-free 4-tuple σ_4^T arises also when case (xxix) fails to hold. Thus thesis (ix) below (13.3') is proved.

To prove thesis (x) below (11.3') we assume that (xxxi) $t \in [0, t_1]$ where $t_1 \doteq T - 2\Delta U_1$; moreover, by thesis (ix), it is not restrictive to assume that $t = t_1$ and $\sigma_4 = \sigma_4^T$.

Using $\bar{\eta}_t(\cdot)$'s choice associated with σ_4 , by A11.1(b), (xxxii) $\bar{\eta}_t(t') = U_1$ where $t' = t + \Delta_{(t)} \geq t = t_1$ – see (11.2). Furthermore, since $\sigma_4 = \sigma_4^T$ is str. jump-free, by A11.1(b) all relations (11.2) hold. Hence, by (xxxi) and (xxxii)₂, $t' \in [t_1, t_1 + 2\Delta_{U_1}[\subset [t_1, T[$, so that (xxxii)₁ and (6.2)₂ yield that (xxxiii) $\bar{\eta}_t(T) < 0$.

Case $t' = t_1$. In it $\Delta_{(t)} = 0$ by (xxxii)₂; then, by (11.2)₁ and (6.2)₂, both $0 = \dot{\eta}(t_1^-) = \dot{u}(t_1^-)$ and $u(t_1^-) = \eta(t_1^-) = \bar{\eta}_{t_1}(t_1) = U_1$ hold for the str. jump-free 4-tuple $\sigma_4 (= \sigma_4^T)$. Hence, as is easy to check, the join σ_4^T of σ_4^t with $\bar{\sigma}_4$ – see⁽³⁷⁾ – remains str. ad. and jump-free when we change it by defining $\bar{\sigma}_4$ by means of (12.6-7) where $\dot{u}(t) = 0 = a$. Thus thesis (x) of A11.1(e) holds in the case $t' = t_1$.

⁽³⁷⁾ One-argument functions can be regarded as as special sets of couples. Then the «join» of $\bar{\sigma}_4$ with σ_4^T , *i.e.* the set-theoretical union of these sets, is clearly a function by σ_4 's initial conditions.

⁽³⁸⁾ In fact by (12.5)₃₋₂ $\bar{u}(t') - U_1 = u(t) - U_1 - \dot{u}(t)^2/2a > 0$ respectively.

The remaining case is equivalent, by (2.12), to (xxxiv) $t_1 < t' < T$. Hence in it (6.7)₃ with $(\tau, \tau', t) = (0, t_1, T)$ and (xxxii)₁ – i.e. $\eta_t(t') = U_1$ – imply that (xxv) $\bar{\eta}_0(T) \leq \eta_{t_1}(T) < U_1$. Thus the case (β) considered in A7.1(b) holds.

The above conclusion renders it natural to try and prove thesis (x) in A11.1(b) for $t' > t$ by using the construction of the solution σ_4 to Problem 6.1 mentioned in thesis (b₁) of A11.1(b) and satisfying all conditions (7.3). Briefly this construction, made in Section 8, is based on the following assertions or conditions:

(x) above (8.2), (xi) above (8.3), (xii) below (8.4), (8.2-3), and (8.4)₁₋₃.

When one sets $T_1 = t'$, they hold for the present str. jump-free choice of $\sigma_4 (= \sigma_4^T)$ by (xxiii)₁, i.e. $\eta_t(t') = U_1$, and by (xxiii), i.e. $\bar{\eta}_t(T) < 0$.

In Section 8, first, σ_4 's restriction to $[0, T_0]$ – see (xi) involving (8.3) – is constructed; and it differs from the same restriction of our σ_4 's choice. However from (8.4) till Remark 8.1 included, σ_4 's restriction to $[T_0, T]$ is constructed in Section 8 in terms of $\xi^{(s)}(T_0^-)$ and $\eta^{(s)}(T_0^-)$ ($s = 0, 1$).

Now we can change the restriction to $[T_0, T]$ of the present choice of σ_4 by reconstructing it in exactly the afore-mentioned way performed in Section 8 (with $T_1 = t'$), obviously referring $\xi^{(s)}(T_0^-)$ and $\eta^{(s)}(T_0^-)$ ($s = 0, 1$) to σ_4 's present choice. Then Remark 8.1 easily shows that the reconstructed restriction to $[T_0, T]$ of σ_4 's present choice also is jump-free⁽³⁹⁾

By (8.6)₁₋₃ the whole reconstructed σ_4 's present choice, say σ_4^* , is easily seen to be ad. and also jump-free. Furthermore (xvi) below the proof of Step 3 shows that the condition $w(\cdot) = 0$, and hence (7.3)₅₋₆, hold for σ_4^* ; and u 's restriction to $[T_0, T]$ has for it the form (8.7). Hence, being $\eta(\cdot) = u(\cdot)$, the remaining relations in (7.3) also hold for σ_4^* . Lastly, by (7.3)₂ and Remark 6.1(a), it solves Problem 6.1. Thus thesis (x) of A11.1(e) is also proved. q.e.d.

13. COMMENTS ON SOLUTIONS TO PROBLEM 9.1

By Theorems A7.1 and A9.1 the properties of (the ad.) solutions to Problem 6.1 easily induce some analogues for Problem 9.1, as is specified below. In particular this happens in connection with Σ 's jump-free motions and especially for Σ implemented as a robot.

REMARK 13.1. For $r \in \{2, 4\}$ let $\sigma_r \doteq (\xi(\cdot), \eta(\cdot), \dots)$ be an ad. r -tuple for Problem 9.1 – see below (9.1) – defined on $[0, T]$. Then the following holds.

(a) In connection with Remark 6.1(a), σ_r solves Problem 9.1 iff, remembering (9.2)₁

⁽³⁹⁾ In more details we use, e.g. (8.2-3) and (8.4) with $T_1 = t'$; hence, for some $\varepsilon \in]0, t'[,$ and an arbitrary $\tau_1 \in]0, \varepsilon[,$ we use the instant $T_0 \doteq t' - \tau_1$. We also use Step 3 (in Section 8) likewise; in particular we extend to $[0, T]$ the restriction to $[0, T_0]$ of σ_4 's present choice by means of (8.7), so that e.g. (8.8) and (8.9)₁ hold. Furthermore we use obvious analogues of assertions (xvi) to (xix) below the proof of Step 3.

and (9.3)₁,

$$(13.1) \quad \text{either } U_1 \leq \bar{\eta}_0(t_*) = \eta(t_*) \text{ or } \bar{\eta}_0(t_*) < U_1 = \eta(T_*);$$

in other words iff (i) $\eta(t) = \max\{\bar{\eta}_0(t_*), U_1\}$ for t such that $\xi(t) = \bar{\xi}$ ⁽⁴⁰⁾

(b) We can choose σ_4 jump-free «on the segment $[0, \bar{\xi}]$ », i.e. with $u(t) = \eta(t)$ whenever $\xi(t) \in [0, \bar{\xi}]$, iff (11.3) holds with T replaced by $t_* [T_*]$ for $\bar{\eta}_0(t_*) \geq \geq [\leq] U_1$ ⁽⁴¹⁾

(c) Assume that $\dot{\eta}_0 \leq M_{\eta_0}$ and $2\Delta_{U_1} \leq t_*$ – see (10.2)₁₋₂ and (9, 2)₁ –, which generally occurs in practice; furthermore let the skier want to implement a str. jump-free solution to Problem 9.1. Then at least in $[0, T_* - 2\Delta_{U_1}]$ he can keep an arbitrary str. jump-free behaviour for which $w_i(T_*) = 0 = \dot{w}_i(T_*)$ ($i = 1, 2$) – see (9.3)₁ ⁽⁴²⁾

(d) Assume that (ii) $\bar{\eta}_0(t_*) < U_1$, (iii) σ_4 is jump-free and str. ad. in $[0, t]$ where $t < T_*$, (iv) $w(t) = 0 = \dot{w}(t)$, and (v) $\dot{\eta}(t) \leq M_{\eta(t)}$. Then $t_* < T_*$, there is a last $\tau \in]T_* - 2\Delta_{U_1}, T_*[$ such that $\bar{\eta}_\tau(T) = U_1$, and by some changes of σ_4 after τ , σ_4 can be turned into a jump-free ad. 4-tuple for Problem 9.1 (that, if preferred, satisfies all conditions (7.3) with T replaced by T_*) ⁽⁴³⁾

(e) Under the assumption $\bar{\eta}_0(t_*) < U_1$, which generally occurs in practice, the Remarks 11.2(c)-(d) on Problem 6.1 can be referred to Problem 9.1 keeping their validities, by simply replacing T with T_* in them – see ⁽⁴²⁾ except its last assertion.

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⁽⁴⁰⁾ Part (a) follows easily from A7.1(a)-(b) and A9.1(a)-(b), remembering that in the case (13.1)₃, by A9.1(b), we have that $t_* < T_*$; hence $\bar{\eta}_0(t_*) < \bar{\eta}_0(T_*) < U_1$.

⁽⁴¹⁾ Indeed, by (6.3)₁, (9.2)₁, and (9.3)₁, if $\bar{\eta}_0(t_*) = U_1$, then $t_* = T_*$. Therefore A9.1(a)-(b) and A11.1(c) easily imply assertion (b).

⁽⁴²⁾ Indeed (iii) above (11.1) can be regarded as valid for $t_1 = 0$; hence by A11.1(b), given $t = 0$, for some $t' = \Delta_{(t)} \geq 0$ all conditions (11.2) hold. Thus $\bar{\eta}_0(t') = U_1$ and $t' \in [0, 2\Delta_{U_1}[\subset [0, t_*]$. Hence $\bar{\eta}_0(t_*) < 0$, so that by A9.1(b), $t_* < T_*$ and the solutions to Problem 9.1 are those to Problem 6.1 for $T = T_*$. At this point Remark 11.1(c) yields Remark 13.1(c).

⁽⁴³⁾ Remember (the preceding) ⁽⁴²⁾ except its last assertion. At this point Remark 13.1(d) follows from (A4.4 and) A11.1(e) (using thesis (ix) in this, if preferred together with thesis (x)).

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