Bohdan Maslowski, Jan Seidler

On sequentially weakly Feller solutions to SPDE's

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **10** (1999), n.2, p. 69–78.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1999_9_10_2_69_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1999.

Rend. Mat. Acc. Lincei
s. 9, v. 10:69-78 (1999)

Calcolo delle probabilità. — *On sequentially weakly Feller solutions to SPDE's.* Nota di
Bohdan Maslowski e Jan Seidler, presentata (*) dal Corrisp. G. Da Prato.

Abstract. — A rather general class of stochastic evolution equations in Hilbert spaces whose transition semigroups are Feller with respect to the weak topology is found, and consequences for existence of invariant measures are discussed.

Key words: Stochastic partial differential equations; Weakly Feller processes; Invariant measures.

Riassunto. — *Su soluzioni debolmente sequenzialmente Feller di equazioni stocastiche alle derivate parziali.* Viene presentata un'ampia classe di equazioni di evoluzione stocastiche in spazi di Hilbert i cui semigruppi di transizione hanno la proprietà di Feller rispetto alla topologia debole; vengono inoltre discusse alcune conseguenze per l'esistenza di misure invarianti.

1. Introduction

Let H be a separable Hilbert space; we will denote by H_w this space when considered as a locally convex space endowed with the weak topology. Let P be a transition probability of a homogeneous Markov process in H, we denote by (P_t) the corresponding transition semigroup on the space of real bounded Borel functions on H (note that H and H_w have the same Borel sets due to separability), and by (P_t^*) the dual semigroup acting on finite Borel measures on H.

The classical Krylov-Bogolyubov method of establishing existence of an invariant measure for P requires, first, to prove that P is Feller, and, second, to find a Borel probability measure ν on H and $T_0 \geq 0$ such that the set of measures

$$M = \{\mu_T;\ T \geq T_0\}, \quad \mu_T \equiv \frac{1}{T}\int_0^T P_t^* \nu(\cdot)\, dt,$$

is tight. Balls in H are weakly compact, so it is often straightforward to establish that M is tight as a set of measures on H_w (this is implied *e.g.* by the boundedness in probability of the underlying Markov process). It remains, however, to prove that (P_t) is weakly Feller, that is, that the operators P_t map the space $\mathscr{C}_b(H_w)$ of all real bounded weakly continuous functions on H into itself.

A particular class of SPDE's defining weakly Feller transition semigroups was investigated by A. Ichikawa (see [7, Theorem 3.1]); he considered equations whose nonlinear terms depend only on a projection of the solution to a finitely dimensional subspace and have finite-dimensional ranges. Markov processes Feller with respect to the weak topology are also discussed in [9], however, without specific applications to stochastic partial differential equations.

(*) Nella seduta del 12 marzo 1999.

In this paper we find a rather general class of semilinear stochastic evolution equations whose transition semigroups map bounded weakly continuous functions into weakly sequentially continuous ones; let us call such semigroups sequentially weakly Feller. In Section 2, we start with proving that any Ornstein-Uhlenbeck process is sequentially weakly Feller and that the same assertion holds for the Markov process defined by a semilinear stochastic equation

$$dX = \big(AX + f(X)\big)\,dt + \sigma(X)Q^{1/2}\,dW$$

provided A generates a *compact* semigroup and the nonlinear terms f, σ are Lipschitz continuous. Next, we show that the assumptions on the drift can be relaxed considerably by means of the Girsanov theorem, the compactness of the semigroup generated by A again playing an important rôle. (In the paper [11], a similar approach is used to establish the strong Feller property).

Finally, in Section 3 the sequential weak Feller property is shown to be sufficient for employing the Krylov-Bogolyubov procedure and some examples to which our results are applicable are provided. In particular, we arrive at an alternative proof of the well known result on existence of invariant measures due to G. Da Prato, D. Gątarek and J. Zabczyk (cf. Example 3.1) as well as at some new existence results (see Example 3.2).

2. Sequentially weakly Feller solutions

At first, let us consider a linear equation

$$dZ = AZ\,dt + Q^{1/2}\,dW \tag{2.1}$$

in a real separable Hilbert space H, where W is a standard cylindrical Wiener process on H, $Q \in \mathscr{L}(H)$ nonnegative and self-adjoint, and $A : \mathrm{Dom}(A) \longrightarrow H$ is an infinitesimal generator of a C_0-semigroup (e^{At}) on H. We assume

(A1) *For any* $T \geq 0$

$$\int_0^T \big\|e^{At}Q^{1/2}\big\|_{\mathrm{HS}}^2\,ds < \infty$$

holds, $\|\cdot\|_{\mathrm{HS}}$ *denoting the norm in the space of Hilbert-Schmidt operators in* H.

Take an arbitrary probability space $(\Omega, \mathscr{F}, \mathsf{P})$ carrying a standard cylindrical Wiener process W on H. Then (A1) implies that, for any $y \in H$, there exists a unique mild solution Z^y to (2.1) satisfying $Z^y(0) = y$. We denote by (R_t) the transition semigroup defined by (2.1), that is

$$R_t\varphi(y) = \int_\Omega \varphi(Z^y(t))\,d\mathsf{P}$$

for any $t \geq 0$, $y \in H$ and all bounded Borel functions $\varphi : H \longrightarrow \mathbb{R}$. In what follows $\mathscr{S}_b(H_w)$ will stand for the set of all real bounded weakly sequentially continuous functions on H.

Proposition 2.1. *Assume* (A1), *then* $R_t\big(\mathscr{S}_b(H_w)\big) \subseteq \mathscr{S}_b(H_w)$ *for any* $t \geq 0$.

REMARK 2.1. This result is essentially due to Ichikawa (see [6, Lemma 3.1; 7, Remark 3.1]; cf. also [14], where an analogous problem is considered in the discrete-time case), we include it here for completeness.

PROOF. Let (x_n) be an arbitrary weakly convergent sequence in H, let x be its weak limit. Since

$$Z^{x_n}(t) - Z^x(t) = e^{At}(x_n - x) \quad \mathsf{P}\text{-almost surely},$$

we obtain that

$$\left\langle Z^{x_n}(t) - Z^x(t), y\right\rangle = \left\langle x_n - x, e^{A^*t}y\right\rangle \underset{n\to\infty}{\longrightarrow} 0 \quad \mathsf{P}\text{-almost surely}$$

for every $y \in H$, which implies

$$\lim_{n\to\infty} \varphi(Z^{x_n}(t)) = \varphi(Z^x(t)) \quad \mathsf{P}\text{-almost surely} \tag{2.2}$$

for any $\varphi \in \mathscr{S}_b(H_w)$ and by the dominated convergence theorem

$$\lim_{n\to\infty} R_t\varphi(x_n) = R_t\varphi(x).$$

The last equality shows that $R_t\varphi \in \mathscr{S}_b(H_w)$. □

Now we turn to the semilinear problem

$$dX = \big(AX + f(X)\big)\,dt + \sigma(X)Q^{1/2}\,dW \tag{2.3}$$

in the space H, where H, A have the same meaning as in the equation (2.1), W is a standard cylindrical Wiener process in another real separable Hilbert space Υ, and $Q \in \mathscr{L}(\Upsilon)$ is a nonnegative self-adjoint operator. We assume

(A2) *The mappings* $f: H \longrightarrow H$ *and* $\sigma: H \longrightarrow \mathscr{L}(\mathrm{Rng}\,Q^{1/2}, H)$ *are Borel and there exist a constant* $K < \infty$ *and a function* $k \in L^1_{\mathrm{loc}}([0,\infty[)$, $k \geq 0$ *such that*

$$\|f(x) - f(y)\| \leq K\|x - y\|,$$

$$\left\|e^{At}\sigma(x)Q^{1/2}\right\|^2_{\mathrm{HS}} \leq k(t)\big(1 + \|x\|^2\big), \tag{2.4}$$

$$\left\|e^{At}[\sigma(x) - \sigma(y)]Q^{1/2}\right\|^2_{\mathrm{HS}} \leq k(t)\|x - y\|^2 \tag{2.5}$$

for every $t \geq 0$, $x, y \in H$.

Here the space $\mathrm{Rng}\,Q^{1/2}$ is equipped with its natural Hilbert structure (see [4, §4.2]) and $\|S\|_{\mathrm{HS}}$ denotes now the Hilbert-Schmidt norm of an operator $S \in \mathscr{L}(\Upsilon, H)$.

REMARK 2.2. Note that (2.4), (2.5) are always satisfied in two important particular cases: if σ is a Lipschitz continuous $\mathscr{L}(\mathrm{Rng}\,Q^{1/2}, H)$-valued function and either Q is nuclear, or Q is arbitrary but

$$\int_0^T \|e^{At}\|^2_{\mathrm{HS}}\,dt < \infty$$

for some $T > 0$. For equations with additive noise, (2.4) and (2.5) reduce to (A1).

Let us fix an arbitrary probability space $(\Omega, \mathscr{F}, \mathsf{P})$ and a standard cylindrical Wiener process W on Υ defined on Ω. For any $y \in H$ there exists a unique mild solution X^y

of (2.3) on Ω such that $X^y(0) = y$. (This follows *e.g.* by a straightforward generalization of Theorem 1 in [12]). Let (P_t) be the transition semigroup of the Markov process solving (2.3),

$$P_t\varphi(y) = \int_\Omega \varphi(X^y(t))\, d\mathsf{P}\,, \quad t \geq 0\,,\ y \in H,\ \varphi : H \longrightarrow \mathbb{R} \ \text{ bounded Borel.}$$

The space of all real bounded continuous (in the norm topology) functions on H will be denoted by $\mathscr{C}_b(H)$.

THEOREM 2.2. *Let the assumption* (A2) *be satisfied. Let the semigroup* (e^{At}) *be compact. Then* $P_t\big(\mathscr{C}_b(H)\big) \subseteq \mathscr{S}_b(H_w)$ *for any* $t > 0$.

PROOF. Let us take arbitrary $\varphi \in \mathscr{C}_b(H)$ and $y_n, y \in H$, $y_n \to y$ weakly; our goal is to prove that

$$\lim_{n\to\infty} P_t\varphi(y_n) = P_t\varphi(y)\,, \quad t > 0. \tag{2.6}$$

Fix a $T > 0$ arbitrarily. Using (A2) we obtain

$$\begin{aligned}\mathsf{E}\|X^{y_n}(t) - X^y(t)\|^2 &\leq 3\big\|e^{At}(y_n - y)\big\|^2 + 3t\int_0^t \mathsf{E}\big\|e^{A(t-s)}\big[f(X^{y_n}(s)) - f(X^y(s))\big]\big\|^2\, ds\, + \\ &\quad + 3\int_0^t \mathsf{E}\big\|e^{A(t-s)}\big[\sigma(X^{y_n}(s)) - \sigma(X^y(s))\big]\,Q^{1/2}\big\|^2_{\mathrm{HS}}\, ds \leq \\ &\leq 3\big\|e^{At}(y_n - y)\big\|^2 + C\int_0^t \big(1 + k(t-s)\big)\mathsf{E}\|X^{y_n}(s) - X^y(s)\|^2\, ds\,,\end{aligned}$$

where C denotes a constant dependent only on T, K and $\|e^{At}\|$. For brevity, set

$$\psi_n(t) = 3\big\|e^{At}(y_n - y)\big\|^2\,, \quad \pi(t) = C\big(1 + k(t)\big)\,, \quad t \geq 0.$$

Applying the generalized Gronwall lemma (see *e.g.* [2, Corollary 8.11]) we get

$$\mathsf{E}\|X^{y_n}(t) - X^y(t)\|^2 \leq \psi_n(t) + \sum_{j=1}^{\infty}(\Pi^j\psi_n)(t)\,, \quad 0 \leq t \leq T,$$

Π being a Volterra integral operator with the kernel π, that is,

$$\Pi h(v) = \int_0^v \pi(v-s)h(s)\, ds\,, \quad v \geq 0\,, \quad h \in L^1_{\mathrm{loc}}(\mathbb{R}_+).$$

Since the semigroup (e^{At}) is compact,

$$\lim_{n\to\infty} \psi_n(t) = 0 \quad \text{for any } t > 0\,,$$

and it is not difficult to show that

$$\lim_{n\to\infty} \mathsf{E}\|X^{y_n}(t) - X^y(t)\|^2 = 0 \quad \text{for any } 0 < t \leq T.$$

By a standard argument this implies

$$\varphi(X^{y_n}(t)) \xrightarrow[n\to\infty]{\mathsf{P}} \varphi(X^y(t)) \tag{2.7}$$

and we complete the proof by invoking the dominated convergence theorem. □

As a following step we show that, roughly speaking, the Girsanov transform preserves the sequential weak Feller property. Let us consider a pair of equations

$$dZ = \big(AZ + g(Z)\big)\, dt + \sigma(Z) Q^{1/2}\, dW, \tag{2.8}$$

$$dX = \big(AX + f(X)\big)\, dt + \sigma(X) Q^{1/2}\, dW, \tag{2.9}$$

where H, A, Q, W are the same as in (2.3), and $f, g : H \longrightarrow H$, $\sigma : H \longrightarrow \longrightarrow \mathscr{L}(\mathrm{Rng}\, Q^{1/2}, H)$ are Borel mappings. We assume

(A3) 1) *There exists a probability space* $(\Omega, \mathscr{F}, \mathsf{P})$ *carrying a standard cylindrical Wiener process* W *on* Υ *and, for any* $y \in H$, *a mild solution* Z^y *to* (2.8) *satisfying* $Z^y(0) = y$.

2) *For any* $y \in H$ *there exists a martingale solution* $\big((\Theta_y, \mathscr{G}^y, p^y), (\mathscr{G}_t^y), W^y, (X^y(t))\big)$ *to* (2.9) *with* $X^y(0) = y$.

3) *Weak uniqueness holds for both* (2.8) *and* (2.9).

The assumption (A3) implies that (2.8), (2.9) define Markov processes; let us denote by (P_t) the transition semigroup corresponding to the equation (2.9),

$$P_t\varphi(y) = \int_{\Theta_y} \varphi(X^y(t))\, dp^y, \quad t \ge 0, \ y \in H, \ \varphi : H \longrightarrow \mathbb{R} \ \text{ bounded Borel.}$$

THEOREM 2.3. *Let the assumption* (A3) *be satisfied, let there exist a Borel function* $u : H \longrightarrow \longrightarrow \Upsilon$ *such that* $f(\cdot) = g(\cdot) + \sigma(\cdot) Q^{1/2} u(\cdot)$. *Set*

$$U(y, t) = \exp\left(\int_0^t \big\langle u(Z^y(s)), \cdot\big\rangle\, dW(s) - \frac{1}{2}\int_0^t \big\|u(Z^y(s))\big\|^2\, ds\right)$$

for $t \ge 0$, $y \in H$. *Suppose that*

(*a*) $\mathsf{E}\, U(y, t) = 1$ *for any* $t \ge 0$ *and* $y \in H$,

(*b*) *for any* $t \ge 0$ *and any* $y_n, y \in H$ *such that* $y_n \longrightarrow y$ *weakly*

$$U(y_n, t) \xrightarrow[n\to\infty]{\mathsf{P}} U(y, t).$$

If, moreover,

$$\varphi(Z^{y_n}(t)) \xrightarrow[n\to\infty]{\mathsf{P}} \varphi(Z^y(t)) \tag{2.10}$$

for all $\varphi \in \mathscr{S}_b(H_w)$, $t \ge 0$ *and any* $y_n, y \in H$ *such that* $y_n \longrightarrow y$ *weakly, then* $P_t\varphi \in \mathscr{S}_b(H_w)$ *for any* $\varphi \in \mathscr{S}_b(H_w)$ *and* $t \ge 0$.

REMARK 2.3. Note that processes considered in Proposition 2.1 and Theorem 2.2 are not only sequentially weakly Feller but they satisfy also (2.10), cf. formulae (2.2), (2.7).

REMARK 2.4. In fact we prove a more general result: if E is a set of bounded Borel functions on H such that (2.10) holds for any $\varphi \in E$ and a $t \ge 0$ then $P_t(E) \subseteq \mathscr{S}_b(H_w)$.

Proof. Fix $t > 0$, $\varphi \in \mathcal{S}_b(H_w)$, and $y_n \in H$, $y_n \to y$ weakly. We aim at proving

$$\lim_{n\to\infty} P_t\varphi(y_n) = P_t\varphi(y).$$

Let us define probability measures $\widetilde{\mathsf{P}}_n$, $\widetilde{\mathsf{P}}$ on $(\Omega, \mathcal{F})$ by $d\widetilde{\mathsf{P}}_n = U(y_n, t)\, d\mathsf{P}$, $d\widetilde{\mathsf{P}} = U(y, t)\, d\mathsf{P}$. Obviously

$$\lim_{n\to\infty} \varphi(Z^{y_n}(t)) = \varphi(Z^y(t)) \quad \text{in } L^1(\mathsf{P})$$

by (2.10). Assumptions (*a*), (*b*) and nonnegativity of $U(y_n, t)$ imply

$$\lim_{n\to\infty} U(y_n, t) = U(y, t) \quad \text{in } L^1(\mathsf{P}).$$

Denoting by $\|\cdot\|_\infty$ the natural sup-norm on $\mathcal{S}_b(H_w)$ we obtain by the Girsanov theorem

$$\begin{aligned}
\left|P_t\varphi(y_n) - P_t\varphi(y)\right| &= \left|\int_\Omega \varphi(Z^{y_n}(t))\, d\widetilde{\mathsf{P}}_n - \int_\Omega \varphi(Z^y(t))\, d\widetilde{\mathsf{P}}\right| \leq \\
&\leq \int_\Omega \left|\{U(y_n, t) - U(y, t)\}\varphi(Z^{y_n}(t))\right| d\mathsf{P} + \\
&+ \int_\Omega \left|U(y, t)\{\varphi(Z^{y_n}(t)) - \varphi(Z^y(t))\}\right| d\mathsf{P} \leq \\
&\leq \|\varphi\|_\infty \int_\Omega \left|U(y_n, t) - U(y, t)\right| d\mathsf{P} + \\
&+ K\int_\Omega \left|\varphi(Z^{y_n}(t)) - \varphi(Z^y(t))\right| d\mathsf{P} + 2\|\varphi\|_\infty \int_{\{U(y,t)>K\}} U(y, t)\, d\mathsf{P}
\end{aligned}$$

for any constant $K > 0$ and our claim follows easily. □

Our next goal is to show that the hypotheses of Theorem 2.3 might be satisfied under reasonable assumptions upon A and the nonlinear terms f, σ.

First, note that if

$$\lim_{n\to\infty} \int_0^T \int_\Omega \left\|u(Z^{y_n}(t)) - u(Z^y(t))\right\|_\Upsilon^2 d\mathsf{P}\, dt = 0 \tag{2.11}$$

for arbitrary $T \geq 0$ and $y_n, y \in H$ such that $y_n \to y$ weakly, then (*b*) follows. If the function u is continuous and of a linear growth, then (2.11) is a consequence of

$$\lim_{n\to\infty} \int_0^T \int_\Omega \left\|Z^{y_n}(t) - Z^y(t)\right\|^2 d\mathsf{P}\, dt = 0 \tag{2.12}$$

(see *e.g.* [10, Theorem 3.1]). This simple observation will be useful in the following examples.

Example 2.1. Let us consider an equation with additive noise

$$dX = \big(AX + f(X)\big)\, dt + Q^{1/2}\, dW \tag{2.13}$$

assuming that $H = \Upsilon$ and (A1) is satisfied, the semigroup (e^{At}) is compact, $\mathrm{Rng}(f) \subseteq \mathrm{Rng}\, Q^{1/2}$, and the function $Q^{-1/2}f : H \longrightarrow H$ is continuous and obeys the linear growth condition, *i.e.*

$$\|Q^{-1/2}f(x)\| \leq K\big(1 + \|x\|\big) \tag{2.14}$$

for a $K < \infty$ and any $x \in H$. Let weak uniqueness hold for (2.13). Then $P_t\big(\mathscr{S}_b(H_w)\big) \subseteq \subseteq \mathscr{S}_b(H_w)$ for any $t \geq 0$, where (P_t) is the transition semigroup defined by the equation (2.13).

Indeed, one can show easily that (2.12) holds for solutions of the equation

$$dZ = AZ\,dt + Q^{1/2}\,dW,$$

since for all $T > 0$ and any weakly convergent sequence $y_n \to y$

$$\lim_{n\to\infty} \int_0^T \int_\Omega \big\|Z^{y_n}(t) - Z^y(t)\big\|^2\,d\mathsf{P}\,dt = \lim_{n\to\infty} \int_0^T \big\|e^{At}(y_n - y)\big\|^2\,dt = 0$$

by compactness of the semigroup (e^{At}). Setting $u = Q^{-1/2}f$ and using (2.14) we can see that assumptions of Theorem 2.3 are satisfied.

Example 2.2. As our second example we will investigate the equation (2.9) in the particular case of a non-degenerate diffusion term; namely, we consider the problem

$$dX = \big(AX + f(X)\big)\,dt + \sigma(X)\,dW \tag{2.15}$$

supposing that $\sigma : H \longrightarrow \mathscr{L}(\Upsilon, H)$ is a Lipschitz continuous function such that $\sigma(z)$ is an invertible operator for any $z \in H$ and

$$\sup_{z\in H} \|\sigma^{-1}(z)\| < \infty.$$

Assume further

$$\int_0^T \big\|e^{At}\big\|_{\mathrm{HS}}^2\,dt < \infty\,, \qquad T > 0\,,$$

then the semigroup (e^{At}) is obviously compact and (the proof of) Theorem 2.2 shows that (2.12) holds for solutions of the equation

$$dZ = AZ\,dt + \sigma(Z)\,dW.$$

Let $f : H \longrightarrow H$ be a bounded continuous function (so weak uniqueness holds for (2.15)). With the choice $u = \sigma^{-1}f$ all assumptions of Theorem 2.3 are satisfied. Taking into account Remark 2.4 we find that $P_t(\mathscr{C}_b(H)) \subseteq \mathscr{S}_b(H_w)$ for all $t > 0$.

3. Invariant measures

We shall close this *Note* by showing that the sequential weak Feller property of a transition semigroup is sufficient for the application of the Krylov-Bogolyubov procedure of deriving existence of an invariant measure from boundedness in probability. Let P be a homogeneous transition probability in H such that $P(\cdot\,,\cdot\,,\Gamma) : \mathbb{R}_+ \times H \longrightarrow \mathbb{R}$ is a Borel function for every Borel set Γ in H. Denote by (P_t) the corresponding transition semigroup on the space of all real bounded Borel functions on H, and by (P_t^*) the adjoint semigroup.

Proposition 3.1. *Suppose that the semigroup (P_t) is sequentially weakly Feller, that is, $P_t\big(\mathscr{C}_b(H_w)\big) \subseteq \mathscr{S}_b(H_w)$. Assume that we can find a Borel probability measure ν on H and $T_0 \geq 0$ such that for any $\varepsilon > 0$ there exists $R > 0$ satisfying*

$$\sup_{T \geq T_0} \frac{1}{T} \int_0^T P_t^* \nu\big(\{\|x\| > R\}\big)\, dt < \varepsilon. \tag{3.1}$$

Then there exists an invariant probability measure for P.

Remark 3.1. In the proof we will need the following simple observation: a bounded function $\psi : H \longrightarrow \mathbb{R}$ belongs to $\mathscr{S}_b(H_w)$ if and only if its restriction $\psi_{|B}$ to any (closed) ball $B \subset H$ is weakly continuous on B. Indeed, it suffices to realize that weakly convergent sequences are bounded and weak topology in any ball is metrizable.

Proof of Proposition 3.1. For any $n \in \mathbb{N}$ define a Borel probability measure μ_n on H by

$$\mu_n = \frac{1}{n} \int_0^n P_s^* \nu(\cdot)\, ds.$$

According to (3.1), the set $M = \{\mu_n;\ n \geq T_0\}$ is tight as a set of measures on H_w, as balls are weakly compact. Since weak compacts in H are metrizable, the set M is relatively sequentially compact in the narrow topology of the space of finite Borel measures on H_w by [8, Theorem 6]. Thus there exist a Borel probability measure $\widetilde{\mu}$ on H and a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}_{n \geq T_0}$ such that $\mu_{n_k} \longrightarrow \widetilde{\mu}$ narrowly, that is

$$\int_H \vartheta\, d\mu_{n_k} \underset{k\to\infty}{\longrightarrow} \int_H \vartheta\, d\widetilde{\mu} \quad \forall\, \vartheta \in \mathscr{C}_b(H_w).$$

We aim at proving that $\widetilde{\mu}$ is an invariant measure, which will follow from

$$\int_H \varphi\, d\widetilde{\mu} = \int_H P_t \varphi\, d\widetilde{\mu} \quad \forall\, t \geq 0 \ \ \forall\, \varphi \in \mathscr{C}_b(H_w). \tag{3.2}$$

We cannot use the standard proof of (3.2) directly, as we do not know a priori whether

$$\int_H P_t \varphi\, d\widetilde{\mu} = \lim_{k\to\infty} \int_H P_t \varphi\, d\mu_{n_k}$$

the function $P_t\varphi$ being only sequentially continuous. Therefore, towards the proof of (3.2) let us fix $\varepsilon > 0$, $t > 0$ and $\varphi \in \mathscr{C}_b(H_w)$ arbitrarily. By (3.1) we can find a ball $B = \{x;\ \|x\| \leq R\}$ in H such that

$$\inf_{k \geq 1} \mu_{n_k}(B) \geq 1 - \varepsilon\,,$$

so also $\widetilde{\mu}(B) \geq 1 - \varepsilon$ by the portmanteau theorem. $P_t\varphi$ is weakly continuous on the weakly compact set B, hence there exists $g \in \mathscr{C}_b(H_w)$ such that $g = P_t\varphi$ on B and

$\|g\|_\infty = \|P_t\varphi\|_\infty \le \|\varphi\|_\infty$. Obviously

$$\left|\int_H P_t\varphi\, d\widetilde{\mu} - \int_H g\, d\widetilde{\mu}\right| = \left|\int_{H\setminus B} P_t\varphi\, d\widetilde{\mu} - \int_{H\setminus B} g\, d\widetilde{\mu}\right| \le 2\|\varphi\|_\infty\, \widetilde{\mu}(H\setminus B) \le 2\varepsilon\|\varphi\|_\infty\,,$$

and

$$\left|\int_H P_t\varphi\, d\mu_{n_k} - \int_H g\, d\mu_{n_k}\right| \le 2\varepsilon\|\varphi\|_\infty\,, \quad k\in\mathbb{N}.$$

Moreover, as well known,

$$\left|\int_H (\varphi - P_t\varphi)\, d\mu_{n_k}\right| = \left|\frac{1}{n_k}\int_0^{n_k}\int_H (\varphi - P_t\varphi)\, dP_s^*\nu\, ds\right| = \left|\frac{1}{n_k}\int_0^{n_k}\int_H (P_s\varphi - P_{t+s}\varphi)\, d\nu\, ds\right| =$$

$$= \left|\int_H \frac{1}{n_k}\left[\int_0^{n_k} P_s\varphi\, ds - \int_t^{n_k+t} P_s\varphi\, ds\right] d\nu\right| \le \frac{2t}{n_k}\,\|\varphi\|_\infty.$$

Altogether,

$$\begin{aligned}\left|\int_H \varphi\, d\widetilde{\mu} - \int_H P_t\varphi\, d\widetilde{\mu}\right| &\le 2\varepsilon\|\varphi\|_\infty + \left|\int_H \varphi\, d\widetilde{\mu} - \int_H g\, d\widetilde{\mu}\right| = \\ &= 2\varepsilon\|\varphi\|_\infty + \lim_{k\to\infty}\left|\int_H \varphi\, d\mu_{n_k} - \int_H g\, d\mu_{n_k}\right| \le \\ &\le 4\varepsilon\|\varphi\|_\infty + \limsup_{k\to\infty}\left|\int_H \varphi\, d\mu_{n_k} - \int_H P_t\varphi\, d\mu_{n_k}\right| \le \\ &\le 4\varepsilon\|\varphi\|_\infty + \lim_{k\to\infty}\frac{2t}{n_k} = 4\varepsilon\|\varphi\|_\infty\,,\end{aligned}$$

and our claim follows as $\varepsilon > 0$ was arbitrary. □

Example 3.1. Let us consider the equation (2.3) assuming that the hypotheses of Theorem 2.2 are satisfied. Let there exist a mild solution X of (2.3), defined on some probability space $(\Omega, \mathscr{F}, \mathsf{P})$, such that

$$\exists T_0 \ge 0\ \ \forall\, \varepsilon > 0\ \exists R < \infty \qquad \sup_{T\ge T_0} \frac{1}{T}\int_0^T \mathsf{P}\big\{\|X(t)\| > R\big\}\, dt < \varepsilon. \tag{3.3}$$

Then there exists an invariant measure for (2.3). This result was obtained in [3, Theorems 4 and 6] (cf. also [5, Theorem 6.1.2]) by a rather different method.

Example 3.2. Let us consider the equation (2.13) under the assumptions of Example 2.1. If there exists a solution X to (2.13) satisfying (3.3) then there exists an invariant measure for this equation. Similar statement hold true for the problem (2.15) under the assumptions of Example 2.2. These results seem to be new; they are closely related to but distinct from Theorem 5.2 in [1].

Example 3.3. I. Vrkoč constructed in [13] a bounded Lipschitz function f in a Hilbert space H such that all solutions of the differential equation

$$\dot{x} = f(x) \tag{3.4}$$

are bounded, nevertheless, there is no invariant measure for (3.4). Accordingly, the transition semigroup defined by (3.4) cannot be sequentially weakly Feller.

ACKNOWLEDGEMENTS

This research has been supported in part by the GA ČR Grant No. 201/98/1454. Thanks are due to P. Holický for useful discussions.

REFERENCES

[1] Z. Brzeźniak - D. Gątarek, *Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces*. The University of Hull, Mathematics Research Reports, 11, 1998, no. 2.
[2] R. F. Curtain - A. J. Pritchard, *Infinite dimensional linear systems theory*. Lecture Notes in Control and Inform. Sci., 8, Springer-Verlag, Berlin 1978.
[3] G. Da Prato - D. Gątarek - J. Zabczyk, *Invariant measures for semilinear stochastic equations*. Stochastic Anal. Appl., 10, 1992, 387-408.
[4] G. Da Prato - J. Zabczyk, *Stochastic equations in infinite dimensions*. Cambridge University Press, Cambridge 1992.
[5] G. Da Prato - J. Zabczyk, *Ergodicity for infinite dimensional systems*. Cambridge University Press, Cambridge 1996.
[6] A. Ichikawa, *Dynamic programming approach to stochastic evolution equations*. SIAM J. Control Optim., 17, 1979, 152-174.
[7] A. Ichikawa, *Semilinear stochastic evolution equations: Boundedness, stability and invariant measures*. Stochastics, 12, 1984, 1-39.
[8] L. LeCam, *Convergence in distribution of stochastic processes*. Univ. Calif. Publ. Statist., 2, 1957, 207-236.
[9] G. Leha - G. Ritter, *Lyapunov-type conditions for stationary distributions of diffusion processes on Hilbert spaces*. Stochastics Stochastics Rep., 48, 1994, 195-225.
[10] R. Lucchetti - F. Patrone, *On Nemytskii's operator and its application to the lower semicontinuity of integral functionals*. Indiana Univ. Math. J., 29, 1980, 703-713.
[11] B. Maslowski - J. Seidler, *Probabilistic approach to the strong Feller property*. To appear.
[12] J. Seidler - I. Vrkoč, *An averaging principle for stochastic evolution equations*. Časopis Pěst. Mat., 115, 1990, 240-263.
[13] I. Vrkoč, *A dynamical system in a Hilbert space with a weakly attractive nonstationary point*. Math. Bohem., 118, 1993, 401-423.
[14] J. Zabczyk, *On optimal stochastic control of discrete-time systems in Hilbert space*. SIAM J. Control, 13, 1975, 1217-1234.

Pervenuta il 20 novembre 1998,
in forma definitiva il 13 gennaio 1999.

Mathematical Institute
Academy of Sciences
Žitná, 25 - 115 67 Praha 1
(Repubblica Ceca)
{maslow,seidler}@math.cas.cz