Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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On the smoothness of viscosity solutions of the prescribed Levi-curvature equation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **10** (1999), n.2, p. 61–68.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1999_9_10_2_61_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1999.

Rend. Mat. Acc. Lincei
s. 9, v. 10:61-68 (1999)

Analisi matematica. — *On the smoothness of viscosity solutions of the prescribed Levi-curvature equation.* Nota di Giovanna Citti, Ermanno Lanconelli e Annamaria Montanari, presentata (*) dal Socio E. Magenes.

Abstract. — In this paper a C^∞-regularity result for the strong viscosity solutions to the prescribed Levi-curvature equation is announced. As an application, starting from a result by Z. Slodkowski and G. Tomassini, the C^∞- solvability of the Dirichlet problem related to the same equation is showed.

Key words: Levi equation; Viscosity solutions; Non-linear vector fields; C^∞-regularity; Boundary value problem.

Riassunto. — *Regolarità delle soluzioni viscose dell'equazione della curvatura di Levi assegnata.* In questa *Nota* viene annunciato un teorema di regolarità C^∞ delle soluzioni viscose, in senso forte, dell'equazione di Levi con assegnata curvatura. Da questo teorema, e da un precedente risultato di Slodkowski e Tomassini, segue la risultibilità C^∞, in senso classico, del problema di Dirichlet relativo alla stessa equazione.

1. Introduction

In this *Note* we are concerned with the regularity properties of the solutions to a boundary value problem for the prescribed Levi curvature equation on a bounded open subset Ω of $\mathbb{R}^3$. Given a real function k defined on $\Omega \times \mathbb{R}$, the equation of the prescribed Levi-curvature k is defined as

$$\mathcal{L}u = k(\xi\,,\,u)(1 + a^2 + b^2)^{3/2}(1 + u_t^2)^{1/2}\,, \tag{1}$$

where

$$\mathcal{L}u := u_{xx} + u_{yy} + 2au_{xt} + 2bu_{yt} + (a^2 + b^2)u_{tt}\,, \tag{2}$$

and $a = a(\nabla u)$, $b = b(\nabla u)$ depend on the gradient of u as follows

$$a\,,\,b : \mathbb{R}^3 \to \mathbb{R} \quad a(p) = \frac{p_2 - p_1 p_3}{1 + p_3^2}\,, \qquad b(p) = \frac{-p_1 - p_2 p_3}{1 + p_3^2}. \tag{3}$$

In (1), (2), $\xi = (x, y, t)$ denotes the point of $\mathbb{R}^3$, u_t is the first derivative of u with respect to t and analogous notations are used for the other first and second order derivatives of u.

As suggested by G. Tomassini, we call equation (1) the prescribed Levi-curvature equation, since, if it has a solution u, then the graph of u has Levi curvature $k(\xi\,,\,u(\xi))$ at every point $(\xi\,,\,u(\xi))$. This notion, first introduced by E. E. Levi in order to characterize the holomorphy domains of $\mathbb{C}^2$, plays an important role in the geometric theory of several complex variables (see for instance [9]).

(*) Nella seduta del 23 aprile 1999.

The aim of this *Note* is to show that the Dirichlet problem associated to equation (1)

$$\begin{cases} \mathcal{L}u = k(\xi, u)(1 + a^2 + b^2)^{3/2}(1 + u_t^2)^{1/2} & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases} \tag{4}$$

has a classical solution u of class C^∞ *in* Ω, under suitable conditions on Ω, the boundary data ϕ and the curvature k (see Corollary 1.1 below).

The quasilinear operator $\mathcal{L}$ in (2) is degenerate elliptic as its characteristic form

$$\begin{aligned} A(p, \zeta) = \zeta_1^2 + \zeta_2^2 + 2a(p)\zeta_1\zeta_3 + 2b(p)\zeta_2\zeta_3 + (a^2(p) + b^2(p))\zeta_3^2 = \\ = (\zeta_1 + a(p)\zeta_3)^2 + (\zeta_2 + b(p)\zeta_3)^2, \end{aligned} \tag{5}$$

is non-negative defined. Furthermore, since the minimum eigenvalue of $A(p, \cdot)$ is equal to zero for every $p \in \mathbb{R}^3$, $\mathcal{L}$ is not elliptic at any point. Hence the theory of boundary value problems for second order quasilinear elliptic equations (see [8]) does not apply to our problem.

When $k \equiv 0$ a first existence and regularity result for (4) was established by Bedford and Gaveau [1] by means of a geometric technique. We briefly recall their result and we refer to the paper for a more precise statement.

THEOREM. *If* $k \equiv 0$, Ω *is a regular pseudoconvex open set,* $\phi \in C^{m+5}(\bar{\Omega})$, $m \in \mathbb{N}$ *and* $\partial\Omega$ *and* ϕ *satisfy some additional geometric conditions, then problem* (4) *has a solution* $u \in C^{m+\alpha}(\Omega) \cap \mathrm{Lip}(\bar{\Omega})$, $0 < \alpha < 1$.

The geometric arguments used in [1] do not work when $k \neq 0$. Slodkowski and Tomassini were able to handle (4) for general k, by using almost completely PDE's methods based on the elliptic regularization of the operator $\mathcal{L}$. For every $\varepsilon > 0$

$$\mathcal{L}_\varepsilon u := \mathcal{L}u + \varepsilon^2 \frac{u_{tt}}{1 + u_t^2} \tag{6}$$

is an elliptic operator since its characteristic form $A_\varepsilon(p, \zeta)$ is positive defined with minimum eigenvalue bounded away from zero in $\{p \in \mathbb{R}^3 : |p| < M\}$ for every $M > 0$. As a consequence, for the classical theory of elliptic equations, if $u \in C^2(\Omega)$ solves

$$\mathcal{L}_\varepsilon u = k(\xi, u)(1 + a^2 + b^2)^{3/2}(1 + u_t^2)^{1/2}, \tag{7}$$

and k is smooth, then $u \in C^\infty(\Omega)$.

DEFINITION 1.1. *We say that a function* $u : \Omega \to \mathbb{R}$ *is a strong viscosity solution to the equation* (1) *if there exist a sequence* (u_n) *in* $C^\infty(\Omega)$ *and a sequence of positive numbers* $\varepsilon_n \to 0$ *such that*

(*i*) (u_n) *pointwise converges to* u *in* Ω,
(*ii*) *there exists* $M > 0$ *such that* $||u_n||_{L^\infty(\Omega)} + ||\nabla u_n||_{L^\infty(\Omega)} \leq M$, $\forall n \in \mathbb{N}$.
(*iii*) $\mathcal{L}_{\varepsilon_n} u_n = H(\xi, u_n, \nabla u_n)$ *in* Ω *for any* $n \in \mathbb{N}$.

Here H *denotes the right-hand side of equation* (1).

Every strong viscosity solution of (1) is locally Lipschitz-continuous in Ω and solves the equation in the viscosity sense of Crandall-Ishii-Lions [4].

We are now in a position to state the existence result for (4) proved by Slodkowski and Tomassini in [10, Theorem 4].

THEOREM. *Let Ω be a strictly pseudoconvex domain with $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$. Let $k \in C^1(\bar{\Omega} \times \mathbb{R})$ satisfy the conditions of Proposition 2 and Theorem 3 in [10]. Then, for every $\phi \in C^{2,\alpha}(\bar{\Omega})$ the Dirichlet problem (4) has a strong viscosity solution $u \in \mathrm{Lip}(\bar{\Omega})$.*

This existence theorem holds for a wide class of curvatures k and requires less regularity hypotheses on $\partial\Omega$ than that of Bedford and Gaveau. However, it leaves open a problem of regularity: the solution found in [10] is merely Lipschitz continuous and only satisfies the Levi-curvature equation in the weak sense of the viscosity; besides, the regularity results for viscosity solutions to non–linear elliptic and parabolic equations in [3] cannot be applied to our case since the operator $\mathcal{L}$ is neither elliptic nor parabolic.

The structure of the Levi equation is well highlighted by some identities first explicitly written in [2], involving the two non-linear vector fields, which appear in the characteristic form of $\mathcal{L}$, defined in (5):

$$X(p) := \partial_x + a(p)\partial_t\,, \qquad Y(p) := \partial_y + b(p)\partial_t\,, \tag{8}$$

where a and b are defined in (3).

For a given function $u : \Omega \to \mathbb{R}$ we will write X instead of $X(\nabla u)$. Analogous abbreviations will be used for Y. Then we have

$$a = Yu\,, \qquad b = -Xu\,, \tag{9}$$

$$\mathcal{L}u = (X^2u + Y^2u)(1 + u_t^2)\,, \tag{10}$$

$$[X, Y] = -\frac{\mathcal{L}u}{1 + u_t^2}\partial_t. \tag{11}$$

The left-hand side of (11) stands for the Lie-bracket of the first order differential operators X and Y defined in (8). By using identities (9) and (10) the prescribed Levi-curvature equation (1) can be written as

$$X^2u + Y^2u = k(\xi\,, u)\frac{(1 + a^2 + b^2)^{3/2}}{(1 + u_t^2)^{1/2}}. \tag{12}$$

This structure has been very recently used by two of the authors in [5] to prove a first regularity result for viscosity solutions:

THEOREM. *Let us suppose $k \in C^1(\Omega \times \mathbb{R})$. Let u be a strong viscosity solution of (1). Then $Xu\,, Yu \in H^1_{\mathrm{loc}}(\Omega)$ and u satisfies (12) pointwise almost everywhere.*

Here $H^1_{\mathrm{loc}}(\Omega)$ denotes the classical Sobolev space of order 1.

Without any extra condition on the curvature k it seems that the previous result cannot be improved. On the other hand the following theorem was known ([2], see also [6]):

THEOREM. *If $k \in C^\infty(\Omega \times \mathbb{R})$ and never vanishes in $\Omega \times \mathbb{R}$, then every $C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ classical solution to (1), with $\alpha > 1/2$, is of class C^∞ in Ω.*

In this paper we fill the gap between these results and prove the following theorem.

THEOREM 1.1. *Let* $k \in C^\infty(\Omega \times \mathbb{R})$ *be such that* $k(\xi, s) \neq 0$ *for every* $(\xi, s) \in \Omega \times \mathbb{R}$. *Then every strong viscosity solution to* (1) *is of class* $C^{2,\alpha}$, *with* $\alpha > 1/2$, *and solves the equation in the classical sense.*

We will sketch the proof of this theorem in the next sections. Together with Theorem 4 in [10] and Theorem 1.1 in [2] our theorem immediately gives the following C^∞-solvability result for (4):

COROLLARY 1.1. *Let* Ω *and* k *satisfy the hypotheses of Theorem* 4 *in* [10]. *Let us also assume* $k \in C^\infty(\Omega \times \mathbb{R})$ *and* $k(\xi, s) \neq 0$ *for any* $(\xi, s) \in \Omega \times \mathbb{R}$. *Then, for every* $\phi \in C^{2,\alpha}(\partial\Omega)$ *the Dirichlet problem* (4) *has a solution* $u \in C^\infty(\Omega) \cap \mathrm{Lip}(\bar{\Omega})$.

We would like to emphasize some important differences between our Corollary 1.1 and the result of Bedford and Gaveau [1]. The interior regularity of the solutions given in [1] for $k = 0$ strictly depends on the regularity of their values at the boundary, and this result cannot be improved, since every C^2 function u depending only on the variable t solves equation (1). On the contrary, if k is of class C^∞ and everywhere different from zero, our solutions are of class C^∞ for every boundary data of class $C^{2,\alpha}$.

The sketch of the proof of Theorem 1.1 is organized in three steps. In Sections 2 and 3 we show some a priori estimates respectively in L^p and C^α for the solutions to the regularized equation (7). In Section 4 we apply these estimates to the viscosity solution u, and prove the stated result.

2. L^p ESTIMATES

In this Section we assume that u is a fixed C^∞ solution of the regularized equation (7). Because of identities (6) and (10), u is a solution of

$$X^2 u + Y^2 u + T_\varepsilon^2 u = k(\cdot\,, u)\frac{(1 + a^2 + b^2)^{3/2}}{(1 + u_t^2)^{1/2}}. \tag{13}$$

where T_ε denotes the following first order differential operator

$$T_\varepsilon(p) := \frac{\varepsilon}{(1 + p_3^2)^{1/2}}\partial_t.$$

We fix two open sets Ω_1 and Ω_2 subsets of Ω such that $\overline{\Omega}_1 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega$, and a function $\phi \in C_0^\infty(\Omega_2)$ such that $\phi \equiv 1$ in Ω_1. We also asssume that there exists a constant $M > 0$, only depending on Ω_1, Ω_2 such that

$$||u||_{L^\infty(\Omega_2)} + ||\nabla u||_{L^\infty(\Omega_2)} + ||a||_{H^1(\Omega_2)} + ||b||_{H^1(\Omega_2)} \leq M. \tag{14}$$

Hereafter we will denote by c a positive constant only depending on M.

In order to state the main steps of the proof it is convenient to introduce some notations.

DEFINITION 2.1. *For the fixed function u we call intrinsec $\mathcal{L}-$gradient the operator*

$$\nabla_{\mathcal{L}} = (X, Y, T_\varepsilon)$$

and denote by $D^{(i)}$, $i = 1, 2, 3$, *its components:* $D^{(1)} = X$, $D^{(2)} = Y$, $D^{(3)} = T_\varepsilon$. *Moreover, we set* $D^{(4)} = \partial_t$. *For every* $i = 1 \cdots 3$, *we say that* D^i *has length 1 while, as suggested by identity* (11), D^4 *has length 2. Then, for any multi-index* $i = (i_1, \cdots, i_p) \in \{1, 2, 3, 4\}^p$ *we call*

$$D^i = D^{i_1} \circ \cdots \circ D^{i_p},$$

a derivation operator of length $l(D^i) = l(D^{i_1}) + \cdots + l(D^{i_p})$.

By differentiating equation (13) we prove that all the derivatives of u and the function

$$v := \arctan u_t$$

are solutions of a linear equation of the following type:

$$X^2 w + Y^2 w + T_\varepsilon^2 w = f \tag{15}$$

with different right–hand sides f. Then we prove the following result.

PROPOSITION 2.1. *Any solution* $w \in C^\infty(\Omega)$ *of* (15) *satisfies the following estimate*

$$\int \Big(|\nabla_{\mathcal{L}} w_t|^2 + |\nabla_{\mathcal{L}} v|^2 w_t^2\Big)\phi^2 \le c \int \Big(w_t^2 |\nabla_{\mathcal{L}} \phi|^2 + \phi^2\Big) + \int \partial_t f w_t \phi^2. \tag{16}$$

Similar inequalities are also satisfied if we replace w_t *with* Xw, Yw *or* $T_\varepsilon w$.

We next use in an essential way the hypothesis $k \neq 0$, and prove the following statement.

PROPOSITION 2.2. *For every function* $w \in C^\infty$ *we have*

$$\int |w_t|^3 \phi^6 \le c \int \Big(|\nabla_{\mathcal{L}} w_t|^2 + (|\nabla_{\mathcal{L}} v|^2 + |\nabla_{\mathcal{L}} w|^2) w_t^2\Big)\phi^6 + c \tag{17}$$

where $c > 0$ *only depends on* M *and* k. *An analogous inequality is also satisfied if we replace* w_t *with* Xw, Yw *or* $T_\varepsilon w$.

Indeed, since k never vanishes, then

$$\int |w_t|^3 \phi^6 \le c \int \frac{|k|(1 + a^2 + b^2)^{3/2}}{(1 + u_t^2)^{1/2}} \partial_t w |w_t| w_t \phi^6$$

(by using (7), (6), (11), and keeping in mind that $u_{tt}/(1 + u_t^2) = v_t$)

$$= -segn(k) \int [X, Y] w |w_t| w_t \phi^6 + segn(k) \varepsilon^2 \int v_t w_t^2 w_t \phi^6. \tag{18}$$

We now split in two integrals the first addend in (18) by replacing the commutator $[X, Y]$ with $XY - YX$. Integrating by parts each of them, after some computations we get the claimed estimate.

The preceeding inequalities underline the crucial role of the function $v = \arctan u_t$ in the regularization procedure. Indeed, if we apply the Propositions 2.1 and 2.2 to the function v itself, we obtain a L^2 estimate for Xv_t and Yv_t, and a L^3 estimate for v_t. Since $v_t = \frac{u_{tt}}{1+u_t^2}$ then, due to Definition 2.1, v_t has to be considered a derivative of length 4 of u, while Xv_t and Yv_t are derivatives of length 5 of the same function. Once proved the summability of these derivatives of the function v, inequalities (16) and (17) can be iterated so as to obtain analogous estimates for any derivation operator of length 5 and 4.

PROPOSITION 2.3. *There exists a constant c, only dependent on M such that*

$$||D^i u||_{L^2(\Omega_1)} + ||D^j u||_{L^3(\Omega_1)} \leq c \tag{19}$$

for any differential operator D^i of length 5 *and* D^j *of length* 4.

3. A PRIORI HÖLDER ESTIMATES

In this Section we still denote by u a solution of the regularized equation (13), satisfying (14). For any fixed $\xi_0 = (x_0, y_0, t_0) \in \Omega_1$ we define two frozen vector fields

$$X_{\xi_0} = \partial_x + (a(\nabla u(\xi_0)) + 2(y - y_0))\partial_t, \quad Y_{\xi_0} = \partial_y + (b(\nabla u(\xi_0)) - 2(x - x_0))\partial_t.$$

These are C^∞ vector fields, and their coefficients are bounded by a constant only dependent on M. Since the commutator $[X_{\xi_0}, Y_{\xi_0}] = -4\partial_t$, and any commutator of higher length is zero, then the Lie algebra generated by X_{ξ_0} and Y_{ξ_0} is an Heisenberg algebra. The space $\mathbb{R}^3$, with the associated group law is an homogeneous Lie group, with homogeneous dimention $N = 4$ (see [7]). We will denote d_{ξ_0} its natural distance and for any $\xi, \xi_0 \in \Omega_2$ we will set

$$d(\xi, \xi_0) = d_{\xi_0}(\xi, \xi_0) + d_\xi(\xi_0, \xi).$$

By identities (9), the inequality (19) can be considered as a summability estimate on the derivatives of length 3 and 4 of the coefficients a and b. As a consequence, we prove the following crucial estimate.

PROPOSITION 3.1. *There exists a positive constant c only dependent on M such that*

$$|a(\xi) - a(\xi_0)| \leq c d(\xi, \xi_0), \quad |b(\xi) - b(\xi_0)| \leq c d(\xi, \xi_0)$$

for every $\xi, \xi_0 \in \Omega_1$. *Here* $a(\xi)$ *and* $b(\xi)$ *stand for* $a(\nabla u(\xi))$ *and* $b(\nabla u(\xi))$ *respectively.*

Once proved that the coefficients a and b of the vector fields X and Y are Lipschitz continuous with respect to d, a Sobolev-Morrey type imbedding theorem follows.

THEOREM 3.1. *There exists a constant c only dependent on M such that*

(*i*) *if* $1 < p < N$, $r = Np/(N-p)$ *then*

$$||w||_{L^r(\Omega_1)} \leq c ||\nabla_{\mathcal{L}} w||_{L^p(\Omega_2)} \quad \forall\, w \in C_0^\infty(\Omega_2),$$

(*ii*) *if* $p > N$, $q > N/2$, *and* $\beta = min(1 - N/p, 2 - N/q)$ *then*

$$|w(\xi) - w(\xi_0)| \leq c d^\beta(\xi, \xi_0)\Big(||\nabla_{\mathcal{L}} w||_{L^p(\Omega_2)} + ||\partial_t w||_{L^q(\Omega_2)}\Big)$$

for every $w \in C_0^\infty(\Omega_2)$ *and for every* $\xi, \xi_0 \in \Omega_1$.

Let us only sketch the proof of the second assertion, essentially based on the following Poincaré type inequality. There exists a positive constant c, only dependent on M, such that for every $\xi \in \Omega_1$, for every d-ball B, such that $cB \subset \Omega_2$

$$|w(\xi) - w_B| \leq c \int_{cB} (d(\xi, \xi'))^{-N+1} |\nabla_{\mathcal{L}} w(\xi')| d\xi' + c \int_{cB} (d(\xi, \xi'))^{-N+2} |\partial_t w(\xi')| d\xi', \tag{20}$$

where w_B denotes the mean value of w on the ball B. From this inequality, the assertion follows by standard techniques.

We next apply our embedding theorem to the derivatives of the function u in order to obtain an estimates of theirs d-Hölder continuity norm.

THEOREM 3.2. *For every* $\alpha \in]0, 1[$ *there exists a constant c only dependent on M such that for every j such that* $l(D^j) = 2$,

$$|D^j u(\xi) - D^j u(\xi_0)| \leq c d^\alpha(\xi, \xi_0) \quad \forall \xi, \xi_0 \in \Omega_2.$$

Indeed, by Theorem 3.1

$$||D^i u||_{L^4(\Omega_1)} \leq c ||\nabla_{\mathcal{L}} D^i u||_{L^2(\Omega_2)}.$$

If D^i is a differential operator of length 4, then, by (19) we deduce that there exists a constant c, only dependent of M such that

$$||D^i u||_{L^4(\Omega_1)} \leq c. \tag{21}$$

Applying again Theorem 3.1 we deduce that for every $p > 1$, for every i' such that $l(D^{i'}) = 3$

$$||D^{i'} u||_{L^p(\Omega_1)} \leq c.$$

In particular

$$||\nabla_{\mathcal{L}} D^j u||_{L^p(\Omega_1)} \leq c.$$

for all p, for every j such that $l(D^j) = 2$. On the other hand, using (21), and keeping in mind the fact that $l(\partial_t) = 2$, we have:

$$||\partial_t D^j u||_{L^4(\Omega_1)} \leq c.$$

Applying the second part of the Theorem 3.1 we get

$$|D^j u(\xi) - D^j u(\xi')| \leq c d^\alpha(\xi, \xi'), \quad \forall \xi, \xi' \in \Omega_1,$$

for any $\alpha \in]0, 1[$, for any j such that $l(D^j) = 2$.

4. $C^{2,\alpha}$ ESTIMATE

In this Section we show how to conclude the proof of Theorem 1.1. Here u denotes a viscosity solution to (1), and (u_n) its approximating sequence. Hence there exists a constant $M_0 > 0$ independent of n such that

$$||u_n||_{L^\infty(\Omega_2)} + ||\nabla u_n||_{L^\infty(\Omega_2)} \leq M_0$$

and, by the result in [5],

$$||a_n||_{H^1(\Omega_2)} + ||b_n||_{H^1(\Omega_2)} \leq M_1 ,$$

where $a_n = a(\nabla u_n)$ and $b_n = b(\nabla u_n)$.

We will also denote by D_n the differential operator defined in terms of u_n as in Definition 2.1 and we call d_n the distance related to u_n, as described at the beginning of Section 3. By Theorem 3.2 there exists a constant c only dependent on $M = M_0 + + M_1$ such that

$$|D^i_n u_n(\xi) - D^i_n u_n(\xi_0)| \leq c d^\alpha_n(\xi , \xi_0) ,$$

for any i such that $l(D^i_n) = 2$. Letting $n \to \infty$ we obtain

$$|D^i u(\xi) - D^i u(\xi_0)| \leq c d^\alpha(\xi , \xi_0) ,$$

for every i such that $l(D^i) = 2$.

From these estimates, arguing as in [2], we finally obtain our main regularity result.

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Pervenuta l'11 febbraio 1999,
in forma definitiva il 20 aprile 1999.

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