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A $\mathcal{U}_q(\mathfrak{sl}(2))$-representation with no quantum symmetric algebra

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Algebra. — *A $\mathcal{U}_q(\mathfrak{sl}(2))$-representation with no quantum symmetric algebra.* Nota (*) di Olivia Rossi-Doria, presentata dal Corrisp. C. De Concini.

Abstract. — We show by explicit calculations in the particular case of the 4-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{sl}(2))$ that it is not always possible to generalize to the quantum case the notion of symmetric algebra of a Lie algebra representation.

Key words: Quantized enveloping algebra; Representation; Symmetric algebra.

Riassunto. — *Una rappresentazione di $\mathcal{U}_q(\mathfrak{sl}(2))$ che non ha algebra simmetrica quantica.* Si dimostra, mediante calcoli espliciti per la rappresentazione irriducibile di dimensione 4 di $\mathcal{U}_q(\mathfrak{sl}(2))$, che non è sempre possibile generalizzare al caso quantico la nozione di algebra simmetrica di una rappresentazione di un'algebra di Lie.

1. Preliminaries

The quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ was introduced by Drinfeld and Jimbo in [1, 2] and [3]. Refer to these papers for the general definition and to Rosso [6] for the representation theory of finite dimensional $\mathcal{U}_q(\mathfrak{g})$-modules. As for the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2))$, recall that it is the Hopf algebra over $\mathbb{C}(q)$ generated by $E, F, K^{\pm 1}$ with the following relations

$$
\begin{aligned}
KEK^{-1} &= q^2 E \\
KFK^{-1} &= q^{-2} F \\
[E, F] &= \frac{K - K^{-1}}{q - q^{-1}}
\end{aligned}
\tag{1.1}
$$

with comultiplication Δ, antipode S and counit ε given by

$$
\begin{aligned}
\Delta(E) &= E \otimes 1 + K \otimes E \\
\Delta(F) &= F \otimes K^{-1} + 1 \otimes F \\
\Delta(K) &= K \otimes K
\end{aligned}
\tag{1.2}
$$

$$
\begin{aligned}
S(E) &= -K^{-1} E \\
S(F) &= -FK \\
S(K) &= K^{-1}
\end{aligned}
\tag{1.3}
$$

$$
\begin{aligned}
\varepsilon(E) &= \varepsilon(F) = 0 \\
\varepsilon(K) &= 1\,.
\end{aligned}
\tag{1.4}
$$

(*) Pervenuta in forma definitiva all'Accademia il 23 ottobre 1998.

The results about finite dimensional representation of $\mathcal{U}_q(\mathfrak{sl}(2))$ are completely analogous to those for the enveloping algebra $\mathcal{U}(\mathfrak{sl}(2))$. In particular one has the following

THEOREM 1. (1) *Let V be an irreducible representation of $\mathcal{U}_q(\mathfrak{sl}(2))$. Then the action of both E and F on V is nilpotent,* $\dim(\ker E) = \dim(\ker F) = 1$. *For any vector $v \in \ker E$, $Kv = \varepsilon q^n v$ with $\varepsilon = \pm 1$ and n the non negative integer such that* $\dim(V) = n + 1$. *The pair (ε, n) is called the highest weight of V.*

(2) *Given a pair (ε, n), with ε as above and n a non negative integer, there exists, up to isomorphism, a unique irreducible representation $V_{(\varepsilon, n)}$ of highest weight (ε, n).*

For the proof see [6] or for example [4, Theorem VI.3.5]. In what follows for the representations of $\mathcal{U}_q(\mathfrak{sl}(2))$ let V_n for $V_{(1, n)}$.

2. THE QUANTUM SYMMETRIC ALGEBRA

We want to give a definition of a quantum analogue of the symmetric algebra of a Lie algebra representation.

Let's start with the classical case. For a representation V of $\mathcal{U}(\mathfrak{g})$ the symmetric algebra $S(V)$ is the quotient of the tensor algebra $T(V)$ by the ideal generated by the antisymmetric component $\Lambda^2(V)$ of the tensor product representation $V \otimes V$. This is exactly the polynomial algebra in the classes $x_1, \dots, x_n$ of the basis vectors $v_1, \dots, v_n$ for V. Since $\Lambda^2(V)$ is a submodule of $V \otimes V$ the Hopf algebra action of $\mathcal{U}(\mathfrak{g})$ on the tensor algebra transfers to the quotient algebra $S(V)$. There is a natural way to characterize the elements of $\Lambda^2(V)$. Since $\mathcal{U}(\mathfrak{g})$ is cocommutative the involutive automorphism σ of $V \otimes V$, given by the switch of the two tensor factors, commutes with the action of $\mathcal{U}(\mathfrak{g})$ on $V \otimes V$. Therefore $V \otimes V$decomposes as a representation of $\mathcal{U}(\mathfrak{g})$ in two subrepresentation of eigenvalues ± 1 for σ. The antisymmetric tensors are the eigenvectors of eigenvalue -1, while the symmetric tensors are the eigenvectors of eigenvalue $+1$.

Now for a representation V of the quantum group $\mathcal{U}_q(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g}$ we define

DEFINITION 2. *The quantum symmetric algebra $S_q(V)$ of V is a $\mathbb{Z}^+$-graded algebra over $\mathbb{C}(q)$ with a graded $\mathbb{C}[q, q^{-1}]$-subalgebra $\overline{S_q(V)}$ such that*

(1) *the degree-one part of $S_q(V)$ is V;*

(2) *$\overline{S_q(V)}$ is free as a graded $\mathbb{C}[q, q^{-1}]$-module and in each degree has the same dimension of the polynomial algebra in* $\dim V$ *variables;*

(3) *the natural map $\overline{S_q(V)} \bigotimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}(q) \to S_q(V)$ is an isomorphism;*

(4) *$\overline{S_q(V)}/(q-1)$ is the classical symmetric algebra of the $\mathcal{U}(\mathfrak{g})$-representation corresponding to V,* i.e. *the polynomial algebra over $\mathbb{C}$ in* $\dim V$ *variables;*

(5) *there is an Hopf algebra action of $\mathcal{U}_q(\mathfrak{g})$ on the algebra $S_q(V)$ such that an integral form of $\mathcal{U}_q(\mathfrak{g})$ acts on $\overline{S_q(V)}$ inducing the usual action of $\mathcal{U}(\mathfrak{g})$ on the symmetric algebra $\overline{S_q(V)}/(q-1)$.*

In order to satisfy condition (5) we have to consider the quotient of the tensor algebra $T(V)$ modulo the ideal generated by the $\mathcal{U}_q(\mathfrak{g})$-submodule of $V \otimes V$ corresponding to the antisymmetric component in the classical case.

The composition $\check{R}$ of the R-matrix with the switch σ of the two tensor factors plays in the quantum case the role of σ and commutes with the action of $\mathcal{U}_q(\mathfrak{g})$ on $V \otimes V$, which therefore decomposes into irreducible submodules that are eigenspaces for $\check{R}$. More precisely

Proposition 2 [5, (1.38)]. *Let $V = U^\lambda$ be the irreducible module with highest weight λ for $\mathcal{U}_q(\mathfrak{g})$. Then $\check{R}$ satisfies the equation*

$$\prod_\nu \left(\check{R} \pm q^{\frac{1}{2}(\nu,\nu+2\rho)-(\lambda,\lambda+2\rho)} \right) = 0$$

where ν ranges over the irreducible summands U^ν of $V \otimes V$ and ρ is half the sum of the positive roots. The sign is positive for the U^ν corresponding to the antisymmetric summands and negative for the U^ν corresponding to the symmetric summands.

As a consequence, for an irreducible representation V, we need to take the quotient of the tensor algebra $T(V)$ by the ideal generated by the eigenvectors for $\check{R}$ of eigenvalues with negative sign, although it is not necessarily true that this verifies all the requested properties for $S_q(V)$. On the contrary we shall give in the next section a counterexample to the existence of the quantum symmetric algebra.

3. The counterexample

Let's consider the irreducible $n+1$-dimensional representation of $\mathcal{U}_q(\mathfrak{sl}(2))$. There is a quantum version of the classical Clebsch-Gordan formula. It gives the decomposition $V_n \otimes V_n \simeq V_{2n} \oplus V_{2n-2} \oplus \cdots \oplus V_0$ into irreducible summands. It is easy to check from the formula for a highest weight vector of V_{2n-2p} (cf. [4, Chapter VII])

$$v^{(2n-2p)} = \sum_{i=0}^{p} (-1)^i \frac{[p]!\,[n-p+i]!\,[n-i]!}{[p-i]!\,[i]!\,[n]!\,[n-p]!} q^{i(n-i+1)} v_i \otimes v_{p-i}\,, \tag{3.1}$$

where $v_i = F^i v_0$ and v_0 is the highest weight vector for V_n, that the V_{2n-2p} with odd p are the ones with negative eigenvalues for $\check{R}$. In the case of V_3 the tensor product $V_3 \otimes V_3$ decomposes into irreducible summands $V_6 \oplus V_4 \oplus V_2 \oplus V_0$, and V_4 and V_0 generate the ideal of relations for the quantum symmetric algebra, if it exists. A basis of V_4 is given by the 2-tensors

$$\begin{aligned}
&v_0 \otimes v_1 - q^3 v_1 \otimes v_0\\
&v_0 \otimes v_2 + (q^{-1} - q^3) v_1 \otimes v_1 - v_2 \otimes v_0\\
&v_0 \otimes v_3 + (q^{-1} + q - q^3) v_1 \otimes v_2 + (q^{-2} - 1 - q^2) v_2 \otimes v_1 - q^{-3} v_3 \otimes v_0\\
&(q + q^{-1}) v_1 \otimes v_3 + (q^{-2} - q^4) v_2 \otimes v_2 - (q + q^{-1}) v_3 \otimes v_1\\
&(q^2 + q^{-2}) v_2 \otimes v_3 - (q + q^5) v_3 \otimes v_2
\end{aligned} \tag{3.2}$$

and V_0 has generator given by $v_0\otimes v_3 - q^3 v_1\otimes v_2 + q^4 v_2\otimes v_1 - q^3 v_3\otimes v_0$. It follows that the quotient algebra is generated on $\mathbb{C}(q)$ by x_0, x_1, x_2, x_3 with the following commutation relations:

$$
\begin{aligned}
x_1x_0 &= q^{-3}x_0x_1\\
x_2x_0 &= x_0x_2 + (q^{-1}-q^3)x_1^2\\
x_2x_1 &= \frac{1+q^2+q^4-q^6}{q^3(q^2+1)}x_1x_2 + \frac{q^3-q^{-3}}{q^3(q^2+1)}x_0x_3\\
x_3x_0 &= \frac{q^4+q^2+1-q^{-2}}{q^3(q^2+1)}x_0x_3 + \frac{1-q^6}{q^2(q^2+1)}x_1x_2\\
x_3x_1 &= x_1x_3 + \frac{1-q^6}{q(q^2+1)}x_2^2\\
x_3x_2 &= q^{-3}x_2x_3.
\end{aligned}
\tag{3.3}
$$

Now it is not difficult to see that this algebra does not satisfy the property (2) of the definition of the quantum symmetric algebra for V_3. Let's take the monomial $x_2x_1x_0$ and write it in terms of ordered monomials. There are two ways of doing it and they give different results. We obtain

$$x_2x_1x_0 = \frac{(q^6-1)}{q^9(q^2+1)}x_0^2x_3 + \frac{(1+q^2+q^4-q^6)}{q^6(q^2+1)}x_0x_1x_2 + \frac{(1-q^4)}{q^4}x_1^3, \tag{3.4}$$

if we commute first x_1 with x_0, and

$$x_2x_1x_0 = \frac{(q^6-1)(q^4+q^2+1-q^{-2})}{q^9(q^2+1)^2}x_0^2x_3 + \frac{-1+q^2+2q^4+4q^6-q^{10}-q^{12}}{q^8(q^2+1)^2}x_0x_1x_2 + \frac{(1+q^2+q^4-q^6)(1-q^4)}{q^4(q^2+1)}x_1^3 \tag{3.5}$$

if we start commuting x_2 with x_1.

Then we find the following linear dependence relation on $\mathbb{C}(q)$ between ordered monomials of degree 3

$$\frac{(q^6-1)^2}{q^{11}(q^2+1)^2}x_0^2x_3 - \frac{(q^6-1)^2}{q^8(q^2+1)^2}x_0x_1x_2 + (q^2-1)^2x_1^3 = 0. \tag{3.6}$$

Remark. Clearly for $q=1$ this is an empty relation.

As a consequence of relation (3.6) the dimension of the degree-three component is less than the dimension of the degree-three component of the polynomial algebra in four variables. We have then finally proved the following

Theorem 3. *For $\mathcal{U}_q(\mathfrak{g})=\mathcal{U}_q(\mathfrak{sl}(2))$ and $V=V_3$ the quantum symmetric algebra $S_q(V)$ does not exist.*

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