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A Note on height pairings on polarized abelian varieties


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**Abstract.** — Let $A$ be an abelian variety defined over a number field $k$. In this short Note we give a characterization of the endomorphisms that preserve the height pairing associated to a polarization. We also give a functorial interpretation of this result.

**Key words:** Abelian varieties; Height pairings; Endomorphisms.

**Riassunto.** — Una Note sugli accoppiamenti associati alle altezze sulle varietà abeliane polarizzate. Sia $A$ una varietà abeliana definita su un campo di numeri $k$. In questa breve Nota diamo una caratterizzazione degli endomorfismi che lasciano invariata la forma bilineare associata all’altezza canonica definita a partire da una polarizzazione. Diamo, inoltre, un’interpretazione funtoriale di questo risultato.

1. Introduction

Let $A$ be an abelian variety defined over a number field $k$. In [4], A. Néron introduced the canonical height pairing: $\langle \ , \ \rangle : \widehat{A}(\overline{k}) \times A(\overline{k}) \rightarrow \mathbb{R}$, between $A$ and its dual abelian variety, $\widehat{A}$. This pairing satisfies the following fundamental properties:

**HP1** The pairing $\langle \ , \ \rangle$ is bilinear.

**HP2** If $f : A \rightarrow B$ is a $k$-homomorphism, then

$$\langle f(b) , a \rangle = \langle b , f(a) \rangle$$

for all $a \in A(\overline{k})$ and $b \in \widehat{B}(\overline{k})$.

A *polarization* on $A$ is an isogeny $\lambda : A \rightarrow \widehat{A}$, such that $\lambda_{\overline{z}} = \varphi_{\mathcal{L}}$ for some ample invertible sheaf on $A_{\overline{z}} = A \times_{\text{spec} k} \text{spec} \overline{k}$ where $\varphi_{\mathcal{L}}(a) = t_{a}^* \mathcal{L} \otimes \mathcal{L}^{-1}$, and $t_{a}$ is the translation by $a$ map (cf. [3]). To any polarization $\lambda$ we can then associate a symmetric bilinear pairing

$$\langle \ , \ \rangle_{\lambda} : A(\overline{k}) \times A(\overline{k}) \rightarrow \mathbb{R} ,$$

$$a , b \longmapsto \langle \lambda(a) , b \rangle ,$$

which is called the *height pairing associated to $\lambda$*.

In this short Note we give a characterization of the endomorphisms that preserve the height pairing associated to a polarization (actually, we prove a slightly more general result see the proposition below). In order to state our functorial interpretation of this result we need to define the category $\mathfrak{hgm}$, of height Galois modules. This is done as follows:

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objects: pairs \( (G, \langle , \rangle_G) \), where \( G \) is an abelian group endowed with an action of \( \Gamma = \text{Gal}(\overline{k}/k) \); \( \langle , \rangle_G \) is a symmetric bilinear real valued pairing on \( G \), which is \( \Gamma \)-equivariant if we let \( \Gamma \) act trivially on \( \mathbb{R} \).

morphisms: \( \Gamma \)-equivariant homomorphisms \( f : G \to H \) such that
\[
\langle f(a), f(a') \rangle_H = \langle a, a' \rangle_G
\]
for all \( a, a' \in G \).

**Theorem.** Let \( k \) be a number field and \( \Gamma = \text{Gal}(\overline{k}/k) \). Let \( \text{pav}_k \) denote the category of polarized abelian varieties defined over \( k \). Then, the functor \( F : \text{pav}_k \to \text{hgm} \), which assigns to \( (A, \lambda) \) the height Galois module \( F(A, \lambda) = (A(\overline{k}), \langle , \rangle_\lambda) \), and to any morphism \( f : A \to B \) the induced morphism \( f : A(\overline{k}) \to B(\overline{k}) \), is fully faithful.

2. Height pairings and homomorphisms

Let \( (A, \lambda), (B, \eta) \), be two polarized abelian varieties defined over \( k \). An homomorphism of polarized abelian varieties is an homomorphism \( f : A \to B \) such that \( \lambda = f \circ \eta \circ f \). We denote by \( \text{Hom}_k((A, \lambda), (B, \eta)) \) the set formed by those homomorphisms from \( (A, \lambda) \) to \( (B, \eta) \) that are defined over \( k \).

Our aim is the following:

**Proposition.** Let \( g : A \to B \) be a morphism defined over \( k \). Suppose that \( \lambda, \eta \) are polarizations on \( A \) and \( B \) respectively. Let \( g = t_u \circ f \), where \( u \in B(k) \), and \( f : A \to B \) is a homomorphism. Then
\[
\langle g(a), g(a') \rangle_\eta = \langle a, a' \rangle_\lambda \quad \forall \ a, a' \in A(\overline{k})
\]
if and only if \( f \in \text{Hom}_k((A, \lambda), (B, \eta)) \) and \( u \) is a torsion point.

**Corollary.** Let \( (A, \lambda) \) be a polarized abelian variety and \( f \) an endomorphism of \( A \). Then
\[
\langle f(a), f(a') \rangle_\lambda = \langle a, a' \rangle_\lambda \quad \forall \ a, a' \in A(\overline{k})
\]
if and only if \( \lambda = f \circ \lambda \circ f \).

We need a preliminary lemma.

**Lemma.** Let \( \lambda \) and \( \eta \) be two polarizations on \( A \). Then \( \langle , \rangle_\lambda = \langle , \rangle_\eta \iff \lambda = \eta \).

**Proof.** Let \( \lambda \) and \( \eta \) be two polarizations on \( A \) such that \( \langle , \rangle_\lambda = \langle , \rangle_\eta \). Then
\[
\langle \lambda(a) - \eta(a), a' \rangle = \langle a, a' \rangle_\lambda - \langle a, a' \rangle_\eta = 0
\]
for all \( a' \) in \( A \). Since the kernel on each side of the Néron pairing is the torsion subgroup of \( A(\overline{k}) \) (see, e.g. [2, Theorem 5.6.3]) we find that for every \( a \in A(\overline{k}) \) there exists \( n \in \mathbb{Z} \), depending on \( a \), such that \( [n]\lambda(a) = [n]\eta(a) \). Let \( C \) be a simple abelian
subvariety of $A$. Then $\lambda(a) = \eta(a)$ for infinitely many $a \in C(\overline{K})$, and thus $\lambda$ and $\eta$ coincide when restricted to $C$. The Poincaré reducibility theorem yields the lemma. □

**Proof of the Proposition.** We start by proving the proposition for homomorphisms. If $a, a' \in A(\overline{K})$, then

$$\langle f(a), f(a') \rangle_\eta = \langle \eta(f(a)) \rangle = \langle (\widehat{f} \circ \eta \circ f)(a), a' \rangle = \langle a, a' \rangle_{\widehat{f} \circ \eta \circ f}.$$  

If $f \in \text{Hom}(\langle A, \lambda \rangle, \langle B, \eta \rangle)$, then $\widehat{f} \circ \eta \circ f = \lambda$, and hence $f$ has the desired property. Conversely, suppose that $\langle f(a), f(a') \rangle_\eta = \langle a, a' \rangle_\lambda$ for all $a, a' \in A(\overline{K})$. Then, by (2.2), we have $\langle , \rangle_{\widehat{f} \circ \eta \circ f} = \langle , \rangle_\lambda$. Therefore, $\widehat{f} \circ \eta \circ f = \lambda$ by the above lemma. Now we deal with the general case. Suppose $g = t_{u^0} f$, where $f \in \text{Hom}(\langle A, \lambda \rangle, \langle B, \lambda \rangle)$, and $u \in B(k)$ a torsion point. The bilinearity of $\langle g(a), g(b) \rangle_\eta$ combined with (2.2), gives

$$\langle g(a), g(b) \rangle_\eta = \langle f(a), f(a') \rangle_\eta = \langle a, a' \rangle_{\widehat{f} \circ \eta \circ f}.$$  

Finally, suppose that $g = t_{u^0} f$ satisfies (2.1). Then $\langle g(a), u \rangle_\eta = 0$ for all $a \in A(\overline{K})$. Using the bilinearity of $\langle , \rangle_\eta$ and (2.2), we find

$$\langle a, a' \rangle_\lambda = \langle f(a), f(a') \rangle_\eta = \langle a, a' \rangle_{\widehat{f} \circ \eta \circ f}.$$  

It then follows from the lemma above that $\widehat{f} \circ \eta \circ f = \lambda$. It remains to show that $u$ is a torsion point. Let $L$ be an ample invertible sheaf on such that $\eta_L = \varphi_{L'}$. Then $L'_0 = L \otimes L^-$ (where $L^- = [-1]^* L$) is ample and symmetric. Since $L \otimes (L^-)^{-1}$ is algebraically equivalent to zero, we have that $\varphi_L = \varphi_{L'}$, so

$$\langle u, u \rangle_{\varphi_{L'_0}} = 2 \langle u, u \rangle_{\varphi_L} = 2 \langle u, u \rangle_\eta = \langle 0, 0 \rangle_\lambda = 0.$$  

But, $L'_0$ being symmetric, $\langle u, u \rangle_{\varphi_{L'_0}}$ is proportional to the canonical height associated to $L_0$, which, for an ample symmetric divisor, vanishes only on torsion points. □

**A functorial interpretation**

The above proposition has a functorial interpretation as we shall now show. Let $\text{pav}_k$, denote the category whose objects are polarized abelian varieties defined over $k$, and let $\text{hgm}$ be the category of height Galois modules, which we defined in the Introduction. Given $(G, \langle , \rangle_G)$ and $(H, \langle , \rangle_H)$, we denote by $\text{Hom}_h(G, H)$ the set of morphisms (in the category of height Galois modules) from $(G, \langle , \rangle_G)$ to $(H, \langle , \rangle_H)$. We define a functor $F : \text{pav}_k \rightarrow \text{hgm}$ as follows: given a polarized abelian variety $(A, \lambda)$ defined over $k$ we let $F(A, \lambda) = (A(\overline{K}), \langle , \rangle_\lambda)$. Given a $k$-morphism $f : (A, \lambda) \rightarrow (B, \eta)$ then $F(f)$ is just the restriction of $f$ to $A(\overline{K})$, which is $\Gamma$-equivariant because $f$ is defined over $k$. Moreover $F(f)$ satisfies $(\ast)$ by the above proposition. The only thing that remains to be verified is that $(A(\overline{K}), \langle , \rangle_\lambda)$ is an object of $\text{hgm}$, i.e. that $\langle , \rangle_\lambda$ is invariant under the action of $\Gamma$. Recall that the canonical height pairing coincides with the canonical height associated to the Poincaré bundle on $\tilde{A} \times A$ (this can be seen by comparing [4, Section 14] and [5, Section 3.4], also cf. [7, Appendix A1,
But the Poincaré bundle is defined over the ground field, and absolute projective heights are invariant under the action of $\Gamma$ (see [6, Lemma 5.10]). Therefore, $\langle \sigma(a), \sigma(a') \rangle$ is invariant under the action of $\Gamma$. Thus

$$\langle \sigma(a), \sigma(a') \rangle = \langle \lambda(\sigma(a)), \lambda(\sigma(a')) \rangle = \langle \lambda(a), \lambda(a') \rangle = \langle a, a' \rangle,$$

where we used that $\lambda$ is defined over $k$.

**Theorem.** Let $k$ be a number field. Then the functor $F : \text{pav}_k \rightarrow \text{hgm}$ is fully faithful.

**Proof.** The faithfulness of $F$ follows directly from the above proposition. To prove that $F$ is full let $A$ and $B$ be two abelian varieties defined over $k$, and suppose that $\tilde{f}$ is in $\text{Hom}_k(A(\bar{k}), B(\bar{k}))$. In particular $\tilde{f}$ is a $\Gamma$-equivariant homomorphism from $A(\bar{k})$ to $B(\bar{k})$. It was shown by Faltings (as a non-trivial consequence of the Tate's conjecture) that the natural injection $\text{End}_k(A) \rightarrow \text{End}_\Gamma(A(\bar{k}))$ is an isomorphism (see [1, Theorem 5, p. 205]). Applying this result to $A \times B$, we find that there exists an $f$ belonging to $\text{Hom}_k(A, B)$ such that $F = \tilde{f}$. By assumption $\tilde{f}$ belongs to $\text{Hom}_k(A(\bar{k}), B(\bar{k}))$, thus $\langle a, a' \rangle = \langle f(a), f(a') \rangle$ for all $a, a' \in A(\bar{k})$. By the above proposition $f$ belongs to $\text{Hom}_k((A, \lambda), (B, \eta))$. □

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