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Regularity results for infinite dimensional diffusions. A Malliavin calculus approach

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Analisi matematica. — *Regularity results for infinite dimensional diffusions. A Malliavin calculus approach.* Nota di STEFANO BONACCORSI e MARCO FUHRMAN, presentata (*) dal Corrisp. G. Da Prato.

ABSTRACT. — We prove some smoothing properties for the transition semigroup associated to a nonlinear stochastic equation in a Hilbert space. The proof introduces some tools from the Malliavin calculus and is based on a integration by parts formula.

KEY WORDS: Transition semigroups; Strong Feller property; Logarithmic derivative; Malliavin calculus.

RIASSUNTO. — *Risultati di regolarità per diffusioni infinito-dimensionali. Una applicazione del calcolo di Malliavin.* Si dimostrano certe proprietà di regolarità per il semigruppo di transizione di una diffusione in dimensione infinita. La dimostrazione introduce tecniche del calcolo di Malliavin e si basa su una formula di integrazione per parti.

1. INTRODUCTION

Let $X(t,x)$ be the solution of the stochastic differential equation

$$\text{(1)} \qquad \begin{cases} dX(t) = \Big(AX(t) + RF(X(t))\Big)\, dt + R\, dW(t) \\ X(0) = x \in H, \end{cases}$$

in a real separable Hilbert space H. W is a U-valued cylindrical Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathcal{F}_t$, $t \geq 0$. We fix the following condition on the coefficients in problem (1).

HYPOTHESIS 1. *The operator* A *is the infinitesimal generator of a* C_0 *semigroup* $S(t)$, $t \geq 0$, *on* H. $R \in L(U, H)$, *the space of linear bounded operators from* U *to* H.
The mapping $F : H \to U$ *is Lipschitz continuous: there exists* $L > 0$ *such that*

$$|F(x) - F(y)| \leq L|x - y|, \qquad x, y \in H.$$

The operators Q_t, *defined by*

$$Q_t x = \int_0^t S(s) R R^* S(s)^* x \, ds, \quad x \in H, \, t \geq 0,$$

are trace class.

Given the conditions in Hypothesis 1, for every $x \in H$ there exists a unique mild solution $X(t,x)$ of equation (1), *i.e.* a predictable process in H satisfying, for $t \geq 0$,

$$\text{(2)} \quad X(t,x) = S(t)x + \int_0^t S(t-s) RF(X(s,x))\, ds + \int_0^t S(t-s) R\, dW(s), \quad \mathbb{P} - a.s.$$

(*) Nella seduta del 12 marzo 1999.

and such that $\sup_{t\in[0,T]} \mathbb{E}|X(t,x)|^p < \infty$, for any $p \in [1,\infty)$ and $T > 0$; X is unique up to modification (see [2, Theorem 9.1]). For simplicity, we will sometimes write $X(t)$ or X_t instead of $X(t,x)$.

If, in addition, there exists $\alpha > 0$ such that

$$\text{Trace} \int_0^t s^{-\alpha} S(s) RR^* S(s)^* \, ds < \infty, \tag{3}$$

then X has a continuous version.

We denote by Z the Gaussian process associated with the case $F = 0$: $Z(t,x) = S(t)x + W_A^R(t)$, where $W_A^R(t) = \int_0^t S(t-r) R \, dW(r)$ is called stochastic convolution. We remark that Q_t is the covariance operator of $W_A^R(t)$.

We will present two main results. First, under the condition that F has a continuous and bounded Fréchet derivative, we will show that an integration by parts formula holds for the law of the H-valued random variable X_t, $t > 0$ being fixed. This is proved in Section 3, by an application of the Malliavin calculus; a short review of basic definitions of the Malliavin calculus is given in Section 2, where we also show that X_t has a Malliavin derivative.

Second, we are interested in conditions implying the strong Feller property of the transition semigroup of the Markov process X (compare the definition in Section 4). It is known that Z is strongly Feller if and only if, for any $t > 0$, $\operatorname{Im} S(t) \subset \operatorname{Im} Q_t^{1/2}$, where Im denotes the image of an operator. Under the same condition we can prove that the strong Feller property holds for X as well, as a consequence of an estimate on the Fréchet derivative of the transition semigroup that may have an interest in itself (Section 4).

Notice that, although the results are already known (cf. [4]), the proof presented here is new and much simpler. In particular, the proof of the integration by parts formula clearly relates properties of the Malliavin derivative of X_t to controllability property of an appropriate control system, and may lead to further extensions.

The results of this paper have been developed while the second author was visiting the University of Trento and have been included in the PhD thesis of the first author [1]. The authors wish to thank Luciano Tubaro for the interesting discussions.

2. Some tools from the Malliavin calculus

We need to recall some facts on the Malliavin calculus. Our approach is the same as in [5, 6].

We take a Hilbert space E (below $E = H$ or $E = \mathbb{R}$), we fix $T > 0$ and, for $g \in L^2([0,T];U)$, we define $W(g) = \int_0^T \langle g(s), dW(s)\rangle$. We denote by $\mathcal{S}$ the set of functions of the form

$$\phi = f(W(g_1), W(g_2), \dots, W(g_n)),$$

where $n \in \mathbb{N}$, $g_1, \dots, g_n \in L^2([0,T];U)$, f is a continuously differentiable real function on $\mathbb{R}^n$, such that f and its partial derivatives $\partial_i f$ have polynomial growth.

Now consider the set $\mathcal{S}_E$ of all functions $F : \Omega \to E$ of the form

$$F = \sum_{i=1}^{n} e_i \phi_i ,$$

where $n \in \mathbb{N}$, $e_i \in E$, $\phi_i \in \mathcal{S}$. $\mathcal{S}_E$ is a subspace of $L^2(\Omega; E)$, the space of E-valued random variables with square summable norm (similar notation is used below). We define an operator D on $\mathcal{S}_E$ setting

$$D_s F = \sum_{i=1}^{n} \partial_i f(W(g_1), W(g_2), \dots, W(g_n))\, e_i \otimes g_i(s), \quad s \in [0, T].$$

We recall that the Hilbert space tensor product $E \otimes U$ can be identified with the space $L_2(U, E)$ of Hilbert-Schmidt operators from U to E. With this agreement, D is an operator from $\mathcal{S}_E \subset L^2(\Omega; E)$ to the space $L^2(\Omega \times [0, T]; L_2(U, E))$, the latter being endowed with the norm

$$\|\Phi\|^2_{L^2(\Omega\times[0,T];L_2(U,E))} = \mathbb{E} \int_0^T \|\Phi(s)\|^2_{L_2(U,E)}\, ds.$$

It is know that D is closable. Still denoting by D its closure, and denoting by $\mathbb{D}^{1,2}(E)$ its domain, D is an operator

$$D : \mathbb{D}^{1,2}(E) \subset L^2(\Omega; E) \to L^2(\Omega \times [0, T]; L_2(U, E)).$$

It is easy to see that the stochastic convolution $W_A^R(t)$ belongs to $\mathbb{D}^{1,2}(H)$ and we have $D_s W_A^R(t) = S(t - s)R$, for $s < t$, whereas $D_s W_A^R(t) = 0$ for $s > t$.

We recall that, if $F \in \mathbb{D}^{1,2}(E)$ is $\mathcal{F}_t$-adapted, then $D_s F = 0$ $\mathbb{P}$-a.s. for $s \in (t, T]$.

When $E = \mathbb{R}$, we write $L^2(\Omega)$ and $\mathbb{D}^{1,2}$ instead of $L^2(\Omega; \mathbb{R})$ and $\mathbb{D}^{1,2}(\mathbb{R})$, and we identify $L_2(U, \mathbb{R})$ with U, so that

$$D : \mathbb{D}^{1,2} \subset L^2(\Omega) \to L^2(\Omega \times [0, T]; U).$$

We denote by

$$\delta : \operatorname{dom} \delta \subset L^2(\Omega \times [0, t]; U) \to L^2(\Omega)$$

the adjoint operator of D. δ is called Skorohod integral and satisfies by definition the equality

$$\mathbb{E}\left[\int_0^T \langle D_s F, k(s)\rangle_U\, ds\right] = \mathbb{E}\left[F\, \delta(k)\right], \quad F \in \mathbb{D}^{1,2},\ k \in \operatorname{dom} \delta. \tag{4}$$

We recall that if k is an adapted process in $L^2(\Omega \times [0, T]; U)$, then $k \in \operatorname{dom} \delta$ and

$$\delta(k) = \int_0^T \langle k(s), dW(s)\rangle.$$

A chain rule for the Malliavin derivative holds: if $F \in \mathbb{D}^{1,2}(H)$, $\varphi : H \to \mathbb{R}$ is bounded and continuous together with its Fréchet derivative, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$D\varphi(F) = \varphi'(F)DF$, *i.e.* for a.e. $(\omega, s) \in \Omega \times [0, T]$, $D_s\varphi(F)$ is the element of U such that

$$\langle D_s\varphi(F), u\rangle = \langle \varphi'(F), [D_sF] \cdot u\rangle, \qquad u \in U.$$

Similar statements hold true for φ taking values in a Hilbert space.

Below we are interested in proving the existence of the Malliavin derivative of X_t, for t fixed. We need the following.

HYPOTHESIS 2. *We assume that* $F : H \to H$ *has first Fréchet derivative* F' *bounded and continuous. Using Hypothesis 1 we get that* $\sup_{x\in H} \|\frac{\mathrm{d}}{\mathrm{dx}} F(x)\| \le L$.

PROPOSITION 1. *Under the conditions in Hypotheses 1 and 2, for every* $t \ge 0$ *the random variable* X_t *belongs to* $\mathbb{D}^{1,2}(H)$ *and its Malliavin derivative* DX_t *satisfies the following equation: for a.e.* $s \in [0, t]$, $\mathbb{P}$*-a.s.,*

$$D_sX_t = S(t-s)R + \int_s^t S(t-r)RF'(X(r))D_sX_r\, dr. \tag{5}$$

PROOF. Let us fix $T > 0$. For an arbitrary H-valued process Y let us denote

$$\Gamma(Y)_t = S(t)x + \int_0^t S(t-r)F(Y_r)\, dr + W_A^R(t), \qquad t \in [0, T].$$

Let us define inductively $X_t^0 = 0$, $X_t^{n+1} = \Gamma(X^n)_t$, $t \in [0, T]$. Then it is known that X^n converges to the solution of the equation (2), more precisely that $\|X_t^n - X_t\|_{L^2(\Omega;H)} \to 0$ as $n \to \infty$, uniformly in t. We will prove inductively that $X_t^n \in \mathbb{D}^{1,2}(H)$ and $\sup_{t\in[0,T]} \psi_n(t) \le K$, where K is a constant independent of n and ψ_n is defined as

$$\psi_n(t) := \mathbb{E}\int_0^t \|D_sX_t^n\|^2_{L_2(U,H)} ds.$$

The inductive statement being trivial for X^0, let us assume that it holds for X^n. By the chain rule and a previous remark, for a.e. $s \le t$,

$$D_s\Big(S(t-r)F(X_r^n)\Big) = S(t-r)F'(X_r^n)D_sX_r^n, \qquad D_sW_A^R(t) = S(t-s)R.$$

Then it is easy to prove that $X_t^{n+1} \in \mathbb{D}^{1,2}(H)$ and for a.e. $s \in [0, t]$, $\mathbb{P}$-a.s.,

$$D_sX_t^{n+1} = S(t-s)R + \int_s^t S(t-r)F'(X_r^n)D_sX_r^n dr.$$

It follows that

$$\begin{aligned} \|D_sX_t^{n+1}\|_{L_2(U,H)} &\le \|S(t-s)R\|_{L_2(U,H)} + \int_s^t \|S(t-r)F'(X_r^n)\|_{L(H,H)}\|D_sX_r^n\|_{L_2(U,H)}\, dr \le \\ &\le \|S(t-s)R\|_{L_2(U,H)} + C_1\int_s^t \|D_sX_r^n\|_{L_2(U,H)}\, dr, \end{aligned} \tag{6}$$

where $C_1 := \sup_{t\in[0,T]} \|S(t-r)\|_{L(H,H)} \sup_{x\in H} \|F'(x)\|_{L(H,H)} < \infty$. Squaring both sides of (6), integrating with respect to s over $[0\,,\,t]$ and taking into account that

$$\int_0^t \|S(t-s)R\|^2_{L_2(U,H)}\,ds = \int_0^t \mathrm{Trace}\,S(t-s)RR^*S(t-s)^*\,ds = \mathrm{Trace}\,Q_t \leq \mathrm{Trace}\,Q_T$$

a standard calculation leads to

$$\int_0^t \|D_sX_t^{n+1}\|^2_{L_2(U,H)}\,ds \leq 2\,\mathrm{Trace}\;Q_T + 2C_1^2\,T\int_0^t\int_0^r \|D_sX_r^n\|^2_{L_2(U,H)}\,ds\,dr.$$

Taking expectation of both sides we arrive at $\psi_{n+1}(t) \leq K_1 + K_2\int_0^t \psi_n(r)dr$, where $K_1\,,\,K_2$ are independent of n. It follows that $\psi_n(t) \leq K_1 e^{K_2 t} \leq K$, where K is a constant independent of n.

Now we have proved that $\|DX_t^n\|_{\mathbb{D}^{1,2}(H)}$ is bounded uniformly in t and n. Since $X_t^n \to X_t$ in $L^2(\Omega; H)$, uniformly in t, a standard argument based on the closedness of the operator D shows that $X_t \in \mathbb{D}^{1,2}(H)$ and $\sup_{t\in[0\,,\,T]}\|DX_t\|_{\mathbb{D}^{1,2}(H)} < \infty$. It is then possible to apply D to both sides of the equality (2), and this gives (5). □

Remark 2. There exists a continuous version of the process D_sX_t, $t \geq s \geq 0$, with values in $L(U, H)$. Indeed, let us fix a version of the process X, and let us fix $\omega \in \Omega$. Then it is easy to see that there exists a unique continuous $L(U, H)$-valued function Y_{ts} such that

$$Y_{ts} = S(t-s)R + \int_s^t S(t-r)RF'(X(\omega\,,\,r))\,Y_{rs}\,dr, \qquad t \geq s \geq 0.$$

Clearly, Y depends on ω and comparing with (5) we see that for every $t \geq 0$ we have $D_sX_t = Y_{ts}$, for a.e. $s \in [0\,,\,t]$, $\mathbb{P}$-a.s.

3. Integration by parts and logarithmic derivatives

We are now ready to state one of our main results. It will be refined below in Proposition 6. In the following $C_b^m(H)$, $m = 0\,,\,1\,,\,\ldots$, denotes the set of real functions on H that are bounded and continuous together with their Fréchet derivatives up to the order m.

Proposition 3. *Under the conditions in Hypotheses 1 and 2, for each* $t > 0$ *and* $h \in \mathrm{Im}\,Q_t^{1/2}$ *there exists a* U*-valued process* $k(s)$, $s \in [0\,,\,t]$, *adapted and mean square integrable, such that the following identity holds for every* $\varphi \in C_b^1(H)$:

$$\mathbb{E}\left(\varphi(X(t))\int_0^t \langle k(s)\,,\,dW(s)\rangle\right) = \mathbb{E}\langle \varphi'(X(t))\,,\,h\rangle. \tag{7}$$

As a first step in the proof of Proposition 3 we prove the following result.

Lemma 4. *For arbitrary* $t \in (0\,,\,T]$ *and* $h \in \mathrm{Im}\,Q_t^{1/2}$ *there exists a square integrable adapted process* $k(s)\,,\,s \in [0\,,\,t]$, *such that:*

$$\int_0^t [D_sX_t]\cdot k(s)\,ds = h. \tag{8}$$

If $h \in \operatorname{Im} Q_t$ then we can take

$$k(s) = R^* S(t-s)^* Q_t^{-1} h - F'(X(s)) Q_s S(t-r)^* Q_t^{-1} h, \qquad s \in [0, t]. \tag{9}$$

PROOF. Let us fix $\omega \in \Omega$. By Remark 2 we can assume that $D_s X_t$ is a continuous function with values in $L(U, H)$, defined for $t \geq s \geq 0$. Let us define

$$y(s) = \int_0^s [D_r X_s] \cdot k(r) dr, \qquad s \in [0, t].$$

Then, applying both sides of the equality (5) to $k(r)$ and integrating over $[0, s]$, we obtain

$$y(s) = \int_0^s S(s-r) Rk(r) dr + \int_0^s S(s-r) RF'(X(r)) y(r) dr, \qquad s \in [0, t].$$

We will say that y is the mild solution of the

$$\begin{cases} \dfrac{d}{ds} y(s) = Ay(s) + RF'(X(s)) y(s) + Rk(s) \\ y(0) = 0 . \end{cases} \tag{10}$$

We are looking for $k \in L^2([0, t]; U)$ such that (8) holds. So we are considering (10) as a control system and our aim is to prove that every $h \in \operatorname{Im} Q_t^{1/2}$ is a reachable state.

Let us consider the following auxiliary control system:

$$\begin{cases} \dfrac{d}{ds} z(s) = Az(s) + Ru(s) \\ z(0) = 0 . \end{cases} \tag{11}$$

Its mild solution is defined to be the function

$$z(s) = \int_0^s S(s-r) Ru(r) dr \qquad s \in [0, t],$$

and the function u is allowed to range over all controls $u \in L^2([0, t]; U)$.

Now we claim that, if z is the solution corresponding to a control u then there exists a control k such that $z(\cdot) = y(\cdot)$. Indeed, if we take $k(\cdot) = u(\cdot) - F'(X(\cdot)) z(\cdot)$, then

$$y(s) = \int_0^s S(s-r) Ru(r)\, dr + \int_0^s S(s-r) RF'(X(r)) [y(r) - z(r)]\, dr,$$

$$y(s) - z(s) = \int_0^s S(s-r) RF'(X(r)) [y(r) - z(r)]\, dr.$$

By Gronwall's lemma, $y(s) = z(s)$ for $0 \leq s \leq t$, and the claim is proved.

However, for the linear autonomous control system (11) it is well known (see *e.g.* [2, Appendix B.3]) that its reachable set at time t coincides with $\operatorname{Im} Q_t^{1/2}$, *i.e.*

$$\operatorname{Im} Q_t^{1/2} = \left\{ h : \exists u \in L^2([0, t]; U) \text{ and } z(t) = h \right\}.$$

Now for $h \in \operatorname{Im} Q_t^{1/2}$ take a control u such that for the corresponding solution we have $z(t) = h$. We set

$$k(s) = u(s) - F'(X(s)) z(s), \qquad s \in [0, t], \tag{12}$$

and for the corresponding solution y we have $y(t) = z(t) = h$, as shown before, so that (8) holds.

Clearly, k depends on the ω chosen at the beginning. Since u (resp. z) is a square integrable (resp. continuous) deterministic function, it follows immediately from (12) that k is a square integrable process adapted to the filtration.

Finally, if $h \in \operatorname{Im} Q_t$ we can give an explicit formula for the control u:

$$u(s) = R^* S(t-s)^* Q_t^{-1} h, \quad s \in [0, t].$$

It follows that

$$z(s) = \int_0^s S(s-r) RR^* S(t-r)^* Q_t^{-1} h \, dr = Q_s S(t-s)^* Q_t^{-1} h. \quad \square$$

We can now give the proof of Proposition 3. Notice that the chain rule for the Malliavin derivative operator implies that, for any process $k \in L^2(\Omega \times [0, T]; U)$:

$$\langle D_s \varphi(X(t)), k(s) \rangle = \langle \varphi'(X(t)), [D_s X(t)] \cdot k(s) \rangle \tag{13}$$

Integrating both sides in $s \in [0, t]$ and taking the expectation in (13) we obtain

$$\mathbb{E} \int_0^t \langle D_s \varphi(X(t)), k(s) \rangle \, ds = \mathbb{E} \, \langle \varphi'(X(t)), \int_0^t [D_s X(t)] \cdot k(s) \, ds \rangle.$$

Now we take for k the process given by the previous lemma. Since k is a square integrable adapted process, it belongs to the domain of the Skorohod integral and we obtain

$$\mathbb{E} \left[\langle \varphi'(X(t)), h \rangle \right] = \mathbb{E} \int_0^t \langle D_s \varphi(X(t)), k(s) \rangle \, ds =$$
$$= \mathbb{E} \left[\varphi(X(t)) \delta(k) \right] = \mathbb{E} \left[\varphi(X(t)) \int_0^t \langle k(s), dW(s) \rangle \right]. \quad \square$$

Our next aim is to give a more explicit expression to the stochastic integral appearing in Proposition 3. To this end we first need a lemma.

Lemma 5. *For* $t \geq s \geq 0$, *we have* $\operatorname{Im} e^{(t-s)A} Q_s^{1/2} \subset \operatorname{Im} Q_t^{1/2}$ *and*

$$\| Q_t^{-1/2} e^{(t-s)A} Q_s^{1/2} \| \leq 1.$$

Proof. By well known arguments (see *e.g.* [2, appendix B]) it suffices to prove

$$\| Q_s^{1/2} e^{(t-s)A^*} x \| \leq \| Q_t^{1/2} x \|, \quad s \in [0, t], \; x \in H.$$

i.e. $e^{(t-s)A} Q_s e^{(t-s)A^*} \leq Q_t$. This follows from the definition of Q_t, since

$$e^{(t-s)A} Q_s e^{(t-s)A^*} = \int_{t-s}^t e^{rA} RR^* e^{rA^*} \, dr = Q_t - Q_{t-s} \leq Q_t. \quad \square$$

To proceed further, let us consider again the stochastic convolution $W_A^R(t)$. Since Q_t is its covariance operator, then for any $h \in \operatorname{Im} Q_t^{1/2}$ we get

$$\mathbb{E}[\langle W_A^R(t)\,,\, Q_t^{-1/2}h\rangle^2] = \|h\|^2.$$

This means that the map $h \to \langle W_A^R(t)\,,\, Q_t^{-1/2}h\rangle$ can be extended to an isometry of H into $L^2(\Omega)$. In the following, this extension will be denoted $\langle W_A^R(t)\,,\, Q_t^{-1/2}[\cdot]\rangle$.

Proposition 6. *Under the conditions in Hypotheses 1 and 2, for each $t > 0$ and $h \in \operatorname{Im} Q_t^{1/2}$ the following identity holds for every $\varphi \in C_b^1(H)$:*

$$\mathbb{E}\,\langle \varphi'(X(t))\,,\, h\rangle = \mathbb{E}\left(\varphi(X(t))\langle W_A^R(t)\,,\, Q_t^{-1/2}[Q_t^{-1/2}h]\rangle\right) + \\ - \mathbb{E}\left(\varphi(X(t))\int_0^t \langle F'(X(s))\,Q_s^{1/2}\left(Q_t^{-1/2}S(t-s)Q_s^{1/2}\right)^* Q_t^{-1/2}h\,,\, dW(s)\rangle\right).$$

Proof. Let us take a sequence $h_n \in \operatorname{Im} Q_t$ such that $Q_t^{-1/2}h_n \to Q_t^{-1/2}h$. Then by Proposition 3 and (9) we have

$$\mathbb{E}\langle \varphi'(X(t))\,,\, h_n\rangle = \mathbb{E}\left(\varphi(X(t))\langle W_A^R(t)\,,\, Q_t^{-1}h_n\rangle\right) + \\ - \mathbb{E}\left(\varphi(X(t))\int_0^t \langle F'(X(s))\,Q_s S(t-s)^* Q_t^{-1}h_n\,,\, dW(s)\rangle\right).$$

Now we let $n \to \infty$. Using the notation introduced before,

$$\langle W_A^R(t)\,,\, Q_t^{-1}h_n\rangle = \langle W_A^R(t)\,,\, Q_t^{-1/2}[Q_t^{-1/2}h_n]\rangle \to \langle W_A^R(t)\,,\, Q_t^{-1/2}[Q_t^{-1/2}h]\rangle\,,$$

in $L^2(\Omega)$, since $\langle W_A^R(t)\,,\, Q_t^{-1/2}[\cdot]\rangle$ is an isometry.

Then we have, by Lemma 5,

$$Q_s S(t-s)^* Q_t^{-1}h_n = Q_s^{1/2}\left(Q_t^{-1/2}S(t-s)Q_s^{1/2}\right)^* Q_t^{-1/2}h_n\,,$$

and we obtain

$$\mathbb{E}\left|\int_0^t \langle F'(X(s))\,Q_s S(t-s)^* Q_t^{-1}h_n\,,\, dW(s)\rangle + \\ - \int_0^t \langle F'(X(s))\,Q_s^{1/2}\left(Q_t^{-1/2}S(t-s)Q_s^{1/2}\right)^* Q_t^{-1/2}h\,,\, dW(s)\rangle\right|^2 = \\ = \mathbb{E}\int_0^t \|F'(X(s))\,Q_s^{1/2}\left(Q_t^{-1/2}S(t-s)Q_s^{1/2}\right)^*(Q_t^{-1/2}h_n - Q_t^{-1/2}h)\|^2 ds \to 0. \quad \square$$

As a consequence of the previous proposition we can draw some conclusions about the existence and summability properties of a logarithmic derivative of the law ν_t of the H-valued random variable $X(t)$. We recall that a probability measure μ on H is said to have a logarithmic derivative in the direction $h \in H$ if there exists $\beta^h \in L^1(H, d\mu)$ such that

$$\int_H \langle \varphi'(y)\,,\, h\rangle\,\mu(dy) = \int_H \varphi(y)\,\beta^h(y)\,\mu(dy)\,,$$

for every $\varphi \in C_b^1(H)$.

PROPOSITION 7. *Under the conditions in Hypotheses 1 and 2, for every* $t > 0$ *and* $h \in \operatorname{Im} Q_t^{1/2}$ *the measure* ν_t *has a logarithmic derivative* β^h *in the direction* h. *Moreover,* $\beta^h \in L^p(H, d\nu_t)$ *for every* $p \in [1, \infty)$.

PROOF. Let us consider the random variable

$$Z = \langle W_A^R(t), Q_t^{-1/2}[Q_t^{-1/2}h]\rangle - \int_0^t \langle F'(X(s))Q_s^{1/2}\left(Q_t^{-1/2}S(t-s)Q_s^{1/2}\right)^* Q_t^{-1/2}h, dW(s)\rangle,$$

and let us define $\beta^h(y) = \mathbb{E}(Z|X(t) = y)$, *i.e.* the conditional expectation of Z given $X(t) = y$. By the Burkholder-Davis-Gundy inequalities the stochastic integral above is in $L^p(\Omega)$ for every $p \in [1, \infty)$. Since $\langle W_A^R(t), Q_t^{-1/2}[Q_t^{-1/2}h]\rangle$ is gaussian, it follows that β^h is well defined and $\beta^h \in L^p(H, d\nu_t)$ for every $p \in [1, \infty)$. Finally we have, by Proposition 6,

$$\begin{aligned}\int_H \langle \varphi'(y), h\rangle \nu_t(dy) &= \mathbb{E}\langle \varphi'(X(t)), h\rangle = \mathbb{E}(\varphi(X(t))Z) = \\ &= \mathbb{E}\left(\varphi(X(t))\beta^h(X(t))\right) = \int_H \varphi(y)\beta^h(y)\mu(dy),\end{aligned}$$

which shows that β^h is indeed the logarithmic derivative of ν_t. □

4. THE STRONG FELLER PROPERTY

We denote by P_t, $t \geq 0$, the transition semigroup associated to (1):

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))], \qquad \varphi \in B_b(H), \tag{14}$$

where $B_b(H)$ denotes the set of real Borel bounded functions on H. We recall that a Markovian semigroup P_t, $t \geq 0$, is said to be a strongly Feller semigroup if, for arbitrary $\varphi \in B_b(H)$ and $t > 0$, $P_t\varphi \in C_b(H)$.

In order to prove the strong Feller property for the semigroup P_t, we are looking for an estimate of the Fréchet derivative $\frac{\mathrm{d}}{\mathrm{dx}}P_t\varphi$ of the form

$$\left|\frac{\mathrm{d}}{\mathrm{dx}}P_t\varphi(x)\right| \leq C_t \sup_{z \in H}|\varphi(z)|, \qquad t > 0, \ x \in H \tag{15}$$

for a constant $C_t > 0$ and valid for all smooth functions φ, say for $\varphi \in C_b^2(H)$. It is known (see [2]) that inequality (15) ensures the strong Feller property for the process. Moreover it is known that the strong Feller property for Z is equivalent to the following condition.

HYPOTHESIS 3. *For any* $t > 0$, $\operatorname{Im} S(t) \subset \operatorname{Im} Q_t^{1/2}$.

We are now ready to prove the main result of this section.

THEOREM 8. *Suppose that the conditions in Hypotheses 1, 2, 3, and* (3), *hold. Then the transition semigroup* P_t *related to equation* (1) *is strongly Feller.*

Proof. In order to prove the theorem it is sufficient to consider the special case $\varphi \in C_b^2(H)$. Actually, it follows from [3, Lemma 7.1.5] that the following conditions are equivalent:

1) $\forall \varphi \in C_b^2(H)$, $\forall x, y \in H$: $|P_t\varphi(x) - P_t\varphi(y)| \leq c\|\varphi\|_0|x-y|$;
2) $\forall \varphi \in B_b(H)$, $\forall x, y \in H$: $|P_t\varphi(x) - P_t\varphi(y)| \leq c\|\varphi\|_0|x-y|$.

We introduce Girsanov's transform. For any $x \in H$ the process

$$\Phi(t, x) = \exp\left(\int_0^t \langle F(Z(s, x)) , dW(s)\rangle - \frac{1}{2}\int_0^t \|F(Z(s, x))\|^2 \; ds\right) , \quad t > 0 ,$$

is a martingale, and under the probability $\Phi(t, x) \cdot \mathbb{P}$, the process

$$L(s) = W(s) - \int_0^s F(Z(r, x))\, dr, \qquad s \in [0 , t] ,$$

is a Wiener process. In particular, for any bounded measurable mapping $\varphi : H \to \mathbb{R}$:

$$(P_t\varphi)(x) = \mathbb{E}[\varphi(X(t, x))] = \mathbb{E}[\Phi(t, x)\varphi(Z(t, x))].$$

Below we denote by $\frac{\mathrm{d}}{\mathrm{dx}}$ the Fréchet derivative operator. Since $\frac{\mathrm{d}}{\mathrm{dx}}Z(t, x) \cdot h = e^{tA}h$ and

$$\begin{aligned}\langle \frac{\mathrm{d}\Phi}{\mathrm{dx}}(t, x) , h\rangle &= \Phi(t, x)\left(\int_0^t \langle \frac{\mathrm{d}F}{\mathrm{dx}}(Z(s, x))e^{sA}h, dW(s)\rangle \; + \right.\\ &\left. - \int_0^t \langle \frac{\mathrm{d}F}{\mathrm{dx}}(Z(s, x))e^{sA}h, F(Z(s, x))\rangle\, ds\right) = \\ &= \Phi(t, x)\int_0^t \langle \frac{\mathrm{d}F}{\mathrm{dx}}(Z(s, x))e^{sA}h, dL(s)\rangle ,\end{aligned}$$

we obtain

$$\begin{aligned}\langle \frac{\mathrm{d}}{\mathrm{dx}}P_t\varphi(x) , h\rangle &= \mathbb{E}\left[\Phi(t, x)\langle \frac{\mathrm{d}\varphi}{\mathrm{dx}}(Z(t, x)) , \frac{\mathrm{d}}{\mathrm{dx}}Z(t, x)\cdot h\rangle\right] + \\ &+ \mathbb{E}\left[\langle \frac{\mathrm{d}\Phi}{\mathrm{dx}}(t, x) , h\rangle\varphi(Z(t, x))\right] = \\ &= \mathbb{E}\left[\Phi(t, x)\langle \frac{\mathrm{d}\varphi}{\mathrm{dx}}(Z(t, x)) , e^{tA}h\rangle\right] + \\ &+ \mathbb{E}\left[\Phi(t, x)\int_0^t \langle \frac{\mathrm{d}F}{\mathrm{dx}}(Z(s, x))e^{sA}h, dL(s)\rangle\varphi(Z(t, x))\right] = \\ &= \mathbb{E}\left[\langle \frac{\mathrm{d}\varphi}{\mathrm{dx}}(X(t, x)) , e^{tA}h\rangle\right] + \\ &\qquad + \mathbb{E}\left[\int_0^t \langle \frac{\mathrm{d}F}{\mathrm{dx}}(X(s, x))e^{sA}h, dW(s)\rangle\varphi(X(t, x))\right].\end{aligned}$$

Hence, from Proposition 6 one gets, for any $h \in H$,

$$\begin{aligned}&\langle \frac{\mathrm{d}}{\mathrm{dx}}P_t\varphi(x) , h\rangle = \mathbb{E}\left[\varphi(X(t, x))\langle W_A^R(t) , Q_t^{-1/2}[Q_t^{-1/2}e^{tA}h]\rangle\right] + \\ &+\mathbb{E}\left[\varphi(X(t, x))\int_0^t \langle \frac{\mathrm{dF}}{\mathrm{dx}}(X(s, x))\left(e^{sA}h - Q_s^{1/2}\left(Q_t^{-1/2}e^{(t-s)A}Q_s^{1/2}\right)^* Q_t^{-1/2}e^{tA}h\right), dW(s)\rangle\right].\end{aligned}$$

The desired inequality (15) follows:

$$\left|\langle \frac{\mathrm{d}}{\mathrm{dx}} P_t\varphi(x)\,,\, h\rangle\right| \le \|\varphi\|_0\,|h|\,\Big(\big\|Q_t^{-1/2}e^{tA}\big\| + \\ + 2L\big\|Q_t^{-1/2}e^{tA}\big\|\left(\int_0^t \|Q_s^{1/2}\|^2\,ds\right)^{1/2} + 2L\left(\int_0^t \|e^{sA}\|^2\,ds\right)^{1/2}\Big) \quad \square$$

In this last part we show how to dispense with Hypothesis 2. The proof will follows the ideas in [7]. Since the constant C_t appearing in estimate (15) depends only on the Lipschitz constant L of the non-linear term $F(x)$, we may assume that F is only measurable and Lipschitz continuous. In [7] it is shown how to construct a sequence of good approximations $F_n(x)$, converging to $F(x)$ such that F_n fulfills Hypothesis 2 and the estimate (15) holds true with the same constant for all n. The desired result follows easily.

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